1. Introduction

Let $G$ be a split, simply connected algebraic group corresponding to an irreducible root system $\Phi$. The group $G$ can be constructed as Chevalley group which is defined over $\mathbb{Z}$. Over a local field $\mathbb{R}$, $\mathbb{Q}_p$ or the ring of adeles $\mathbb{A} = \mathbb{A}_\mathbb{Q}$, the group $G$ has a unique non-trivial 2-fold central extension denoted by $G$:

$$1 \to \mu_2 \to G \to G \to 1.$$ 

An irreducible representation of $G$ (local or global) is called genuine if the central subgroup $\mu_2$ acts via the unique non-trivial character on the representation. The central extension $G(\mathbb{A})$ splits over the group of rational points $G(\mathbb{Q})$. Thus it is natural to study the space $L^2_{\text{gen}}(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ where the subscript gen indicates that we consider only the functions $f$ such that $f(\epsilon g) = \epsilon f(g)$ for every $\epsilon$ in $\mu_2$. This problem is a natural continuation of the study of classical modular forms of half integral weight. One purpose of this paper is to define Eisenstein series on $G(\mathbb{A})$ and to construct residual representation(s) $\Theta$ which, if $G = \text{SL}_2$, correspond to the classical theta series $1 + 2 \sum_{n>0} q^{n^2}$ or its anti-holomorphic analogue. Along the way, we study principal series representations of groups $G(\mathbb{Q}_p)$ where $p$ is any prime.

In order to explain our results here, let $T$ be a maximal split torus in $G$. Then its inverse image $T$ in $G$ is not necessarily commutative. The Weyl group acts by conjugation on irreducible genuine representations of $T(\mathbb{Q}_p)$, a natural question is whether there are Weyl group invariant representations. A need for such representations is obvious: If $V$ is a genuine representation of $T(\mathbb{Q}_p)$ then we can define a family of representations $i(\chi) = V \otimes \chi$ by twisting with unramified characters of the torus $T(\mathbb{Q}_p)$. If $V$ is Weyl group invariant, then the conjugation action of the Weyl group on $i(\chi)$ reduces to the conjugation action on the character $\chi$. In this way, at least, one can express some basic results on principal series in a neat way. For example, if $G = \text{Sp}_{2n}$ then a Weyl group invariant $V$ can be constructed using the Weil index $[W] [Rao]$. On the other hand, in [Sa2] an explicit construction of such representations is given for simply laced groups. However, the corresponding Weyl group invariance was obtained by a somewhat tedious check using relations in the Steinberg group. In this paper we present a more natural construction of those representations of $T(\mathbb{Q}_p)$. The Weyl group invariance will follow from a simple global argument. More precisely, our result is based on an observation that the analogous problem for real groups already has a solution as given by Adams, Barbasch, Paul, Vogan and

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Trapa in [A-V]. Let $K_\infty$ be a maximal compact subgroup of $G(\mathbb{R})$. Recall that $T(\mathbb{R})$ has a decomposition $MA$, where $M$ is the centralizer of $A$ in $K_\infty$. The group $K_\infty$ has certain small genuine representations, called pseudo-spherical representations, whose property is that they reduce irreducibly to $M$. The Weyl group invariance of such representations of $M$ is obvious. Next, we consider the space $L_2^{\text{gen}}(AT(\mathbb{Q})\backslash T(\mathbb{A}))$ of automorphic representations of $T(\mathbb{A})$.

Given a pseudo-spherical type $\delta$, one easily sees that there is only one automorphic representation $\pi = \otimes \pi_v$ of $T(\mathbb{A})$ such that $\pi_\infty \cong \delta$ and $\pi_p$ is unramified for all primes $p$. The uniqueness of $\pi$ and the Weyl group invariance of $\delta$ immediately imply the Weyl group invariance of all $\pi_p$. If $G = \text{Sp}_{2n}$ then one easily sees that our construction gives a Weil index.

We use $\pi$ to define local principal series representations, the corresponding Eisenstein series and a global residual representation $\Theta$ of Eisenstein series. Moreover, if $p \neq 2$ we use the central character $\gamma_p$ of $\pi_p$ to normalize Hecke operators in the Iwahori Hecke algebra $\mathcal{H}_-$ of $G(\mathbb{Q})$. Following [Sa2] this Hecke algebra is isomorphic to the Iwahori Hecke algebra of an algebraic group $G^I$. This isomorphism allows us to (Shimura) lift genuine representations of $G(\mathbb{Q}_p)$ with Iwahori fixed vectors to the linear group $G^I(\mathbb{Q}_p)$. We show that the Shimura lift sends unitary representations to unitary representations. For example, the local component $\Theta_p$ of $\Theta$ lifts to the trivial representation of $G^I(\mathbb{Q}_p)$. In particular, if $G \neq \text{SL}_2$ it follows that $\Theta_p$ is isolated in the unitary dual of $G(\mathbb{Q}_p)$. We emphasize once again that the representation $\Theta$ and the isomorphism of Hecke algebra depend on the choice of the pseudo spherical type $\delta$.

Acknowledgment. This work has been motivated by the pioneering works of Gelbart and Sally [GS] and of Kazhdan and Patterson [KP] on this subject. We would like to thank Dan Ciubotaru, Goran Muić and Peter Trapa for help and interest in this work. The first author would like to thank the hospitality of the University of Utah while part of this paper was written. The second author is supported by an NSF grant DMS-0551846.

2. An Adèlic group

Let $\Phi$ be a root system with simple roots $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. Let $(\alpha|\beta)$ denote the inner product on $\Phi$ normalized such that $(\alpha|\alpha) = 2$ for a long root. We set $\alpha^\vee := \frac{2\alpha}{(\alpha|\alpha)}$ and $\langle \alpha, \beta \rangle := (\alpha|\beta^\vee)$. We extend $\langle \cdot, \cdot \rangle$ to a pairing between the root lattice and the coroot lattice $\Lambda$.

Let $\mathfrak{g}$ be the corresponding simple Lie algebra over $\mathbb{Q}$. Fix a Chevalley basis in $\mathfrak{g}$. It defines a simply connected group Chevalley group $G$. It is an algebraic group defined over $\mathbb{Z}$. For any field $F$ the group $G(F)$ is generated by one-parameter subgroups $U_\alpha \simeq F$ for every root $\alpha$ in $\Phi$. More precisely, the choice of Chevalley basis fixes an isomorphism of $F$ and $U_\alpha$, $t \mapsto \mathfrak{g}_\alpha(t)$ for every $t \in F$. For example, if $G = \text{SL}_2$ then these elements are

$$
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 \\
0 & t
\end{pmatrix}.
$$
Define elements
\[
\begin{align*}
\lambda_\alpha(t) &= e_\alpha(t) e_{-\alpha}(-t^{-1}) e_\alpha(t) \\
\mu_\alpha(t) &= \frac{w_\alpha(t)}{w_\alpha(-1)}.
\end{align*}
\]
If \( G = \text{SL}_2 \) then these elements are
\[
\begin{pmatrix}
0 & t \\
-t^{-1} & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
t & 0 \\
0 & t^{-1}
\end{pmatrix}.
\]
By a result of Steinberg, the group \( G(F) \) is abstractly generated by the one-parameter groups \( U_\alpha \) modulo the relations
\[
[\lambda_\alpha(t), \lambda_\beta(u)] = \prod_{i,j \geq 1} \lambda_{i\alpha + j\beta}(c_{ij}t^i u^j) \quad \text{if } \alpha + \beta \text{ is a root}
\]
and
\[
\mu_\alpha(s) \lambda_\alpha(t) = \lambda_\alpha(st)
\]
where \( c_{ij} \) are integers depending on \( \alpha, \beta \). See [St].

Now assume that \( F = \mathbb{R} \) or \( \mathbb{Q}_p \). Let \((\cdot, \cdot)\) be the Hilbert symbol\(^1\) over \( F \). It defines a two fold central extension \( G(F) \)
\[
1 \to \mu_2 \to G(F) \to \mathbb{G}(F) \to 1
\]
by replacing the relation (2) by
\[
\mu_\alpha(s) \lambda_\alpha(t) = \lambda_\alpha(st) \cdot (s, t)^{-1}(\alpha \mid \alpha^\vee).
\]
Note that, if \( \alpha \) is a short root in \( B_n, C_n \) or \( F_4 \), then \((\alpha^\vee \mid \alpha^\vee) = 4 \) and there is no Hilbert symbol in the formula. Important to us will be the subgroup \( G_\alpha \) generated by the subgroups \( U_\alpha \) and \( U_{-\alpha} \). Let \( G_\alpha \cong \text{SL}_2 \) be the corresponding algebraic group in \( G \). Then \( G_\alpha \) is central extension of degree \( m_\alpha \) which is 2 except when \( \alpha \) is a short root in \( B_n, C_n \) or \( F_4 \) and then \( m_\alpha = 1 \).

The group \( G_\alpha(\mathbb{Z}_p) \) is a (preferred) hyperspecial maximal compact subgroup of \( G(\mathbb{Q}_p) \). It stabilizes the Chevalley lattice and is generated by \( e_\alpha(t) \) with \( t \) in \( \mathbb{Z}_p \). By reducing modulo \( p \) we have an exact sequence
\[
1 \to K^1_p \to G(\mathbb{Z}_p) \to G(\mathbb{F}_p) \to 1.
\]
The central extension splits over \( G(\mathbb{Z}_p) \) for \( p \neq 2 \). Note that the splitting is unique. Indeed two splittings would differ by a homomorphism from \( G(\mathbb{Z}_p) \) to \( \mu_2 \). Such a homomorphism is clearly trivial on the pro \( p \)-group \( K^1_p \), and then it must be trivial on \( G(\mathbb{F}_p) \) since it is a simple group. (Both arguments rely on the fact that \( p \neq 2 \).) Let \( K_p \) denote the lift of \( G(\mathbb{Z}_p) \). Let \( K_2 \) denote the full inverse image of \( G(\mathbb{Z}_2) \).

