MOTIVES WITH GALOIS GROUP OF TYPE $\mathbb{G}_2$: AN EXCEPTIONAL THETA-CORRESPONDENCE

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Contents

Introduction

Serre has asked if there are motives M with motivic Galois group of type G_2 [Se3; pg 386]. This paper is the first step in a project to construct such a motive M, of rank 7 and weight 0, over the base field $\mathbb{Q}$.

Let G be the anisotropic form of G_2 over $\mathbb{Q}$, and let $\pi = \hat{\otimes}_v \pi_v$ be an automorphic representation of the adelic group $G(\mathbb{A})$. At almost all primes p, the local representation π_p is unramified and has Satake parameter s_p, a semi-simple conjugacy class in the dual group $\hat{G}(\mathbb{C}) = G_2(\mathbb{C})$. Let $\hat{\mathbb{V}}$ be the irreducible 7-dimensional representation of $\hat{G}(\mathbb{C})$. The unramified representation π_p is determined by the characteristic polynomial of s_p on $\hat{\mathbb{V}}$:

$$L(\pi_p, \hat{\mathbb{V}}, X) = \det(1 - s_p X|\hat{\mathbb{V}})^{-1}.$$

By giving a rational structure on the space of modular forms for G, we show that the coefficients of all the polynomials $\det(1 - s_p X|\hat{\mathbb{V}})$ lie in a totally real number field $E \subset \mathbb{C}$. Under the additional hypothesis that at least one local component of π is the Steinberg representation, we conjecture the existence of a motive $M = M(\pi)$ of weight 0 over $\mathbb{Q}$ with coefficients in E, whose local L-function at unramified primes p is given by the formula

$$L_p(M, s) = L(\pi_p, \hat{\mathbb{V}}, p^{-s}).$$

1991 *Mathematics Subject Classification*. Primary 11F70 , Secondary 22E35 and 50.

The first step in our construction of M is to lift π to an automorphic, cuspidal representation π' of the split group $G' = PGSp_6$. Such a lifting, at least at the level of L-packets is predicted by Langlands functoriality. Indeed, we have an inclusion of dual groups

$$\hat{G}(\mathbb{C}) = G_2(\mathbb{C}) \hookrightarrow Spin_7(\mathbb{C}) = \hat{G}'(\mathbb{C})$$

which realizes G_2 as the stabilizer of a non-isotropic vector in the 8-dimensional spin representation of $Spin_7$. We are able to construct a lifting $\pi \to \pi'$ with the property predicted by the functoriality in many cases, using an exceptional theta-correspondence. The group $G \times G'$ appears as a dual pair in the adjoint group H of type E_7 and rank 3 over $\mathbb{Q}$.

The bulk of this paper is devoted to a study of the theta-correspondence which results from the restriction of the minimal representation of H to the subgroup $G \times G'$. The local results were suggested by the work of Huang, Pandžić, and Savin [HPS] on the quaternionic form of E_7 of rank 4 over $\mathbb{R}$. The global results on cuspidality and non-vanishing were inspired by the work of Ginzburg, Rallis, and Soudry [GRS2] in the split case.

The second step in our construction of M is to use the lifted representation π' of G' to define a motive M' in the cohomology of a Siegel modular variety. Specifically, assume that there is a finite, non-empty set S of primes such that π_p is unramified for all $p \notin S$, and π_p is the Steinberg representation for all $p \in S$. Let X be the 6-dimensional Siegel modular variety over $\mathbb{Q}$, classifying principally polarized abelian varieties of dimension 3 with an Iwahori level structure at all $p \in S$. If π_∞ has highest weight $k_1\omega_1 + k_2\omega_2$, where ω_1 is the weight of the 7-dimensional representation and ω_2 is the weight of the 14-dimensional adjoint representation, let $\mathcal{F}$ be the local system on X corresponding to the rational representation of G' with highest weight $k_2\omega_1+k_1\omega_2+k_2\omega_3 = (k_1+2k_2, k_1+k_2, k_2)$. Then the π'_f-isotypic component $M' \subseteq H^6_c(X, \mathcal{F})(3)$ should have rank 8 and coefficients in E. As π' is lifted from G, M' should decompose as the sum of M and a Hodge class, arising from Hilbert modular 3-folds in X. Several difficulties remain in proving this, but we hope to treat these geometric questions in a future paper.

