# MATH 5320 - SOLUTIONS TO FIRST EXAM

1) (5 pts) Prove that $I = \{x + yi \in \mathbb{Z}[i] \mid x + y \equiv 0 \pmod 2\}$ is an ideal in $\mathbb{Z}[i]$.

Solution: The fact that $I$ is closed under addition is trivial. Let $x + yi \in I$ and $a + bi \in \mathbb{Z}[i]$. Then $(a + bi)(x + yi) = (ax - by) + (ay + bx)i$. As $+ = -$ modulo 2,

$$(ax - by) + (ay + bx) \equiv (ax + by) + (ay + bx) = (a + b)(x + y) \equiv (a + b) \cdot 0 \equiv 0 \pmod 2.$$

2) (5 pts) Let $\varphi : \mathbb{Z}[x] \to \mathbb{C}$ be the map defined by $f(x) \mapsto f(i)$. Let $I$ be the kernel of $\varphi$. Prove that $I$ is principal, i.e. find a generator $g(x)$ and prove that any element in $I$ is a multiple of $g(x)$.

Solution: Let $g(x) = x^2 + 1$, Then, since $g(x)$ is monic,

$$f(x) = h(x)g(x) + ax + b$$

for some $h(x) \in \mathbb{Z}[x]$ and $a, b \in \mathbb{Z}$. If $f(x) \in I$ then, after substituting $x = i$ in the above equation, we get $0 = ai + b$. Since $i$ and 1 are complex numbers linearly independent over $\mathbb{R}$, $a = b = 0$. Thus $f(x)$ is a multiple of $g(x)$.

3) (5 pts) Let $F$ be a field. Prove that the ring $F[x]/(x(1 - x))$ is isomorphic to $F \times F$.

First solution: Let $\varphi : F[x] \to F \times F$ be the map $\varphi(f(x)) = (f(0), f(1))$. This is a homomorphism. It is surjective, since $\varphi(a(1 - x) + bx) = (a, b)$. Let $f(x)$ be in the kernel of $\varphi$. Write

$$f(x) = x(1 - x)q(x) + ax + b$$

Then $f(0) = f(1) = 0$ imply that $b = 0$ and $a + b = 0$, i.e. the kernel of $\varphi$ consists of multiples of $x(1 - x)$. Thus $\varphi$ induces an isomorphism of $F[x]/(x(1 - x))$ and $F \times F$ by the first isomorphism theorem.

Second solution: Write $R = F[x]/(x(1 - x))$. Note that $x(1 - x) = 0$ is equivalent to $x = x^2$. Hence $x$ and $1 - x$ are non-trivial idempotents in $R$, and $R = A \times B$ where $A = (x)/(x(1 - x)) \cong F[x]/(1 - x) \cong F$ and $B = (1 - x)/(x(1 - x)) \cong F[x]/(x) \cong F$.

4) (5 pts) Let $F$ be a field. Prove that the ring $F[x]/(x^2)$ has no non-trivial idempotents.

Solution: Any element in $F[x]/(x^2)$ can be written, uniquely, as $a + bx$ for $a, b \in F$. Thus, if we write $e = a + bx$, then $e^2 = a^2 + 2abx$, and $e^2 = e$ implies $a = a^2$ and $b = 2ba$. These two equations have two solutions: $(a, b) = (0, 0)$ or $(1, 0)$, i.e. $e = 0$ or 1.

Second solution: Write $R = F[x]/(x^2)$. Note that $R$ has only one non-trivial ideal that corresponds to $(x) \subseteq F[x]$. If $R$ has non-trivial idempotents, then $R \cong A \times B$ and $R$ has two non-trivial ideals: $0 \times B$ and $A \times 0$. This is a contradiction.

5)( 5 pts) Determine the ring $\mathbb{Z}[x]/(2x+1, 10)$.

Solution: The idea is to divide $2x+1$ by 10. This can be done provided we take an appropriate multiple of $2x+1$:

$$5(2x+1) = 10x + 5.$$

Thus $5 \in (2x+1, 10)$ and the generator 10 can be replaced by 5. It follows that

$$\mathbb{Z}[x]/(2x+1, 10) = \mathbb{Z}[x]/(2x+1, 5) \cong \mathbb{F}_5[x]/(2x+1) \cong \mathbb{F}_5$$

where the last isomorphism is obtained by evaluating polynomials at 2, the root of $2x+1$ in $\mathbb{F}_5$.