Let \( U_\alpha \) be the subgroup of \( G(F) \) generated by \( e_\alpha(t) \). Then \( U_\alpha \cong U_\alpha \) and the splitting is unique since \( F \) is 2-divisible.

\(^1\)For reference: Hilbert symbol over \( \mathbb{Q}_2 \) is given by \((2^u v, 2^v) = (-1)^r \) where \( u, v \in 1 + 2\mathbb{Z}_2 \) and \( r = (\frac{u-1}{2}) \cdot (\frac{v-1}{2}) + \alpha^2 - 1 + \beta^2 - 1 \). The symbol over \( \mathbb{Q}_p \) is \((p^u v, p^v) = (-1)^r \cdot (\frac{u}{p})^3 (\frac{v}{p})^\alpha \) where \( u, v \in \mathbb{Z}_p^\times \) and \( r = \alpha \beta \cdot (\frac{1}{2}) \).
Proposition 2.1. If \( p \) is odd then \( K_p \) contains \( e_\alpha(t) \) for all \( t \in \mathbb{Z}_p \) and, therefore, \( h_\alpha(t) \) for all \( t \in \mathbb{Z}_p^\times \).

Proof. Note that \( U_\alpha \) and \( K_p \) give two splittings of \( U_\alpha(\mathbb{Z}_p) \). They differ by a quadratic character of \( \mathbb{Z}_p \). Since \( \mathbb{Z}_p \) is 2-divisible if \( p \neq 2 \), the character is trivial. \( \square \)

Let \( S \) be any finite set of places containing \( \{\infty, 2\} \). Let

\[
\mu_S = \left\{ (\epsilon_1, \ldots, \epsilon_{|S|}) \in \mu_2^{|S|} : \epsilon_1 \cdots \epsilon_{|S|} = 1 \right\}.
\]

Define

\[
G_S = \left( \prod_{v \in S} G(\mathbb{Q}_v) \right) / \mu_S \times \prod_{v \not\in S} K_v.
\]

If \( S \subseteq S' \) then \( G_S \subseteq G_{S'} \). We define \( G(\mathbb{A}) \) as a direct limit of all \( G_S \). We have a central extension

\[
1 \rightarrow \mu_2 \rightarrow G(\mathbb{A}) \rightarrow G(\mathbb{A}) \rightarrow 1.
\]

For every \( \alpha \in \Phi \) and \( t \in \mathbb{Q} \), \( e_\alpha(t) \) can be viewed as an element in \( G(\mathbb{A}) \) by diagonally embedding. This is well-defined by Proposition 2.1. These elements clearly satisfy relations (1). Moreover, corresponding \( h_\alpha(t) \)'s satisfy relations (2) by quadratic reciprocity. In particular, we have an explicit splitting of the extension over \( G(\mathbb{Q}) \).

Maximal compact \( K_\infty \). There is an automorphism \( \sigma \) of \( G(\mathbb{R}) \) such that \( \sigma : e_\alpha(t) \mapsto e_{-\alpha}(-t) \) for every root \( \alpha \) and \( t \in \mathbb{R} \) (see Thm. 16 in [St]). The fixed points of \( \sigma \) on \( G(\mathbb{R}) \) is a maximal compact subgroup \( K_\infty \). Similarly there is an automorphism \( \sigma \) of \( G(\mathbb{R}) \) and its fixed points \( K_\infty \) is a maximal compact subgroup of \( G(\mathbb{R}) \).

3. The torus

Let \( \mathcal{T} \subseteq G \) be the maximal split torus. If \( R \) is a ring then \( \mathcal{T}(R) \) is generated by \( h_\alpha(t) \) with \( t \in R^\times \). If \( \Lambda \) is the coroot lattice then \( \mathcal{T}(R) \simeq \Lambda \otimes R^\times \) with the isomorphism given by

\[
h_\alpha(t) \mapsto \alpha^\vee \otimes t.
\]

Let \( T(F) \subset G(F) \) be the inverse image of \( \mathcal{T}(F) \). Then \( T(F) \) is generated by \( h_\alpha(t) \). It is a Heisenberg group. Thus, the following commutator formula is crucial to us throughout the paper:

\[
[h_\alpha(s), h_\beta(t)] = (s, t)^{\alpha \vee |\beta^\vee}.
\]

The goal of this section is to describe the structure of \( T(F) \) for \( F = \mathbb{R} \) and \( F = \mathbb{Q}_p \), and define pseudo-spherical representations of \( T(\mathbb{R}) \) and \( T(\mathbb{Q}_2) \), and unramified representations of \( T(\mathbb{Q}_p) \) for \( p \) odd.
Case $F = \mathbb{Q}_p$, with $p$ odd. Define $T_p = T(\mathbb{Q}_p) \cap K_p$. Then by Proposition 2.1, $T_p$ is generated by $h_\alpha(t)$ for all $t \in \mathbb{Z}_p^\times$ and is isomorphic to $\overline{T(\mathbb{Z}_p)}$ by $h_\alpha(t) \mapsto h_\alpha(t)$. Note that the symbol $(\cdot, \cdot)$ is tame here, i.e. $h_\alpha(s)h_\alpha(t) = h_\alpha(st)$ for all $s, t \in \mathbb{Z}_p^\times$. Let $T_p^2$ be the set of squares in $T_p$. Critical to us are the genuine representations of $T(\mathbb{Q}_p)$ which are trivial on $T_p^2$. A genuine representation of $T(\mathbb{Q}_p)$ is unramified if it has a non-zero vector fixed by $T_p^1$.

Case $F = \mathbb{R}$. We note that $(-1, -1) = -1$. In this case $\overline{T(\mathbb{R})} = M\Lambda$ where $M \simeq \Lambda \otimes \mathbb{R}^+$. Then $T(\mathbb{R}) = MA$ where $M$ is generated by $h_\alpha(-1)$ and contains the kernel $\mu_2$ of the central extension. On the other hand $A$ is generated by $h_\alpha(t)$ for $t \in \mathbb{R}^+$ and $A \simeq \mathbb{A}$. Note also that $A$ is in the the center of $T(\mathbb{R})$. Thus it is natural to concentrate on genuine representations of $M$. Let $M_s$ be the subgroup of $M$ generated by $h_\alpha(-1)$ for all roots $\alpha$ such that $m_\alpha = 1$. Since $h_\alpha(-1)h_\alpha(-1) = 1$ for such roots, $M_s$ does not contain the central subgroup $\mu_2 < M$. An irreducible genuine representation of $M$ trivial on the normal subgroup $M_s$ is called a pseudo-spherical representation. An important feature of pseudo-spherical representations of $M$ is that they are invariant under the conjugation action of the Weyl group. See Lemma 4.11(3) in [A-V].

Case $F = \mathbb{Q}_2$. This is the most interesting case. The Hilbert symbol is ramified. The group $\mathbb{Z}_2^\times$ has a filtration

$$\mathbb{Z}_2^\times = 1 + 2\mathbb{Z}_2 \supseteq 1 + 4\mathbb{Z}_2 \supseteq 1 + 8\mathbb{Z}_2.$$  

Note that $1 + 8\mathbb{Z}_2 = (\mathbb{Z}_2^\times)^2$. In particular $1 + 8\mathbb{Z}_2$ is in the kernel of the Hilbert symbol. Since $\mathbb{Z}_2^\times/(1 + 8\mathbb{Z}_2) \simeq (\mathbb{Z}/8\mathbb{Z})^\times = \{\pm 1, \pm 5\}$, all values of the symbol are easily obtained from the following table.

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>-1</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Observe that the kernel of the symbol $(\cdot, \cdot)$ when restricted to $\mathbb{Z}_2^\times$ is $1 + 4\mathbb{Z}_2$. Let $T_2^i < T(\mathbb{Z}_2)$ be the subgroup isomorphic to $\mathbb{Z}_2^i \simeq \Lambda \otimes (1 + 2^{1+i}\mathbb{Z}_2)$. Let $T(\mathbb{Z}_2) \subset G(\mathbb{Q}_2)$ be the inverse image of $T(\mathbb{Z}_2)$. Since the Hilbert symbol is trivial on $1 + 4\mathbb{Z}_2$, for every $i \geq 1$ elements $h_\alpha(t)$ for $t \in 1 + 2^{1+i}\mathbb{Z}_2$ generate a subgroup $T_2^i \subset T(\mathbb{Z}_2)$ isomorphic to $T_2^i$. Note that $T_2^1$ is contained in the center of $T(\mathbb{Z}_2)$, while $T_2^2$ is contained in the center of $T(\mathbb{Q}_2)$.

Since $(-1, -1)_2 = (-1, -1)_\infty = -1$, the subgroup of $T(\mathbb{Z}_2)$ generated by $h_\alpha(-1)$ is isomorphic to $M$ of the real case! Moreover, since the non-trivial coset of $1 + 4\mathbb{Z}_2$ in $1 + 2\mathbb{Z}_2 = \mathbb{Z}_2^\times$ is represented by $-1$, we have an isomorphism $T(\mathbb{Z}_2) \simeq M \times T_2^1$.