I Spaces of modular forms

In this chapter we develop the arithmetic theory of modular forms for semi-simple groups over $\mathbb{Q}$ with $G(\mathbb{R})$ compact. At the end of this chapter we specialize to the case when G is of type G_2. In particular, we construct two interesting modular forms for the anisotropic form of G_2. It will be shown in Chapter V, that they lift non-trivially to $G' = PGSp_6$.

1. Groups.

Let G be a semi-simple algebraic group over $\mathbb{Q}$ with $G(\mathbb{R})$ compact. To simplify some of the exposition, we will further assume that G is simply connected, and is an inner form of a split group over $\mathbb{Q}$. Then G is split over $\mathbb{Q}_p$ for almost all primes p, and -1 is an element of the Weyl group of G. Also, the center $Z(G)$ is killed by 2.

Let W be an irreducible algebraic representation of G over $\mathbb{Q}$, which is absolutely irreducible (i.e. remains irreducible over $\bar{\mathbb{Q}}$). Then W is orthogonal, and the G-invariant symmetric bilinear form on W is definite over $\mathbb{R}$. We fix a G-invariant, positive-definite inner product

$$<,>: W \times W \to \mathbb{Q}, \tag{1.1}$$

which is unique up to scaling by $\mathbb{Q}^+$.

2. Modular forms.

Let $\hat{\mathbb{Q}} = \mathbb{Q} \otimes \hat{\mathbb{Z}}$ be the ring of finite adèles of $\mathbb{Q}$, so $\mathbb{A} = \mathbb{R} \times \hat{\mathbb{Q}}$ is the ring of adèles. Let K be an open compact subgroup of the locally compact group $G(\hat{\mathbb{Q}})$. Since $G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})$ is compact ([B2]), the double coset space

$$G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/K \tag{2.1}$$

is finite. Let $\{g_\alpha\}$ represent the distinct double cosets, and for each α, define the finite group

$$\begin{aligned} \Gamma_\alpha &= G(\mathbb{Q}) \cap g_\alpha K g_\alpha^{-1} \\ &= \{\gamma \in G(\mathbb{Q}) \mid \gamma g_\alpha K = g_\alpha K\}. \end{aligned} \tag{2.2}$$

Associated to $K \subset G(\hat{\mathbb{Q}})$ and an irreducible representation W of G over $\mathbb{Q}$, we have a rational vector space

$$\begin{aligned} A &= A(K, W) \\ &= \{F : G(\hat{\mathbb{Q}})/K \to W \mid F(\gamma g) = \gamma F(g), \text{ all } \gamma \in G(\mathbb{Q})\}. \end{aligned} \tag{2.3}$$

This is the space of modular forms of "level K" and "weight W" for G.

A function F in A is clearly determined by the values $F(g_\alpha)$ on the double coset representatives, and $F(g_\alpha)$ lies in the subspace of W fixed by Γ_α. This observation gives a proof of the following.

Proposition 2.3. *The space A is finite-dimensional and the map taking F to the elements $F(g_\alpha)$ in W^{Γ_α} is a linear isomorphism $A \cong \oplus_\alpha W^{\Gamma_\alpha}$.*

We can use the proposition to define an inner product on A with values in $\mathbb{Q}$, by the formula

$$< F, F' >_A = \sum_\alpha w_\alpha^{-1} < F(g_\alpha), F'(g_\alpha) >, \tag{2.4}$$

where $w_\alpha = Card(\Gamma_\alpha)$. This is independent of the choice of coset representatives, as the pairing $<,>$ on W is $G(\mathbb{Q})$-invariant.

3. Hecke operators.

The Hecke algebra of K is the convolution algebra of locally constant, compactly supported functions

$$\begin{aligned} \mathcal{H}_K =& \mathcal{H}(G(\hat{\mathbb{Q}})//K) \\ =& \{f : K\backslash G(\hat{\mathbb{Q}})/K \to \mathbb{Q}\}, \end{aligned} \tag{3.1}$$

using Haar measure giving K volume 1. This has, as additive basis, the characterisitic functions $char(KtK)$ of double cosets, and acts $\mathbb{Q}$-linearly on A as follows. Let $F \in A$. Writing

$$KtK = \cup_i t_i K \tag{3.2}$$

where the number of single cosets is finite, we have the formula

$$char(KtK)|F(g) = \sum_i F(gt_i). \tag{3.3}$$

The following adjoint formula shows that A is a semi-simple $\mathcal{H}_K$-module.

