MATH 5320 - MIDTERM EXAM, SOLUTIONS

1) (5 pts) Use the Eiseinstein Criterion to prove that $f(x) = x^4 + 1$ is irreducible.

Solution: $f(x)$ is irreducible if and only if $f(x+1)$ is. $f(x+1) = x^4 + 4x^3 + 6x^2 + 4x + 2$, and the criterion applies with $p = 2$.

2) (5 pts) Let $\varphi : \mathbb{Z}[x] \to \mathbb{C}$ be the map defined by $f(x) \mapsto f(\frac{1}{2} + i)$. Let I be the kernel of φ. Prove that I is principal, i.e. find a generator $g(x)$ and prove that any element in I is a multiple of $g(x)$.

Solution: $(x - (\frac{1}{2} + i))(x - (\frac{1}{2} - i)) = x^2 - x + \frac{5}{4}$. Thus $g(x) = 4x^2 - 4x - 5$ is the smallest degree, primitive polynomial in $\mathbb{Z}[x]$ such that $\frac{1}{2} + i$ is a root. Let $f(x) \in \mathbb{Z}[x]$. Then

$$f(x) = h(x)g(x) + ax + b$$

for some $h(x) \in \mathbb{Q}[x]$ and $a, b \in \mathbb{Q}$. If $f(x) \in I$ then, after substituting $x = \frac{1}{2} + i$ in the above equation, we get $0 = a(\frac{1}{2} + i) + b$. Since $\frac{1}{2} + i$ and 1 are linearly independent over $\mathbb{Q}$, $a = b = 0$. Thus $f(x) = h(x)g(x)$. Since $g(x)$ is primitive, $h(x) \in \mathbb{Z}[x]$ by Gauss's lemma.

3) (5 pts) Find the quotient $\mathbb{Z}^3/N$ (in the normal form) where N is a submodule generated by the columns of the matrix

$$\begin{pmatrix} 4 & 2 & 8 \\ 4 & 4 & 10 \\ 4 & 6 & 12 \end{pmatrix}$$

Solution: The elementary divisors are $2, 2, 0$. Thus $\mathbb{Z}^3/N \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$.

4) (5 pts) Prove that the ring $\mathbb{Z}[i]$ is euclidean with respect to the norm $N(x + yi) = x^2 + y^2$, i.e. for every $\alpha, \beta \in \mathbb{Z}[i]$, with $\beta \neq 0$, show that there exists $\gamma, \delta \in \mathbb{Z}[i]$, such that $\alpha = \gamma\beta + \delta$, and $N(\delta) < N(\beta)$.

Solution: Let $\Sigma = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ be the Voronoi polygon consisting of complex numbers z such that 0 is the closest lattice point. The polygon Σ is contained in the unit circle centered at 0 and $\mathbb{Z}[i]$-translates of Σ cover $\mathbb{C}$. Let $\gamma' = \frac{\alpha}{\beta}$. Then there exists $\gamma \in \mathbb{Z}[i]$ such that $\gamma' \in \gamma + \Sigma$. Let $\delta = \alpha - \gamma\beta$. Since $\gamma + \Sigma$ is contained in the unit circle centered at γ, $N(\gamma' - \gamma) < 1$. Then

$$N(\delta) = N(\beta)N(\gamma' - \gamma) < N(\beta).$$

5) (5 pts) See Noetherian ring in the book.