1) State the quadratic reciprocity law. Then calculate \((\frac{122}{1307})\).

Solution: Note that \(122 = 2 \cdot 61\), \(127 \equiv 7 \pmod{8}\) and \(61 \equiv 5 \pmod{8}\). In particular 2 is a square modulo 127.

\[
\left(\frac{122}{127}\right) = \left(\frac{2}{127}\right) \cdot \left(\frac{61}{127}\right) = \left(\frac{61}{61}\right) = 1
\]

2) Let \(n\) be a positive integer. Let \(p\) be an odd prime divisor of \(n^2 + 1\). What is \(p\) modulo 4?

Solution: \(p\) dividing \(n^2 + 1\) implies \(n^2 \equiv -1 \pmod{p}\). Hence \(-1\) is a square modulo \(p\). Hence \(p \equiv -1 \pmod{4}\).

3) Use the descent procedure to find a solution of the equation \(x^2 + y^2 = 61\) starting with \(11^2 + 1^2 = 2 \cdot 61\).

Solution: \(11 \equiv 1 \pmod{2}\) and \(1 \equiv 1 \pmod{2}\)

\[
\frac{11 + i}{1 + i} = 5 + 6i
\]

giving \(5^2 + 6^2 = 61\).

4) Prove by induction the closed formula for the sum of the first \(n\) powers of 2. (The first is \(2^0 = 1\):)

\[
1 + 2 + \cdots + 2^{n-1} = 2^n - 1.
\]

Solution: If \(n = 1\) then \(1 = 2^1 - 1\), so the formula is true for \(n = 1\). Assume the formula true for \(n\). Adding \(2^n\) to both sides yields

\[
1 + 2 + \cdots + 2^{n-1} + 2^n = 2^n - 1 + 2^n.
\]

Since \(2^n + 2^n = 2^n(1 + 1) = 2^n \cdot 2 = 2^{n+1}\), the formula is true for \(n + 1\).

5) Show that, if \((u, v)\) is a solution of \(x^2 - 3y^2 = -2\) then

\[
\begin{align*}
    u_1 &= 2u + 3v \\
    v_1 &= u + 2v
\end{align*}
\]

is another solution of \(x^2 - 3y^2 = -2\).

Solution:

\[
(2u + 3v)^2 - 3(u + 2v)^2 = 4u^2 + 12uv + 9v^2 - 3(u^2 + 4uv + 4v^2) = u^2 - 3v^2 = -2.
\]
NB: Solutions of the Pell equation \( x^2 - 3y^2 = 1 \) “act” on the solutions of \( x^2 - 3y^2 = -2 \). The first solution of \( x^2 - 3y^2 = 1 \) is \( 2 + \sqrt{3} \), and if \( u + v\sqrt{3} \) is a solution of \( x^2 - 3y^2 = -2 \), then

\[
(2 + \sqrt{3})(u + v\sqrt{3}) = (2u + 3v) + (u + 2v)\sqrt{3}
\]

is another solution of \( x^2 - 3y^2 = -2 \).

6) Recall that numbers \( P_n = \frac{3n^2 - n}{2} \) are pentagonal, while \( T_m = \frac{m(m+1)}{2} \) are triangular. The equation \( P_n = T_m \), after substituting \( n = (x + 1)/6 \) and \( m = (y - 1)/2 \), becomes the equation \( x^2 - 3y^2 = -2 \). Use the previous problem to generate solutions of this equation to find three pentagonal-triangular numbers. (\( P_1 = T_1 = 1 \) is one.)

Solution: The first pentagonal-triangular number \( P_1 = T_1 = 1 \) corresponds to the solution \((5, 3)\) of \( x^2 - 3y^2 = -2 \). The previous exercise generates the following solutions of \( x^2 - 3y^2 = -2 \):

\[
(19, 11), (71, 41), (265, 153), (989, 571) \ldots
\]

Only every other of these solutions can be converted into integral \((n, m)\): \((71, 41)\) into \((12, 20)\) and \((989, 571)\) into \((165, 285)\). It follows that

\[
P_{12} = T_{20} = 210 \quad \text{and} \quad P_{165} = T_{285} = 40755
\]

are two additional pentagonal-triangular numbers.