Proposition 3.4.

$$< char(KtK)|F, F' >_A = < F, char(Kt^{-1}K)|F' >_A .$$

Proof. This is standard. See, for example, [Shm].

Over the algebraically closed field $\mathbb{C}$, the isotypic decomposition of $A \otimes \mathbb{C}$ as an $\mathcal{H}_K \otimes \mathbb{C}$ module is given by the theory of automorphic forms. Let m be an element in

$$Hom_{G(\mathbb{R})}(W \otimes \mathbb{C}, L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K)). \tag{3.6}$$

If w is an element in $W \otimes \mathbb{C}$, the function $m_w = m(w)$ is smooth. In particular, for any g in $G(\mathbb{A})$, the complex number $m_w(g)$ is well defined. We define an element $F = F(m)$ in $A \otimes \mathbb{C}$ by the formula

$$< F(g_f), w >= m_w(1 \times g_f), \tag{3.6}$$

where g_f is in $G(\hat{\mathbb{Q}})$.

Proposition 3.7. *The map $m \mapsto F(m)$ gives a linear isomorphism of $\mathcal{H}_K \otimes \mathbb{C}$-modules*

$$Hom_{G(\mathbb{R})}(W \otimes \mathbb{C}, L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K)) \cong A \otimes \mathbb{C}.$$

Proof. We construct an inverse map. Let F be in $A \otimes \mathbb{C}$. If g is an element in $G(\mathbb{A})$, write $g = g_\infty \times g_f$ where g_∞ is in $G(\mathbb{R})$, and g_f is in $G(\hat{\mathbb{Q}})$. We define an element $m = m(F)$ in $Hom_{G(\mathbb{R})}(W \otimes \mathbb{C}, L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K))$ by the formula

$$m_w(g) =< g_\infty w, F(g_f) > .$$

The map $F \mapsto m(F)$ is the inverse of the map $m \mapsto F(m)$.

4. Spherical operators.

Assume that $K = \prod K_p$ in $G(\hat{\mathbb{Q}})$. If

$$\mathcal{H}_{K_p} = \{f_p : K_p\backslash G(\mathbb{Q}_p)/K_p \to \mathbb{Q}\} \tag{4.1}$$

is the local Hecke algebra, we obtain a map of $\mathbb{Q}$-algebras $\mathcal{H}_{K_p} \to \mathcal{H}_K$ taking f_p to the function $f = f_p \otimes char(\prod_{l \neq p} K_l)$ on $G(\hat{\mathbb{Q}})$. In fact, we obtain an isomorphism of the restricted tensor product of local algebras (with respect to the unit element) and $\mathcal{H}_K$

$$\hat{\otimes}_p \mathcal{H}_{K_p} \cong \mathcal{H}_K. \tag{4.2}$$

We say the prime p is unramified for K if $G(\mathbb{Q}_p)$ is split and $K \cap G(\mathbb{Q}_p) = K_p$ is a hyperspecial maximal compact subgroup. In this case, the Satake isomorphism gives an identification [Ct; pg 148]

$$\mathcal{H}_{K_p} \otimes_{\mathbb{Q}} \mathbb{Q}[p^{1/2}] \cong R(\hat{G}) \otimes_{\mathbb{Z}} \mathbb{Q}[p^{1/2}], \tag{4.3}$$

where $\hat{G}$ is the dual Langlands group and $R(\hat{G})$ its representation ring. The half-integral powers enter only in terms $p^{<\lambda,\rho>}$, where λ is a co-character of a maximal torus $T \subset G\otimes\mathbb{Q}_p$, and ρ is half the sum of the positive roots relative to a Borel subgroup containing T. Since we have assumed that G is simply connected, $<\lambda,\rho>$ is always an integer, and the Satake transform gives an isomorphism of $\mathbb{Q}$-algebras

$$\mathcal{H}_{K_p} \cong R(\hat{G}) \otimes \mathbb{Q}. \tag{4.4}$$

Let $\mathbb{T}$ be the commutative $\mathbb{Q}$-subalgebra of $\mathcal{H}_K$ generated by the local algebras $\mathcal{H}_{K_p}$ for all unramified p. Then by (4.4)

$$\mathbb{T} \cong \hat{\otimes}_{p \text{ unram}} R(\hat{G}) \otimes \mathbb{Q}. \tag{4.5}$$

Proposition 4.6. *The elements of* $\mathbb{T}$ *give commuting self-adjoint operators on* A.