As in the real case, let $M_s$ be the subgroup of $M$ generated by $h_\alpha(-1)$ for all roots $\alpha$ such that $m_\alpha = 1$. Then $M_sT_2^1$ is a commutative subgroup of $T(\mathbb{Q}_2)$. Note that this group is generated by $h_\alpha(t)$ where $t$ is in $1 + 4\mathbb{Z}_2$ if $\alpha$ is long and $t$ is in $\mathbb{Z}_2^\times$ if $\alpha$ is short. We
say that a genuine representation of $T(\mathbb{Q}_2)$ is pseudo-spherical if it has a vector invariant under $M_s T_f^1$.

**Weyl groups.** Assume that $F = \mathbb{R}$ or $\mathbb{Q}_p$. Let $W_F$ denote the subgroup of $G(F)$ generated by $w_{\alpha}(1)$ for all simple roots $\alpha$. Let $T_F(\mathbb{Z})$ denote the subgroup generated by $h_{\alpha}(-1)$ for all simple roots $\alpha$. Let $W$ denote the Weyl group of $\overline{G}(\mathbb{Q})$. Then we have an exact sequence

$$1 \rightarrow T_F(\mathbb{Z}) \rightarrow W_F \rightarrow W \rightarrow 1.$$  

Conjugation action of $W_F$ on $T(F)$ does not descend to that of $W$ because $T_F(\mathbb{Z})$ does not lie in the center of $T(F)$. Suppose $(\pi, V)$ is an representation of $T(F)$ and $w \in W_F$. Let $V^w$ denote the representation defined by $t \mapsto \pi(w^{-1}tw)$. Note that the isomorphism class of $V^w$ depends only on the projection of $w$ into the Weyl group $W$. In other words, we have a conjugation action of the Weyl group on the set of isomorphism classes of irreducible representations of $T(F)$. The following lemma implies that the classes of pseudo-spherical and unramified representations are preserved under the conjugation action of the Weyl group.

**Proposition 3.1.** The following subgroups of $T(F)$ are $W_F$-invariant under conjugation:

1. $T_p$ if $F = \mathbb{Q}_p$ and $p$ is an odd prime.
2. $T_2^1$ and $M_s$ if $F = \mathbb{Q}_2$.
3. $A$ and $M_s$ if $F = \mathbb{R}$.

**Proof.** We need the following formula, Lemma 37(c) in [St],

$$w_{\alpha}(1)h_{\beta}(t)w_{\alpha}(-1) = h_\kappa(t) \cdot (c, t)^{\frac{1}{2}(\beta^\vee | \beta^\vee)}$$

were $\kappa = w_{\alpha}(\beta)$ and $c = \pm 1$ which depends on structure coefficients for the Chevalley basis. In order to prove the proposition we need to show that the sign after $h_\kappa(t)$ is trivial for $h_{\alpha}(t)$ generating the relevant groups. If $h_{\alpha}(t)$ is in $T_p$, $T_2^1$ or $A$ then $(c, t) = 1$ by elementary properties of the Hilbert symbol. Finally, recall that $M_s$ is generated by $h_{\beta}(-1)$ where $\beta$ is a root such that $(\beta^\vee | \beta^\vee) = 4$. Thus the sign is trivial in here, too. \hfill \Box

4. **Representations of $T(F)$**

Suppose $H$ is subgroup of $G$ which is the inverse image of an abelian subgroup $\overline{H}$ in $\overline{G}$. Let $Z(H)$ be the center of $H$ and suppose the index of $Z(H)$ in $H$ is finite. Recall that an irreducible representation of $H$ (resp. $Z(H)$) is called genuine if it is nontrivial on the kernel $\mu_2$ of the covering map. We denote the sets of irreducible genuine finite dimensional representations of $H$ and $Z(H)$ by $\text{Irr}_{\text{gen},f}(H)$ and $\text{Irr}_{\text{gen},f}(Z(H))$ respectively. Proposition 2.2 in [A-V] says:

**Proposition 4.1.** Given $H$ and $Z(H)$ as above. Then there is a one-to-one correspondence between $\text{Irr}_{\text{gen},f}(H)$ and $\text{Irr}_{\text{gen},f}(Z(H))$ given by sending an irreducible genuine representation of $H$ to its central character. Moreover, the dimension of every genuine irreducible representation is equal to the square root of the index of $Z(H)$ in $H$. \hfill \Box

We apply this proposition to the group $M$, which is the inverse image of $\overline{M}$. In order to describe the center $Z(M)$ of $M$ we need to consider the commutator map on $M$, which
induces a (symmetric) $\mu_2$-valued pairing on $\Lambda \cong \Lambda \otimes \{\pm 1\} \cong \Lambda/2\Lambda$. Since the commutator is given by

$$[h_\alpha(-1), h_\beta(-1)] = (-1, -1)_{\frac{\alpha\gamma|\beta\gamma}}$$

the pairing is (the same as) the bilinear form $(\cdot|\cdot)$ reduced modulo 2. The kernel is given by the lattice $\Lambda \cap 2\Lambda^*$ where $\Lambda^*$ is the dual lattice. In particular, the index of $\mu_2$ in $Z(M)$ is equal to the index $[\Lambda \cap 2\Lambda^* : 2\Lambda]$ and the index of $Z(M)$ in $M$ is equal to the index $[\Lambda : \Lambda \cap 2\Lambda^*]$. By Proposition 4.1 we have proved the following:

**Proposition 4.2.** The number of irreducible generic representations of $M$ is equal to the index $[\Lambda \cap 2\Lambda^* : 2\Lambda]$. The dimension of each such representation is a square root of the index of $[\Lambda : \Lambda \cap 2\Lambda^*]$.

In the following table we give the index of $\Lambda \cap 2\Lambda^*$ in $\Lambda$ in the simply laced case and $G_2$:

<table>
<thead>
<tr>
<th>$\Phi$</th>
<th>$\Lambda_{2n-1}$</th>
<th>$\Lambda_{2n}$</th>
<th>$\Delta_{2n-1}$</th>
<th>$\Delta_{2n}$</th>
<th>$\Delta_{4}$</th>
<th>$\Delta_{4}$</th>
<th>$\Delta_{4}$</th>
<th>$\Delta_{4}$</th>
<th>$G_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$4^{n-1}$</td>
<td>$4^n$</td>
<td>$4^{n-1}$</td>
<td>$4^{n-1}$</td>
<td>$4$</td>
<td>$4$</td>
<td>$4$</td>
<td>$4$</td>
<td>4</td>
</tr>
</tbody>
</table>

The index for types $B_l$, $C_l$ and $F_4$ is the same as the index for $A_{l-1}$, $A_1$ and $A_2$, respectively. In other words, it is the same as the index for the subsystem generated by simple long roots.

In order to discuss genuine irreducible representation $V$ of $T(\mathbb{Q}_p)$, we need to describe the center of $T(\mathbb{Q}_p)$. We need some notation at this point. We fix a positive a simple root $\Delta$ and an enumeration of the simple roots $\Delta = \{\alpha_1, \ldots, \alpha_t\}$. If $\lambda = n_1\alpha_1 + \ldots + n_t\alpha_t$ is an element in the coroot lattice $\Lambda$, then we define

$$\eta(\lambda) := h_{\alpha_1}(p^{n_1}) \cdots h_{\alpha_t}(p^{n_t}) \in T(\mathbb{Q}_p).$$

We shall use $\eta_p$ instead of $\eta$ if there is need to distinguish between primes. Note that the order of multiplication is important as the $h_{\alpha_i}(p^{n_i})$’s may not commute with one another. Indeed the commutator is given by

$$[\eta(\lambda), \eta(\lambda')] = (p, p)^{\langle \lambda | \lambda' \rangle},$$

which is trivial if $p = 2$ and $p \equiv 1 \pmod{4}$. If $\Lambda'$ is a subset of $\Lambda$, then we set $\eta(\Lambda') := \{\eta(\lambda) : \lambda \in \Lambda'\}$.

Case $p$ is odd: Note that we have a decomposition $T(\mathbb{Q}_p) = T_p \cdot \eta(\Lambda) \cdot \mu_2$. The commutator of $h_\alpha(p)$ in $\eta(\Lambda)$ and $h_\beta(t)$ in $T_p$ is

$$[h_\alpha(p), h_\beta(t)] = (p, t)^{\langle \alpha | \beta \rangle}.$$

Since $(p, t)_p = 1$ if and only if $t$ is a square in $\mathbb{Z}_p^*$, it follows that the commutator defines a pairing of $\Lambda \times T_p/T_p^2 \cong \Lambda \times \Lambda/2\Lambda$ which is simply the restriction of the bilinear form $(\cdot|\cdot)$ modulo 2. This shows that the centralizer of $T_p$ in $\eta(\Lambda)$ is $\eta(\Lambda \cap 2\Lambda^*)$ and the centralizer of $\eta(\Lambda)$ in $T_p$ is the group $C_p$ containing $T_p^2$ and such that $C_p/T_p^2 \cong (\Lambda \cap 2\Lambda^*)/2\Lambda$. It follows that the center of $T(\mathbb{Q}_p)$ is $Z_p = C_p \cdot \eta(\Lambda \cap 2\Lambda^*) \cdot \mu_2$. Note that the index of $Z_p$ in $T(\mathbb{Q}_p)$ is $[\Lambda : \Lambda \cap 2\Lambda^*]^2$. The next proposition follows from Proposition 4.1.

