DETERMINANTS

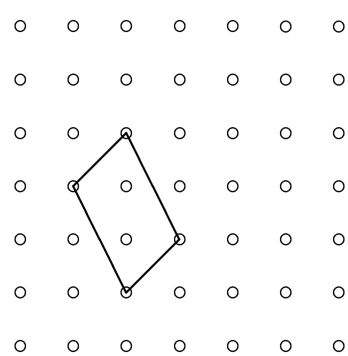
In this note we shall explain why the determinant of of a 2×2 matrix is equal to (oriented) area of the parallelogram spanned by the rows of the matrix. The explanation is based on the fact that adding a multiple of one row to another row does not change the determinant. Indeed

$$\begin{vmatrix} a & b \\ c+ra & d+rb \end{vmatrix} = a(d+rb) - b(d+rb) = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

On the other hand, row operations do not change the area of the corresponding parallelogram as we shall see on the following example. Consider the determinant

$$\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 3.$$

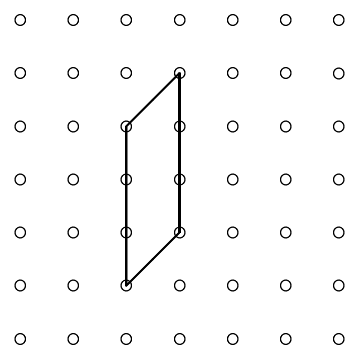
The rows of this 2×2 matrix span the following parallelogram:



We shall now perform row operations. First, add the first row to the second. As we know, this does not change the determinant

$$\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = 3.$$

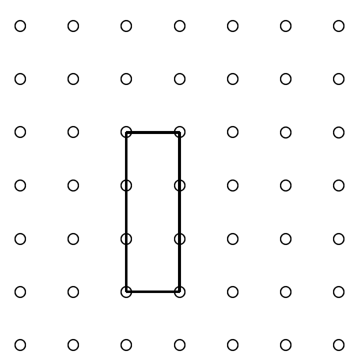
The corresponding parallelogram is given by the next picture. Note that the parallelogram above and the parallelogram below have the same basis (given by the vector $(1,1)$ and the same height!. In particular the operation did not change the are of the parallelogram.



The last row operation is to subtract one third of the second row from the first row. Again, this operation does not change determinant

$$\begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 3.$$

The corresponding parallelogram is given by the next picture. Note that this and the previous parallelogram have the same basis (this time given by the vector $(0, 3)$) and the same height!. In particular, this operation again did not change the are of the parallelogram. Since this last parallelogram is in fact a rectangle, its area is clearly equal to the determinant $= 3$.



Summarizing, on one hand row operations preserve determinant, on the other hand they preserve the area of the corresponding parallelogram. Once the matrix is in a diagonal form (this means that off-diagonal entries are 0), then the corresponding parallelogram is in fact a rectangle, and the fact that determinant is equal to its area is obvious.