Proof. We have
$$Kt_pK = Kt_p^{-1}K$$
for all t_p in $G(\mathbb{Q}_p)$. Indeed, if T is a split torus in $G(\mathbb{Q}_p)$, then $G(\mathbb{Q}_p) = K_pTK_p$, by the Cartan decomposition. Since -1 is in the Weyl group of G, there is an element n in K_l such that $nt_pn^{-1} = t_p^{-1}$. The proposition now follows from Prop. 3.4.

Let F be an eigenvector for $\mathbb{T}$ acting on the space $A \otimes \mathbb{C}$. By (4.4), F gives rise to an element

$$s_p \in Hom(R(\hat{G}), \mathbb{C}) \tag{4.7}$$

for all unramified primes p. But the spectrum of $R(\hat{G})\otimes\mathbb{C}$ consists of the set of semi-simple conjugacy classes in $\hat{G}(\mathbb{C})$. Thus we have shown

Proposition 4.8. *If* F *is an eigenvector for* $\mathbb{T}$ *in* $A \otimes \mathbb{C}$, *the eigenvalues determine a collection* $\{s_p\}$ *of semi-simple conjugacy classes in* $\hat{G}(\mathbb{C})$, *indexed by the unramified primes* p *for* K.

We note that by Prop. 4.6 each such eigenvector in $A \otimes \mathbb{C}$ is actually defined over a totally real number field.

5. The Steinberg subspace.

Assume that G is split and quasi-simple over $\mathbb{Q}_p$, and K_p a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$. We now consider the space $A = A(K, W)$ of forms of weight W and level K in the special case when $K = I_p \times K'$ where I_p is an Iwahori subgroup of $G(\mathbb{Q}_p)$ contained in K_p. The Iwahori Hecke algebra $\mathcal{H}_{I_p} = \mathcal{H}(G(\mathbb{Q}_p)//I_p)$ then has a distinguished rational character, corresponding to the Steinberg representation St_p of $G(\mathbb{Q}_p)$. This character sends the standard Iwahori-Matsumoto generators [Lu] of $\mathcal{H}_{I_p}$ to -1.

Let

$$A(St_p) \subseteq A \tag{5.1}$$

be the $\mathbb{Q}$-subspace of modular forms on which $\mathcal{H}_{I_p}$ acts by the Steinberg character. This is an $\mathcal{H}_{K'}$-submodule of A. Our aim in this section is to show it is a reasonably large subspace of A, so that there are many automorphic representations with local component St_p. To first order, we will show that:

$$\frac{\dim A(St_p)}{\dim A} = 1 - \frac{r+1}{p} + O(\frac{1}{p^2}), \qquad r = rank(G). \tag{5.2}$$

The algebra $\mathcal{H}_{I_p}$ contains the finite sub-algebra $\mathcal{H}(K_p//I_p)$. Let ϵ_p be the restriction to $\mathcal{H}(K_p//I_p)$ of the Steinberg character of $\mathcal{H}_{I_p}$. This character corresponds to the Steinberg representation of the finite Chevalley group $G(p)$ [Ca]. The space $A(St_p)$ is contained in the subspace

$$A(\epsilon_p) \subseteq A \tag{5.3}$$

on which the finite algebra acts by the character ϵ_p. To first order, we will show that

$$\frac{\dim A(\epsilon_p)}{\dim A} = 1 - \frac{r}{p} + O(\frac{1}{p^2}), \qquad r = rank(G). \tag{5.4}$$

Let $\mathcal{F}$ be a maximal simplex in the building of G over $\mathbb{Q}_p$, which is fixed by the Iwahori subgroup I_p, and for each non-empty face $\mathcal{F}_i$ of $\mathcal{F}$ let $I_p(i) \supseteq I_p$ be the stabilizer of $\mathcal{F}_i$. Let $A(i)$ be the subspace of A of forms of weight W and level $K(i) = I_p(i) \times K'$.