**Proposition 4.3.** There is a bijection between genuine irreducible representation $V$ of $T(\mathbb{Q}_p)$ and genuine characters $\gamma$ of $Z_p$, the center of $T(\mathbb{Q}_p)$. Moreover any such representation $V$ has the dimension equal to the index $[\Lambda : \Lambda \cap 2\Lambda^*]$. □
If \( \gamma \) is a genuine character of \( Z_p \), the corresponding representation of \( T(\mathbb{Q}_p) \) will be henceforth denoted by \( V(\gamma) \). Let \( \mathcal{I} \) be the set of isomorphism classes of genuine representations \( V \) of \( T(\mathbb{Q}_p) \) with nonzero \( T_p^2 \)-fixed vectors. Define an equivalence relation on \( \mathcal{I} \) where two representations \( V \) and \( V' \) are equivalent if \( V' \) is isomorphic to a twist of \( V \) by an unramified character of the algebraic torus \( \mathbb{T}(\mathbb{Q}_p) \).

**Proposition 4.4.** Two genuine representations \( V(\gamma) \) and \( V(\gamma') \) in \( \mathcal{I} \) are equivalent if and only if \( \gamma|_{C_p} = \gamma'|_{C_p} \). The number of equivalence classes is equal to the index \([\Lambda \cap 2\Lambda^*: 2\Lambda]\).

*Proof.* Since \( Z_p = C_p \cdot \eta(\Lambda \cap 2\Lambda^*) \cdot \mu_2 \), it easily follows that any two genuine characters of \( Z_p \) which coincide on \( C_p \) are unramified twists one of another. It follows that the equivalence classes are parametrized by characters of the finite group \( C_p/T_p^2 \). Since the order of this group is \([\Lambda \cap 2\Lambda^*: 2\Lambda]\), we have proved the first two statements. If \( V(\gamma) \) is unramified, that is, it contains a vector fixed by \( T_p \), then the central character must be trivial on \( C_p \). The proposition is proved. \( \square \)

**Case \( p = 2 \):** The set \( T^1(\mathbb{Q}_2) := T^1_2 \cdot \eta(\Lambda) \cdot \mu_2 \) is a normal subgroup of \( T(\mathbb{Q}_2) \) and commutes with \( M \), as it can be seen from the values of the Hilbert symbol \((\cdot, \cdot)_2 \). Thus
\[
T(\mathbb{Q}_2) = (M \times T^1(\mathbb{Q}_2))/\mu_2.
\]
It follows that any genuine representation of \( T(\mathbb{Q}_2) \) is a tensor product of genuine representations of \( M \) and \( T^1(\mathbb{Q}_2) \). Moreover, we have the following key proposition which reduces the study of representations of \( T(\mathbb{Q}_2) \) to that of \( M \) and \( T(\mathbb{Q}_2) \) for \( p \) odd.

**Proposition 4.5.** Assume that \( p \equiv 1 \pmod{4} \). Pick a non-square \( \zeta \) in \( \mathbb{F}_p^\times \). The map given by \( h_\alpha(2) \mapsto h_\alpha(p) \) and \( h_\alpha(5) \mapsto h_\alpha(\zeta) \) induces an isomorphism
\[
T^1(\mathbb{Q}_2)/T^2_2 \cong T(\mathbb{Q}_p)/T^2_p.
\]
*Proof.* This is obvious since the tame symbol \((\cdot, \cdot)_p \) takes the following values

|  \| \ |  \ |  \ |  \ |  \\
|---|---|---|---|
|  \ |  \ |  \ |  \\
|  \ |  \ |  \ |  \\
|  \ |  \ |  \ |  \\
|  \ |  \ |  \ |  \\

\( \square \)

## 5. Modular forms on \( T(\mathbb{A}) \)

We are interested in studying Eisenstein series on \( G(\mathbb{A}) \). To that end we need to understand the space \( \mathcal{A} = \mathcal{I}^\text{gen}_2(\mathbb{T}(\mathbb{Q}) \backslash T(\mathbb{A})) \). It is natural to look for maximally unramified representations first. Recall that \( T_p = K_p \cap T(\mathbb{Q}_p) \) if \( p \) is odd and \( T_2 \) is generated by \( h_\alpha(t) \) for all simple roots \( \alpha \) and \( t \in 1 + 4\mathbb{Z}_2 \).

**Proposition 5.1.** Let \( \mathcal{A}_0 \) be the space of all right \( T^1_2 \prod_{p \neq 2} T_p \)-invariant functions in \( \mathcal{A} \). Note that this is naturally an \( M \times M \) module where the two factors sit in \( T(\mathbb{R}) \) and \( T(\mathbb{Z}_2) \). As such it is isomorphic to the genuine part of the regular representation of the finite group \( M \):
\[
\mathcal{A}_0 \cong \mathcal{I}^\text{gen}_2(M).
\]
Proof. In the proof, \( h_{\alpha,\mathbb{Q}}(t), h_{\alpha,\infty}(t) \) and \( h_{\alpha,p}(t) \) denote elements of the global group \( T(\mathbb{Q}) \), and the local groups \( T(\mathbb{R}) \) and \( T(\mathbb{Q}_p) \), respectively. Let \( I \) be the group of invertible adeles. In view of the decomposition

\[
I = \mathbb{Q}^\times \cdot \mathbb{R}^+ \times \prod_p \mathbb{Z}_p
\]

the space \( \mathcal{A}_0 \) is indeed isomorphic to \( L_{\text{gen}}^2(M) \) where \( M \) is here considered as a subgroup of \( T(\mathbb{Z}_2) \). In order to finish the proof we need to determine the action of \( h_{\alpha,\infty}(-1) \) for this identification. Let \( f \) be in \( \mathcal{A}_0 \). Since \( f \) is left \( \mathbb{T}(\mathbb{Q}) \) and right \( T_p \)-invariant, \( p \neq 2 \), for every \( m \) in \( T(\mathbb{Z}_2) \) we have

\[
f(mh_{\alpha,\infty}(-1)) = f(h_{\alpha,\mathbb{Q}}(-1)^{-1} mh_{\alpha,\infty}(-1)) = f(h_{\alpha,2}(-1)^{-1} m).
\]

Recall that \( M_s \subseteq M \) is generated by \( h_{\alpha,2}(-1) \) for all roots \( \alpha \) such that \( m_\alpha = 1 \). In particular it is a central subgroup. Now let \( \mathcal{A}_{00} \) be the subspace of \( \mathcal{A}_0 \) consisting of \( M_s \)-invariant functions. Let \( \bar{M} = M/M_s \) be the quotient group. By the Peter-Weyl theorem, we have

\[
\mathcal{A}_{00} \cong L_{\text{gen}}^2(\bar{M}) = \oplus_\delta \delta \otimes \delta^\vee
\]

where the sum is taken over irreducible genuine representations \( \delta \) of \( \bar{M} \) or, equivalently over the pseudo-spherical representations of \( M \). Thus we have the following corollary:

**Corollary 5.2.** Let \( \delta \) be a pseudo-spherical representation of \( M \). Then there exists a unique representation \( \pi \subseteq L_{\text{gen}}^2(\mathbb{A}T(\mathbb{Q})\backslash T(\mathbb{A})) \) such that \( \pi_{\infty} \cong \delta \) and \( \pi_p \) is unramified at all primes. The isomorphism class of \( \pi_p \) is invariant under the conjugation of the Weyl group.

Proof. The uniqueness is obvious. Now consider a Weyl group conjugate \( \pi^w \). Note that \( \pi^w \) is again unramified at all primes. Since \( \delta^w \cong \delta \) it follows that \( \pi^w \cong \pi \) by the uniqueness of \( \pi \).

Let \( \pi \) be the global representation as in the previous corollary. We would like to determine the local components \( \pi_p \). To that end we need to determine the the corresponding central characters. A large part of the center acts trivially on \( \pi \), independent of the choice of \( \delta \):

**Proposition 5.3.** Let \( p \) be any prime. For any \( t \in \mathbb{Q}_p^\times \) the central element \( h_{\alpha,p}(t^{m_\alpha}) \) acts trivially on \( \mathcal{A}_{00} \).

Proof. Since \( \mathcal{A}_{00} \) is \( (M_sT_2^1) \prod_{p \neq 2} T_p \) right invariant it suffices to check this for \( t = p \). Assume first that \( p \) is odd. Let \( f \) be in \( \mathcal{A}_{00} \). Note that \( f \) is right \( h_{\alpha,q}(p^{m_\alpha}) \)-invariant for every \( q \neq p \). Indeed, \( h_{\alpha,q}(p^{m_\alpha}) \) is contained in \( T_q \) if \( q \neq 2 \) and in \( M_sT_2^1 \), if \( q = 2 \). (This is clear if \( m_\alpha = 1 \), otherwise it follows from \( p^2 \equiv 1 \) (mod 4) for every odd \( p \).) Using left \( h_{\alpha,q}(p^{m_\alpha}) \)-invariance of \( f \) we have

\[
f(mh_{\alpha,p}(p^{m_\alpha})) = f(h_{\alpha,q}(p^{m_\alpha})^{-1} mh_{\alpha,p}(p^{m_\alpha})) = f(m).
\]

Now assume that \( p = 2 \). Then, analogously,

\[
f(mh_{\alpha,2}(2^{m_\alpha})) = f(h_{\alpha,q}(2^{m_\alpha})^{-1} mh_{\alpha,2}(2^{m_\alpha})) = f(m).
\]
In order to determine the central character of $\pi_p$, we need to determine the action of the full center of $T(Q_p)$ on $\delta \otimes \delta^\vee \subseteq A_{00}$. Observe that $(p,p)_p = (p,p)_2 = (-1)^{(p-1)/2}$ for any odd prime. This allows us to define a homomorphism
\[
\varphi : \eta_p(\Lambda) \cdot \mu_2 \to T(\mathbb{Z}_2)
\]
by sending $h_{\alpha,p}(p)$ to $h_{\alpha,2}(p)$. The restriction of $\varphi$ to $\eta_p(\Lambda \cap 2\Lambda^*)$ has the image in the center of $T(\mathbb{Z}_2)$. Thus, if $\gamma_\infty$ is the central character of $\delta$, then the composite
\[
\gamma_p = \gamma_\infty \circ \varphi
\]
defines an unramified central character for $T(Q_p)$. We also define $\gamma_2$ to be any $\gamma_2(\eta_2(\lambda)) = 1$ for any $\lambda$ in $\Lambda \cap 2\Lambda^*$.

