

EIGENVECTORS

A non-zero vector v is said to be an eigenvector of for the linear transformation L if $L(v) = \lambda \cdot v$. The scalar λ is called the eigenvalue.

How do we find eigenvectors and the corresponding eigenvalues? To explain, assume that L is represented by a 2×2 matrix, and vectors, as usual, by 2×1 column matrices. Then the equation $L(v) = \lambda \cdot v$ can be written as

$$\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$\begin{cases} (5 - \lambda)x + 3y = 0 \\ 3x + (5 - \lambda)y = 0 \end{cases}$$

which can be again written in matrix notation as

$$\begin{pmatrix} 5 - \lambda & 3 \\ 3 & 5 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In other words, x times the first column plus y times the second column is equal 0 vector. Clearly, this is only possible if both column-vectors line on the same line. It follows that determinant is zero:

$$\begin{vmatrix} 5 - \lambda & 3 \\ 3 & 5 - \lambda \end{vmatrix} = (5 - \lambda)^2 - 9 = 0.$$

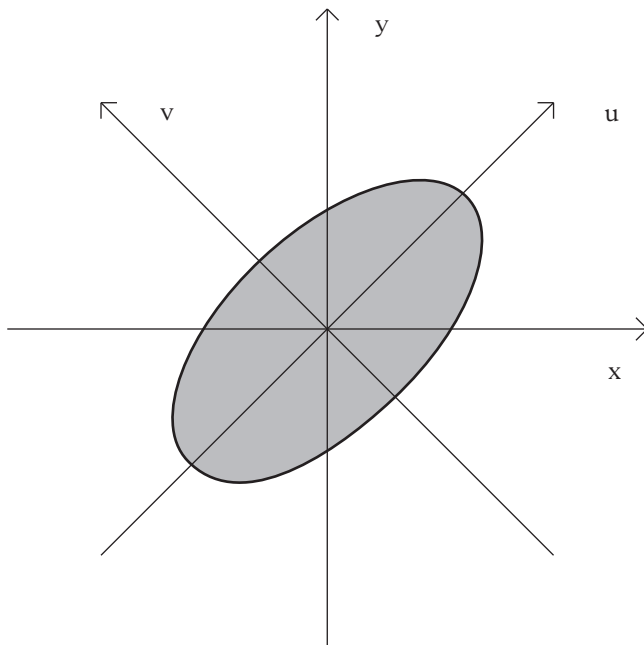
In particular, eigenvalues are zeroes of the *characteristic* polynomial $(5 - \lambda)^2 - 9$. One easily checks that the two zeros are 8 and 2. How do we find the corresponding eigenvectors? If we substitute $\lambda = 8$, then the pair of linear equations gives $-3x + 3y = 0$ twice. This can be easily solved by picking any value for y . For example, if we pick $y = 1$, then $x = 1$. Similarly, if we substitute $\lambda = 2$, then we get $3x + 3y = 0$. Here $y = 1$ and $x = -1$ as a solution. Summarizing, we have found eigenvalues and eigenvectors:

$$\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 8 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

1. CONICS

Ellipse, hyperbola and parabola (conics) are simplest curves. As you well know, they are described by quadratic equations. For example, $x^2 + 4y^2 = 4$ defines an ellipse with vertices $(\pm 2, 0)$ and $(0, \pm 1)$. On the other hand, if we change a sign in front of x^2 or y^2 , then we have an equation of hyperbola. But what happens if we introduce a mixed term xy in the equation? For example, what is the curve defined by

$$5x^2 - 6xy + 5y^2 = 8?$$



It turns out that the theory of eigenvalues and eigenvectors furnishes a simple way to understand this curve. First, note that the expression $5x^2 - 6xy + 5y^2$ can be in matrix notation written as

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Here the 2×2 matrix has the same eigenvectors and eigenvalues (albeit switched) as the one discussed in the previous section. Let u, v be the coordinate system whose axes are passing through the eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

This coordinate system is obtained by rotating the x, y coordinate system by 45° in the counter-clockwise direction. In particular, the two systems are related by

$$\begin{cases} x = u/\sqrt{2} - v/\sqrt{2} \\ y = u/\sqrt{2} + v/\sqrt{2}. \end{cases}$$

(The coefficients on the right hand sides of the two equations form the 2×2 matrix of rotation by 45° degrees.) Substituting these expressions for x and y into the original equation yields

$$2u^2 + 8v^2 = 8.$$

This clearly is equation of the ellipse with vertices $(\pm 2, 0)$ and $(0, \pm 1)$ in u, v coordinate system. It follows that the equation $5x^2 - 6xy + 5y^2 = 2$ defines an ellipse which is obtained by rotating the ellipse $x^2 + 4y^2 = 4$ for 45° in the counter clockwise direction.

Summarizing, not only the eigenvectors determine the axes of the ellipse, but the ratio of the eigenvalues ($8 : 2 = 4 : 1$) determines its focus.