MATH 4400 - EXAM II SOLUTIONS

1) State the quadratic reciprocity law. Then calculate $\left(\frac{122}{1307}\right)$.

Solution: Note that $122 = 2 \cdot 61$, $127 \equiv 7 \pmod 8$ and $61 \equiv 5 \pmod 8$. In particular 2 is a square modulo 127.

$$\left(\frac{122}{127}\right) = \left(\frac{2}{127}\right) \cdot \left(\frac{61}{127}\right) = \left(\frac{61}{127}\right) = \left(\frac{127}{61}\right) = \left(\frac{5}{61}\right) = \left(\frac{61}{5}\right) = \left(\frac{1}{5}\right) = 1$$

2) Let n be a positive integer. Let p be an odd prime divisor of $n^2 + 1$. What is p modulo 4?

Solution: p dividing n^2+1 implies $n^2 \equiv -1 \pmod p$. Hence -1 is a square modulo p. Hence $p \equiv -1 \pmod 4$.

3) Use the descent procedure procedure to find a solution of the equation $x^2+y^2 = 61$ starting with $11^2 + 1^2 = 2 \cdot 61$.

Solution: $11 \equiv 1 \pmod 2$ and $1 \equiv 1 \pmod 2$

$$\frac{11+i}{1+i} = 5 + 6i$$

giving $5^2 + 6^2 = 61$.

4) Prove by induction the closed formula for the sum of the first n powers of 2. (The first is $2^0 = 1$.):

$$1 + 2 + \cdots + 2^{n-1} = 2^n - 1.$$

Solution: If $n = 1$ then $1 = 2^1 - 1$, so the formula is true for $n = 1$. Assume the formula true for n. Adding 2^n to both sides yields

$$1 + 2 + \cdots + 2^{n-1} + 2^n = 2^n - 1 + 2^n.$$

Since $2^n + 2^n = 2^n(1 + 1) = 2^n \cdot 2 = 2^{n+1}$, the formula is true for $n + 1$.

5) Show that, if (u, v) is a solution of $x^2 - 3y^2 = -2$ then

$$\begin{cases} u_1 = 2u + 3v \\ v_1 = u + 2v \end{cases}$$

is another solution of $x^2 - 3y^2 = -2$.

Solution:

$$(2u + 3v)^2 - 3(u + 2v)^2 = 4u^2 + 12uv + 9v^2 - 3(u^2 + 4uv + 4v^2) = u^2 - 3v^2 = -2.$$

NB: Solutions of the Pell equation $x^2 - 3y^2 = 1$ "act" on the solutions of $x^2 - 3y^2 = -2$. The first solution of $x^2 - 3y^2 = 1$ is $2 + \sqrt{3}$, and if $u + v\sqrt{3}$ is a solution of $x^2 - 3y^2 = -2$, then

$$(2 + \sqrt{3})(u + v\sqrt{3}) = (2u + 3v) + (u + 2v)\sqrt{3}$$

is another solution of $x^2 - 3y^2 = -2$.

6) Recall that numbers $P_n = \frac{3n^2-n}{2}$ are pentagonal, while $T_m = \frac{m(m+1)}{2}$ are triangular. The equation $P_n = T_m$, after substituting $n = (x+1)/6$ and $m = (y-1)/2$, becomes the equation $x^2 - 3y^2 = -2$. Use the previous problem to generate solutions of this equation to find three pentagonal-triangular numbers. ($P_1 = T_1 = 1$ is one.)

Solution: The first pentagonal-triangular number $P_1 = T_1 = 1$ corresponds to the solution $(5, 3)$ of $x^2 - 3y^3 = -2$. The previous exercise generates the following solutions of $x^2 - 3y^2 = -2$:

$$(19, 11), (71, 41), (265, 153), (989, 571) \ldots$$

Only every other of these solutions can be converted into integral (n, m): $(71, 41)$ into $(12, 20)$ and $(989, 571)$ into $(165, 285)$. It follows that

$$P_{12} = T_{20} = 210 \text{ and } P_{165} = T_{285} = 40755$$

are two additional pentagonal-triangular numbers.