**Proposition 5.4.** Let $\pi$ be the unique unramified representation such that $\pi_\infty \cong \delta$ in Corollary 5.2. Let $\gamma_p$ be the central character defined by (4). Then $\pi_2 \cong \delta^\vee \otimes V(\gamma_2)$ and $\pi_p \cong V(\gamma_p)$ for $p$ odd.

**Proof.** The proof is completely analogous to the proof of Proposition 5.3. We leave details to the reader. \qed

For uniformity, we set $\gamma_\infty$ to be the central character of $\pi_\infty = \delta$ extended trivially to $A$. We set $V(\gamma_\infty)$ to be the representation $\delta$ extended trivially to $A$.

6. **Principal series representations of $G(Q_v)$**

In this section we define principal series representations of $G(Q_v)$ where $v = \infty$ or $p$. Let $B = TU$ denote the Borel subgroup of $G$ where $U$ is generated by $e_\alpha(t)$ for all positive roots $\alpha$. Let $\bar{U}$ be the group generated by $e_\alpha(t)$ for all negative roots $\alpha$.

Fix a pseudo-spherical representation $\delta$ of $M$. It gives rise to a global representation $\pi$ of $T(\mathbb{A})$, such that $\pi_\infty \cong \delta$ as in Corollary 5.2. Let $\chi$ be an unramified character of $T(Q_v)$. (If $v = \infty$ an unramified character is a character trivial on $\mathcal{M}$.) Let $i(\chi)$ be the twist of $\pi_v$ by $\chi$. We consider the normalized induced representation
\[
I(\chi) = \text{Ind}_{B}^{G}(i(\chi)).
\]

Let $\alpha$ be a simple root. A character $\chi$ is called $\alpha$-dominant if $\chi(L_\alpha(t)) = |t|^s$ with $\Re(s) > 0$. A character $\chi$ is called dominant if it is $\alpha$-dominant for all simple roots. For every $w$ in $W_{Q_v}$ we have an intertwining maps $A_w : I(\chi) \to I(\chi^w)$ defined by
\[
A_w(f)(g) = \int_{U \cap \mathcal{U} w^{-1}} f(w^{-1}ug)du.
\]

**Proposition 6.1.** The operator $A_w$ is absolutely convergent if $\chi$ is dominant.

**Proof.** Let $\ell(w)$ denote the length of the projection of $w$ into the Weyl group. The proof of the proposition is on induction on the length $\ell(w)$. We consider the case of $\ell(w) = 1$. Then $w$ corresponds to a simple root $\alpha$, so we shall denote it by $w_\alpha$.

**Lemma 6.2.** Let $\alpha$ be a simple root and $\chi$ an unramified $\alpha$-dominant character of $T$. Then $A_{w_\alpha}$ is absolutely convergent.
Proof. The proof of this Lemma is a reduction to $SL_2$. Let $s \in \mathbb{C}$ such that $\chi(h_\alpha(t)) = |t|^s$. Then $R(s) > 0$ since $\chi$ is $\alpha$ dominant. In the formula for $A_{w_\alpha}(f)$ we can assume that $g = 1$, by replacing $f$ by its translate if necessary. Note that $U \cap w_\alpha U w_\alpha^{-1} = U_\alpha$, thus the question of convergence is answered by working in $G_\alpha$. The restriction of $f$ to $G_\alpha$ belongs to the induced representation $Ind_{B_\alpha}^{G_\alpha} i(\chi)$ where $B_\alpha = G_\alpha \cap B$. Note that $B_\alpha = T_\alpha U_\alpha$, where $T_\alpha$ is a (commutative) group generated by elements $h_\alpha(t)$. Decompose $i(\chi) = \oplus \mu_i$ as a sum of characters of $T_\alpha$. It follows that $Ind_{B_\alpha}^{G_\alpha} i(\chi) = \oplus I_i$ where $I_i$ are principal series representation induced from the characters $\mu_i$. Recall that $i(\chi) = \pi_p \otimes \chi$. Since Proposition 5.3 describes the action of $h_\alpha(t)$ on $\pi_p$ it follows that

$$|\mu_i(h_\alpha(t))| = |t|^{R(s)}$$

for every $i$. Thus, if we write $f = \oplus f_i$ with $f_i$ in $I_i$ then $|f_i|$ belongs to a principal series representation $I(\Re(s))$ of $G_\alpha \cong SL_2$ induced from the character $h_\alpha(t) \mapsto |t|^{\Re(s)}$. The convergence of the integral for $|f_i|$ can be easily calculated. Indeed, since $|f_i|$ is bounded on the maximal compact subgroup, it suffices to check the convergence for $|f_i|$ equal to a spherical vector. Then, if $v = \infty$, the integral is

$$\int_\mathbb{R} \left( \frac{1}{1 + x^2} \right)^{\frac{\Re(s)+1}{2}} dx$$

while, if $v = p$, the integral is (essentially)

$$\sum_{i=n}^{\infty} \frac{1}{t^{\Re(s)}}.$$

Both of these converge if $\Re(s) > 0$. \hfill \Box

Now we can easily finish the proof of the proposition. Assume that $\chi$ is dominant and $A_w$ is absolutely convergent for some $w$ in $W$. If $\ell(w_\alpha w) = \ell(w) + 1$ then $\chi^w$ is $\alpha$-regular. In particular the composite $A_{w_\alpha} \circ A_w$ is absolutely convergent. It is equal to $A_{w_\alpha w}$ by Fubini’s theorem. The proposition is proved. \hfill \Box

Let $m_\alpha$ be the degree of the central extension $G_\alpha$ of $G_\alpha \cong SL_2$. This number is equal to 2 except when $\alpha$ is short root in the root systems $C_n$, $B_n$ and $F_4$. A character $\chi_0 : T(\mathbb{Q}_v) \to \mathbb{R}^+$ such that $\chi_0(h_\alpha(t)) = |t|^{\frac{1}{m_\alpha}}$ for every simple root $\alpha$ is called an exceptional character. Note that $\chi_0$ is unique and dominant. In particular, the induced representation $I(\chi_0)$ has a unique quotient denoted by $\Theta(\gamma_v)$, called an exceptional representation.

Remark. For $G(\mathbb{Q}_v)$ of type $C_n$, the exceptional representation $\Theta(\gamma_v)$ is an even component of the oscillator representation $[W]$. The representation $\pi_v = V(\gamma_v) = \gamma_v$ is one dimensional and it is the Weil index [Rao].

If $v = p$ then the representation $\Theta(\gamma_p)$ can be better described. By the geometric lemma in [BZ], the semi simplification of the (unnormalized) Jacquet module $I(\chi)U$ is

$$I(\chi)_U \cong \oplus_{w \in W} [\rho_U \cdot i(\chi^w)]$$

where $\rho_U$ is the modular character with respect to $U$. 

Proposition 6.3. Let \( \chi_0 \) be the exceptional character and \( w_0 \) the longest element in the Weyl group. Then \( \Theta(\gamma_U) \cong \rho_{\mu} \cdot i(\alpha_0^{w_0}) \).

Proof. Let \( \alpha \) be a simple root. Let \( L_\alpha \) be the Levi factor of a parabolic subgroup containing \( B \) such that \([L_\alpha, L_\alpha] = G_\alpha \). In order to prove the proposition, by a result of Rodier [Ro], it suffices to show that \( \text{Ind}_{T_{U_\alpha}}^{L_{U_\alpha}} i(\chi_0) \) reduces for every simple root \( \alpha \). Let us restrict this representation to \( G_\alpha \). Decompose \( i(\chi) = \oplus \mu_i \) as a sum of characters of \( T_\alpha \). It follows that \( \text{Ind}_{T_{U_\alpha}}^{L_{U_\alpha}} i(\chi) \cong \oplus I_i \) where \( I_i \) are principal series representations of \( G_\alpha \), parabolically induced from the characters \( \mu_i \). If \( m_\alpha = 1 \), then by Proposition 5.3, \( \mu_i(h_\alpha(t)) \cong |t| \). It follows that each \( I_i \) has the Steinberg representation as a submodule and the trivial representation as a quotient. Since \( G_\alpha \) is a normal subgroup of \( L_\alpha \), the sum of all Steinberg submodules is a proper submodule for \( L_\alpha \). A similar argument works if \( m_\alpha = 2 \). Then, by Proposition 5.3, \( \mu_i(h_\alpha(t^2)) \cong |t| \). It follows, by Theorem 2 in [GS], that each \( I_i \) reduces with a discrete series representation as a submodule and a quotient isomorphic to an even component of an oscillator representation. Again, the sum of discrete series representations is an \( L_\alpha \)-submodule. The proposition is proved. \( \square \)

Assume that \( p \) is odd. Let \( v^o \) be a non-zero element in \( i(\chi) \) fixed by \( T_p \). Note that \( v^o \) is unique up to a non-zero scalar. Then the representation \( I(\chi) \) contains a unique \( K_p \)-fixed vector \( f_\chi^o \) normalized by \( f_\chi^o(1) = v^o \). The action of the intertwining operators on the spherical vector has been computed in [Sa2].

Proposition 6.4. Assume that \( p \neq 2 \). Let \( \alpha \) be a simple root. Then

\[
A_{w_\alpha}(f_\chi^o) = \frac{1 - p^{-1}(\chi(h_\alpha(p^{m_\alpha})))}{1 - \chi(h_\alpha(p^{m_\alpha}))} f_\chi^{w_\alpha}. 
\]

Note that the formula for \( A_{w_\alpha}(f_\chi^o) \) depends on the projection of \( w_\alpha \) into the Weyl group \( W \). Thus, for a general element in \( W_{Q_p} \) we have the following corollary.

Corollary 6.5. Let \( w \) be in \( W \) and \( w \) a preimage of \( w \) in \( W_{Q_p} \). Then

\[
A_w f_\chi^o = \prod_{\alpha > 0, w(\alpha) < 0} \frac{1 - p^{-1}(\chi(h_\alpha(p^{m_\alpha})))}{1 - \chi(h_\alpha(p^{m_\alpha}))} f_\chi^{w}. \quad \square
\]

7. Eisenstein series

Recall that \( B = TU \) denote the Borel subgroup of \( G \) where \( U \) is generated by \( e_\alpha(t) \) for all positive root \( \alpha \). In the same fashion, we define the Borel subgroup \( B = TU \) of \( G \).

We identify \( A^l \cong \mathcal{T}(A) \) by \( (x_1, \ldots, x_l) \mapsto \prod_{i=1}^l h_\alpha(x_i) \). For \( s = (s_1, \ldots, s_l) \in \mathbb{C}^l \), we define the Hecke character \( \chi_{s} \) of \( \mathcal{T}(\mathbb{Q}) \setminus \mathcal{T}(\mathbb{A}) \) by \( \chi_{s}(h_\alpha(x_i)) = |x_i|^{s_\alpha} \) for every simple root \( \alpha \). Here \( |x| = \prod_v |x|_v \). We extend this to a function on \( \mathcal{G}(A) \) by \( \chi_{s}(utk) = \chi_{s}(t) \) where \( u \in U(A) \), \( t \in T(A) \) and \( k \in \prod_v K_v \mathcal{G}(\mathbb{Z}_p) \). The square root of the modular function is given by \( \rho = \chi_{(1, \ldots, 1)} \) where \( 1 = (1, \ldots, 1) \).

Similarly for a place \( v \) of \( \mathbb{Q} \), we define a character \( \chi_{s,v} \) of \( \mathcal{T}(\mathbb{Q}_v) \) by \( \chi_{s,v}(h_\alpha(t)) = |t|_v^{s_\alpha} \) for all every simple root \( \alpha \). We extend this to a function on \( \mathcal{G}(\mathbb{Q}_v) \) by \( \chi_{s,v}(utk) = \chi_{s,v}(t) \) where \( u \in U(\mathbb{Q}_v) \), \( t \in T(\mathbb{Q}_v) \) and \( k \in K_v \).
Let \( \pi \) be as in Corollary 5.2. Let \( K = K_\infty \prod_p K_p \). Let \( \mathcal{J} \) denote the space of functions on \( G(\A) \) satisfying the following conditions:

1. \( f(ubag) = f(g) \) for \( u \in U(\A), b \in B(\Q), a \in A, \ g \in G(\A) \).
2. \( f \) is \( K \)-finite and for each \( k \in K \), the function \( t \mapsto f(tk) \) is a function in \( \pi \).

Let \( I(\chi_\alpha) \) denote the representation of \( G(\A) \) on functions of the form \( g \mapsto f(g) \chi_{s+1}(g) \) where \( f \in \mathcal{J} \). We have

\[
I(\chi_\alpha) = \text{Ind}_{B(\A)}^{G(\A)} \pi \chi_\alpha = \left( \text{Ind}_{B(\R)}^{G(\R)} a_\infty \chi_{s,\infty} \right) \bigotimes_p \text{Ind}_{B(\Q_p)}^{G(\Q_p)} a_p \chi_{s,p}
\]

where all the induced representations are normalized inductions. We form an Eisenstein series:

\[
E(g, s, f) = \sum_{x \in B(\Q) \setminus G(\Q)} f(xg) \chi_{s+1}(g)
\]

where \( g \in G(\A), s \in \C^i, f \in \mathcal{J} \). The above sum converges absolutely and uniformly on compact sets if \( \Re(s_i) > 1 \) for all \( i \). The Eisenstein series can be continued meromorphically to \( \C^i \), see [MW]. We define of constant term of the above Eisenstein series by

\[
E(g, s, f)_U = \int_{U(\Q) \setminus U(\A)} E(ug, s, f) du.
\]

A standard computation in the domain of convergence of \( E(g, s, f) \) gives

\[
E(g, s, f)_U = \sum_{w \in W} (A_w(s)f)(g)
\]

where

\[
(A_w(s)f)(g) = \int_{(U(\Q) \cap wU(\Q)w^{-1}) \setminus (U(\A) \cap wU(\A)w^{-1})} f(w^{-1}ug) \chi_{s+1}(w^{-1}ug) du
\]

and \( w \in W_\Q \) is an (arbitrary) element such that \( \text{pr}(w) = w \). Suppose \( S \) is a finite set of primes including \( 2 \) and \( \infty \) and \( f = (\bigotimes_{v \in S} f_v) \otimes (\bigotimes_{p \not\in S} f_p^0) \), then by Corollary 6.5

\[
(A_w(s)f)(g) = \left( \bigotimes_{v \in S} A_{w,v}(s)f_v \right) \otimes \left( c_S(w, s) \bigotimes_{p \not\in S} f_p^0 \right)
\]

where

\[
c_S(w, s) = \prod_{p \not\in S} \prod_{\alpha > 0, \omega(\alpha) < 0} \frac{1 - p^{-1}(\chi_{s,p}(h_\alpha(p^{m_\alpha}))))}{1 - \chi_{s,p}(h_\alpha(p^{m_\alpha})))} = \prod_{\alpha > 0, \omega(\alpha) < 0} \frac{\zeta_S(m_\alpha \alpha(s))}{\zeta_S(1 + m_\alpha \alpha(s))}
\]

and \( \zeta_S(z) = \prod_{p \not\in S} (1 - p^{-z})^{-1} \) is the partial Riemann zeta function. The number \( \alpha(s) = \sum_{i=1}^l n_is_i \) if \( \alpha = \sum_{i=1}^l n_i \alpha_i \) is a sum of simple roots. Therefore as \( s \) tends to \( s_0 = (m_{\alpha_1}^{-1}, \ldots, m_{\alpha_l}^{-1}) \), each term \( (\prod_{i=1}^l (s_i - m_{\alpha_i}^{-1})) A_w(s)f \) vanishes except the term where \( w = w_0 \) is the longest element of \( W \). Furthermore if we set \( S = \{2, \infty\} \), then \( A_{w,v}(s_0)f \) for \( v \in S \) are nonzero intertwining operators so we may arrange \( f \) such that \( (\prod_{i=1}^l (s_i - m_{\alpha_i}^{-1})) A_w(s)f \) is nonzero.
For \( f \in J \), we define
\[
\theta_f(g) = \lim_{s \to s_0} \left( \prod_{i=1}^{l} (s_i - m^{-1}_{\alpha_i}) \right) E(g, s, f).
\]
Then
\[
\int_{U(Q) \setminus U(A)} \theta_f(ug) du = A_{w_0}(s_0)(f)
\]
and, by the criterion of Jacquet (see [J] and [MW]), \( \theta_f(g) \) is a square integrable in \( L^2(G(Q) \setminus G(A)) \). Let \( \Theta \) denote the span of \( \{ \theta_f : f \in J \} \). We now recall the exceptional representation \( \Theta(\gamma_v) \) defined in Section 6.

**Theorem 7.1.** The span \( \Theta \) lies in \( L^2(G(Q) \setminus G(A)) \). It is an irreducible automorphic representation of \( G(A) \) and it is isomorphic to \( \otimes_v \Theta(\gamma_v) \).

**Proof.** For every \( f \in J \), the map \( f \chi_{s_0+1} \mapsto \theta_f \) defines a nonzero intertwining operator from the induced representation to \( L^2(G(Q) \setminus G(A)) \). Thus the image \( \Theta \) must decompose as a direct sum of irreducible representations. On the other hand, at each local place \( v \) the exceptional representation \( \Theta(\gamma_v) \) is a unique quotient of the local induced representation. This implies that \( \Theta \cong \otimes_v \Theta(\gamma_v) \), as desired. \( \square \)

**Corollary 7.2.** The exceptional representation \( \Theta(\gamma_v) \) is unitarizable. \( \square \)

In a terminology of [A-V], \( \Theta(\gamma_\infty) \) corresponds to the trivial representation of a split group \( G^d(R) \) which will be introduced in the next section. The unitarity of \( \Theta(\gamma_\infty) \) was proved and studied for classical groups of type \( B_n \) in [Kn], [LS] and [T]. The unitarity for other groups may be new.

### 8. Iwahori-Hecke Algebras

We will fix an odd prime \( p \) in this section. We fix an Iwahori subgroup \( I \) of \( K_p \) such that \( I \) contains \( U_\alpha(Z_p) \) for all positive \( \alpha \) and \( I \cap T(Q_p) = T_p \). We recall that \( \mu_2 \) is the kernel of the covering map \( pr : G(Q_p) \to G(Z_p) \). Let \( \mathcal{H}_- = \mathcal{H}_-(G(Q_p)) \) denote the algebra of all compactly supported \( I \)-bi-invariant functions on \( G(Q_p) \) such that \( f(\epsilon g) = \epsilon f(g) \) for all \( \epsilon \in \mu_2 \). The multiplicative structure of on \( \mathcal{H}_- \) is defined by convolution of functions,
\[
(f' \cdot f'')(g) = \int_G f'(h)f''(h^{-1}g)dh
\]
where \( dh \) is a Haar measure on \( G \) so that the volume of \( \mu_2 \times I \) is one. We call \( \mathcal{H}_- \) the Iwahori-Hecke algebra of \( G \). The following is Proposition 6.1 in [Sa2].

**Proposition 8.1.** Let \( N' \) denote the normalizer in \( G \) of \( T_p \). Then the support of the Hecke algebra is \( \text{supp}(\mathcal{H}_-) = IN'I \).

One can easily describe \( N' \). Recall that, if \( N(Q_p) \) is the normalizer of \( T(Z_p) \) in \( G(Q_p) \), then the quotient of the two is isomorphic to the affine Weyl group \( \Lambda \ltimes W \). The group \( N' \) is smaller than the inverse image of \( N(Q_p) \). Recall that \( \eta_p(\lambda) \) centralizes (or normalizes) \( T_p \) if and only if \( \lambda \) is in
\[
\Lambda' := \Lambda \cap 2\Lambda^*.
\]
In particular, we have an exact sequence
\[ 1 \to \mu_2 \times T_p \to N' \xrightarrow{\phi} \Lambda' \rtimes W \to 1, \]
where \( \phi \) is defined by sending \( w_\alpha(1) \) to the reflection \( w_\alpha \) in \( W \) and \( \eta_p(\lambda) \) to \( \lambda \) in \( \Lambda' \).

We now define a normalization of elements in the Hecke algebra. Let \( \pi_p \) be an unramified, Weyl group invariant, irreducible genuine representation of \( T(\mathbb{Q}_p) \) as in Corollary 5.2. Let \( \gamma_p \) be the central character of \( \pi_p \). Recall that \( \eta_p(\lambda) \) is in the center of \( T(\mathbb{Q}_p) \) for every \( \lambda \) in \( \Lambda' \). In particular, \( \gamma_p(\eta_p(\lambda)) \) is well defined for every \( \lambda \) in \( \Lambda' \). The Weyl group invariance of the central character of \( \pi_p \) implies that we can extend \( \gamma_p \) to \( N' \) by setting
\[ \gamma_p(w_\alpha(1)) = 1. \]
Thus, \( \gamma_p \) is a character of \( N' \) which is trivial on \( T_p \). For \( w \) in \( \Lambda' \rtimes W \), we define \( e_w \in \mathcal{H}_- \) by its values for every \( x \) in \( N' \), as follows:
\[ e_w(IxI) = \begin{cases} \gamma_p(x) & \text{if } \phi(x) = w \\ 0 & \text{otherwise} \end{cases}. \]

We note some elementary properties of elements \( e_w \). Let \( \ell(w) \) denote the usual length function on the affine Weyl group \( \Lambda \rtimes W \) as in [Sa2]. If \( \ell(w_1w_2) = \ell(w_1) + \ell(w_2) \), for two elements in \( \Lambda' \rtimes W \), then \( e_{w_1w_2} = e_{w_1} \cdot e_{w_2} \). (A key for this property is the multiplicativity property of \( \gamma_p \).)

Let \( \mathcal{L} \) denote the \( \mathbb{C} \)-span of \( e_\lambda \) where \( \lambda \) is dominant in \( \Lambda' \). Note that \( \ell(\lambda) = \langle \rho, \lambda \rangle \) for dominant \( \lambda \). It follows that \( \ell(\lambda + \lambda') = \ell(\lambda) + \ell(\lambda') \) for dominant \( \lambda, \lambda' \) in \( \Lambda' \). Hence \( e_\lambda \cdot e_{\lambda'} = e_{\lambda + \lambda'} \). In particular, \( \mathcal{L} \) is a commutative subalgebra in \( \mathcal{H}_- \).

Let \( H \) denote the subalgebra consisting of functions supported on \( \mu_2 \times K_p \). It has basis \( \{e_w : w \in W\} \). If \( \alpha \) is a simple root and \( w_\alpha \) is the corresponding simple reflection, then we denote \( e_{w_\alpha} \) by \( e_\alpha \). These elements satisfy the following relations:
\begin{enumerate}
  \item \( (e_\alpha - p)(e_\alpha + 1) = 0 \) and
  \item \( e_\alpha \cdot e_\beta \cdot e_\alpha \ldots = e_\beta \cdot e_\alpha \cdot e_\beta \ldots \) where the number of factors on each side is equal to the order \( m_{\alpha\beta} \) of the element \( w_\alpha w_\beta \) in \( W \).
\end{enumerate}
Conversely \( H \) is the \( \mathbb{C} \)-algebra generated by the set of \( e_\alpha \) for all simple roots \( \alpha \) satisfying the above two relations. One easily sees that
\[ \mathcal{H}_- = H \cdot \mathcal{L} \cdot H. \]

An important result is that for a positive \( \lambda \in \Lambda', e_\lambda \) is an invertible element in \( \mathcal{H}_- \). This implies that if \( V \) is an admissible genuine \( G \)-module generated by the subspace \( V^I \), then every submodule \( V_1 \) of \( V \) is also generated by its subspace \( V_1^I \).

Given \( \lambda \in \Lambda' \), we write \( \lambda = \lambda_1 - \lambda_2 \) where \( \lambda_1, \lambda_2 \) are positive in \( \Lambda' \). We define
\[ t_\lambda = p^{-\frac{1}{2}(\rho, \lambda)} e_{\lambda_1} \cdot e_{\lambda_2}^{-1}. \]
This definition does not depend on the choice of \( \lambda_1 \) and \( \lambda_2 \). We state the main results of [Sa1] and [Sa2]. (Note that we have already explained the first three relations.)

**Theorem 8.2.** Let \( \alpha, \beta \) be two simple roots, and \( \lambda, \lambda' \in \Lambda' \). Then \( e_\alpha, e_\beta, t_\lambda \) and \( t_{\lambda'} \) satisfy the following relations:
Definition of $\mathcal{O}_d$. We will define an algebraic split group $\mathcal{O}_d(Q_p)$. In order to do this, it suffices to define its coroots $\Psi^\vee$ and its co-character lattice $\Lambda_c$. We recall that $\Lambda$ is the coroot lattice of $G$ and we define

$$\Psi^\vee := \left\{ \frac{m_\alpha}{2} \alpha^\vee \in \Lambda \otimes \mathbb{R} | \alpha \in \Phi^\vee \right\}$$

and $\Lambda_c := \frac{1}{2} \Lambda'$. Note that the root system $\Psi$ is dual to the root system $\Phi$. The isogeny class of $\mathcal{O}_d$ is determined by the lattice $\Lambda_c$. Let $\Lambda_{cr}$ be the $\mathbb{Z}$-span of co-roots in $\Psi^\vee$. The group $\mathcal{O}_d$ is a split algebraic group obtained by taking a quotient of the split, simply connected algebraic group corresponding to $\Psi$ by the central subgroup isomorphic to $\Lambda_c/\Lambda_{cr}$. It is an elementary 2-group. Its order is equal to the number of pseudo-spherical representations of $M$. The following table lists all cases when this 2-group is non-trivial:

<table>
<thead>
<tr>
<th>$\Phi$</th>
<th>$\mathcal{A}_{2n-1}$</th>
<th>$\mathcal{D}_{2n-1}$</th>
<th>$\mathcal{D}_{2n}$</th>
<th>$\mathcal{C}_n$</th>
<th>$\mathcal{B}_{2n}$</th>
<th>$\mathcal{E}_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Psi$</td>
<td>$\mathcal{A}_{2n-1}$</td>
<td>$\mathcal{D}_{2n-1}$</td>
<td>$\mathcal{D}_{2n}$</td>
<td>$\mathcal{B}_n$</td>
<td>$\mathcal{C}_{2n}$</td>
<td>$\mathcal{E}_7$</td>
</tr>
<tr>
<td>$</td>
<td>\Lambda_c: \Lambda_{cr}</td>
<td>$</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

The Iwahori-Hecke algebra $\mathcal{H}(\mathcal{O}_d)$ of $\mathcal{O}_d$ is similarly generated by $t_\lambda$ and $e_w$ where $\lambda \in \Lambda_c$ and $w \in W$.

Let $f(x) \in \mathcal{H}(G)$ (resp. $\mathcal{H}(\mathcal{O}_d)$). We define $f^*(x) = \overline{f(x^{-1})}$. Hence $*: \mathcal{H}_- \to \mathcal{H}_-$ (resp. $*: \mathcal{H}(\mathcal{O}_d) \to \mathcal{H}(\mathcal{O}_d)$) satisfies $(f^*)^* = f$ and $f^* \cdot g^* = (g \cdot f)^*$, ie it is an algebra anti-involution. We have $e_{-\lambda}^* = e_{\lambda}$ and $e_w^* = e_{w^{-1}}$ in $\mathcal{H}_-$. Similarly $e_{-\lambda}^* = e_{-\lambda}$ and $e_w^* = e_{w^{-1}}$ in $\mathcal{H}(\mathcal{O}_d)$.

**Theorem 8.3.** (i) There is an algebra homomorphism $A: \mathcal{H}(\mathcal{O}_d) \to \mathcal{H}_-$ given by $A(t_\lambda) = t_{2\lambda}$ and $A(e_w) = e_w$ for $\lambda \in \Lambda_c$ and $w \in W$.

(ii) The algebra isomorphism $A$ commutes with anti-involutions $*$ on $\mathcal{H}(\mathcal{O}_d)$ and $\mathcal{H}_-$.

**Proof.** Part (i) follows by comparing relations in $\mathcal{H}(\mathcal{O}_d)$ in [Lu] and those for $\mathcal{H}_-$ in Theorem 8.2. For (ii) we first have $A(e_{-w^{-1}}) = A(e_{w^{-1}}) = e_{w^{-1}} = e_w^*$. By the decomposition $\mathcal{H}_- = H \cdot L \cdot H$, it suffices to show that $A(e_{\lambda}) = (A(e_{-\lambda}))^*$ for a dominant co-character.
\( \lambda \). To that end, let \( w \) be the unique element such that \( w(\Delta) = -\Delta \). Then \( \mu = -\lambda^w \) is again-dominant. Since
\[
\begin{cases}
\ell(\mu w) = \ell(\mu) + \ell(w) \\
\ell(-w \lambda) = \ell(w) + \ell(-\lambda)
\end{cases}
\]
we have \( \ell_{w \lambda} = \ell_{-w \lambda} = \ell_{w \mu} \), and a similar statement for elements in \( \mathcal{H}_- \). Now we have \( A(\ell_{w \mu}) = A(\ell_{-w \mu}) = A(e_w^{-1} e_{w} e_w) = e_w^{-1} A(e_w) e_w = p^{-\ell(\mu)/2} e_w^{-1} e_{2 \mu} e_w = p^{-\ell(\mu)/2} e_{-2 \lambda} = p^{-\ell(\lambda)/2} e_{2 \lambda} = A(\ell_{\lambda})^* \) as required. \( \square \)

9. REPRESENTATIONS WITH IWAHORI FIXED VECTORS

Let \( I \) and \( I' \) denote the Iwahori subgroups of \( G \) and \( \mathcal{G}^i \), respectively which give rise to the isomorphic Iwahori Hecke algebras \( \mathcal{H}_- \) and \( \mathcal{H} = \mathcal{H}(\mathcal{G}^i) \) in Theorem 8.3. Let \( \mathcal{R}(\mathcal{H}_- \) and \( \mathcal{R}(\mathcal{H}) \) denote the categories of finite dimensional representations of the Iwahori-Hecke algebras \( \mathcal{H}_- \) and \( \mathcal{H} \) respectively.

Let \( \mathcal{R}^I(\mathcal{G}) \) denote the category of admissible smooth genuine representations \( V \) of \( G \) such that \( V^I \) generates \( V \) as a \( G \)-module. Similarly we let \( \mathcal{R}^I'(\mathcal{G}^i) \) denote the category of admissible smooth representations \( V \) of \( \mathcal{G}^i \) such that \( V^{I'} \) generates \( V \) as a \( \mathcal{G}^i \)-module.

By [Bo] and [BZ], the functor \( V \mapsto \tilde{V} \) is an equivalence of categories from \( \mathcal{R}^I(\mathcal{G}) \) to \( \mathcal{R}(\mathcal{H}) \). Let \( C_c(\mathcal{G}^i/\mathcal{I}') \) denote locally constant, compactly supported, complex valued functions on \( \mathcal{G}^i/\mathcal{I}' \). This is a right \( \mathcal{H} \)-module. Then the inverse functor is given by \( E \mapsto I(E) := C_c(\mathcal{G}^i/\mathcal{I}') \otimes_{\mathcal{H}} E \).

Similarly the functor \( V \mapsto V^I \) is an equivalence of categories from \( \mathcal{R}^I(\mathcal{G}) \) to \( \mathcal{R}(\mathcal{H}_-) \). Let \( C_{c,-}(G/I) \) denote locally constant, compactly supported, complex valued functions on \( G/I \) such that \( f(\epsilon x I) = \epsilon f(x I) \) for \( \epsilon \in \mu_2 \), \( x \in G \). This is a right \( \mathcal{H}_- \)-module. Then the inverse functor is given by \( E \mapsto I(E) := C_{c,-}(G/I) \otimes_{\mathcal{H}_-} E \).

We recall the isomorphism \( A : \mathcal{H} \rightarrow \mathcal{H}_- \) in Theorem 8.3. This establishes an equivalence of categories between \( \mathcal{R}(\mathcal{H}) \) and \( \mathcal{R}(\mathcal{H}_-) \). Hence the following four categories are equivalent:

\[
\mathcal{R}^I(\mathcal{G}) \simeq \mathcal{R}(\mathcal{H}) \simeq \mathcal{R}(\mathcal{H}_-) \simeq \mathcal{R}^I(\mathcal{G}).
\]

Suppose \( V \) is a representation in \( \mathcal{R}^I(\mathcal{G}) \), then we call the corresponding representation in \( \mathcal{R}^I(\mathcal{G}) \) the local Shimura lift of \( V \). For example, the Shimura lift of \( \Theta(\gamma_p) \) is the trivial representation.

**Hermitian representations.** We gather some facts from [BM1] and [BM2]. Let \( (\pi, E) \) be a finite dimensional representation of \( \mathcal{H} \). We say that \( E \) is a **Hermitian** representation of \( \mathcal{H} \) if there exists a Hermitian form \( \langle \, , \rangle \) on \( E \) such that
\[
\langle \pi(f)v_1, v_2 \rangle = \langle v_1, \pi(f^*)v_2 \rangle
\]
for \( v_1, v_2 \in E \) and \( f \in \mathcal{H} \). We said that \( E \) is a **unitary** representation of \( \mathcal{H} \) if the Hermitian form is positive definite. Similarly we define Hermitian representations and unitary representations of \( \mathcal{H}_- \).

Let \( V \) be a representation in \( \mathcal{R}^I(\mathcal{G}) \) (resp. \( \mathcal{R}^I(\mathcal{G}) \)). Suppose \( \langle \, , \rangle \) is a non-degenerate \( \mathcal{G}^i \)-invariant (resp. \( G \)-invariant) Hermitian form on \( V \). Then the restriction of the Hermitian form on \( V^I \) gives a Hermitian representation of \( \mathcal{H} \) (resp. \( \mathcal{H}_- \)). Similarly, a unitary
representation $V$ gives rise to a unitary representation of the Iwahori-Hecke algebra $\mathcal{H}$ (resp. $\mathcal{H}_-$).

Conversely if $E$ is a Hermitian representation of $\mathcal{H}$ (resp. $\mathcal{H}_-$), then $I(E)$ exhibits an $G^l$-invariant (resp. $G$-invariant) Hermitian form. Moreover, if $E$ is a unitary representation of $\mathcal{H}$ then $I(E)$ is a unitary representation of $G^l$. This non-trivial statement is due to Barbasch and Moy (see [BM1] and Thm 8.1 in [BM2]). This, combined with the equivalence of the four categories in (5) (with the middle isomorphism preserving the anti-involution $*$) gives:

**Theorem 9.1.** If $V$ is an irreducible unitary representation in $\mathcal{R}^l(G)$, then its local Shimura lift to $G^l(Q_p)$ is unitary.

Note that the Shimura lift of the exceptional representation $\Theta(\gamma_p)$ is the trivial representation of $G^l(Q_p)$. We have proved unitarizability of $\Theta(\gamma_p)$ by global methods.

**Corollary 9.2.** Assume that $G \neq SL_2$. Then the unitary representation $\Theta(\gamma_p)$ is isolated in the unitary dual $G(Q_p)$.

**Proof.** Recall that the space of (equivalence classes of) smooth irreducible representation of $G(Q_p)$ is equipped with a Fell topology [Ta]. To every irreducible representation $\Pi$ we can attach a point in the support $\Omega$ of the Bernstein center of $G(Q_p)$. (The support is a disjoint union of complex varieties of dimension less than or equal to the rank of $G(Q_p)$). Tadic in [Ta, Theorem 5.7], shows that this map is continuous and closed. Thus, the question whether $\Theta_p$ is isolated with respect to Fell’s topology is equivalent to the same question for the Bernstein center. Since our isomorphism of Hecke algebras gives an equivalence of categories, $\Theta_p$ must be isolated in the unitary dual since the trivial representation is isolated in the unitary dual of $G^l(Q_p)$.

**Remark:** Theorem 9.1 completes a part of [Hu]. Indeed, a key to Theorem 9.1 is that the isomorphism of Hecke algebras preserves $*$-structures. This was claimed but not verified in [Hu]. In retrospect, a verification of this statement at that time was impossible since normalizations of Hecke operators were not properly defined in [Sa1].

**References**


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