Finally, we let $A(1)$ be the subspace of A corresponding to the trivial 1-dimensional representation of G. Then $\dim A(1) = 1$ when $W = \mathbb{Q}$, and $\dim A(1) = 0$ otherwise.

Proposition 5.5. *In the Grothendieck group of finite-dimensional $\mathcal{H}_{K'}$-modules:*

$$A(St_p) + (-1)^r A(1) = \sum_{\mathcal{F}_i} (-1)^{\operatorname{codim} \mathcal{F}_i} A(i)$$

$$A(\epsilon_p) = \sum_{\mathcal{F}_i \supseteq \mathcal{F}_0} (-1)^{\operatorname{codim} \mathcal{F}_i} A(i)$$

where $\mathcal{F}_0$ is the hyperspecial vertex fixed by K_p.

Proof. The first follows from the Casselman's formula [BW]:

$$\sum_{\mathcal{F}_i} (-1)^{\operatorname{codim} \mathcal{F}_i} \dim(\pi^{I_p(i)}) = \begin{cases} 1 & \pi = St_p \\ (-1)^r & \pi = 1 \\ 0 & \text{otherwise} \end{cases}$$

for the Euler characteristic of the continous cohomology of a unitary irreducible representation π of $G(\mathbb{Q}_p)$. The second formula follows from the standard resolution of the Steinberg character of the finite Chevalley group $G(p)$ [Ca; pg 187]. Note: if we include the empty face $\mathcal{F}_\emptyset$, with the stabilizer $K(\emptyset) = G$, then the formula for $A(St_p)$ can be written without the correction term $A(1)$.

We can compute $\dim A(i)$ if $G(\mathbb{Q})$ acts freely on $G(\hat{\mathbb{Q}})/I_p(i) \times K'$. In this case:

$$\dim A = (I_p(i) : I_p) \dim A(i). \tag{5.6}$$

We say that K' is p-neat if $G(\mathbb{Q})$ acts freely on $G(\hat{\mathbb{Q}})/I_p(i) \times K'$ for all i, and that it is weakly neat if $G(\mathbb{Q})$ acts freely on $G(\hat{\mathbb{Q}})/I_p(i) \times K'$ for all $I_p(i) \subseteq I_p(0) = K_p$. The Proposition and (5.6) give a formula for $\dim A(St_p)$ and $\dim A(\epsilon_p)$:

Corollary 5.7. *If K' is weakly neat, we have*

$$\dim A = \prod_{i=1}^{r} \frac{p^{e_i+1}-1}{p-1} \dim A(0)$$

and

$$\dim A(\epsilon_p) = \prod_{i=1}^{r} p^{e_i} \dim A(0).$$

If K' is neat we also have

$$\dim A(St_p) = \prod_{i=1}^{r} (p^{e_i}-1) \dim A(0) - (-1)^r \dim A(1).$$

Here $e_1, e_2, \dots, e_r$ are the exponents for the Weyl group of G [Bu; pg 118].

Proof. This is standard inclusion-exclusion [Se1; pgs 664-667]. For example,

$$\prod_{i=1}^{r} \frac{p^{e_i+1}-1}{p-1} = (K_p : I_p) = (G(p) : B(p))$$

$$\prod_{i=1}^{r} p^{e_i} = p^{\dim U(p)} = \sum_{I_p(i) \subseteq K_p} (-1)^{\operatorname{codim} \mathcal{F}_i} (K_p : I_p(i))$$

where $B(p)$ is a Borel subgroup of $G(p)$ and $U(p)$ the unipotent radical of $B(p)$.

6. The case $G = Aut(\mathbb{O})$.

We work out some details of the theory presented in this chapter for the simplest case of a group G with a model over $\mathbb{Z}$ with $G(\mathbb{R})$ compact, and $G(\mathbb{Q}_p)$ split for all p. Namely G is the simple group of type G_2 over $\mathbb{Q}$ defined as the automorphism group of Cayley's octonion algebra $\mathbb{O}$ [J3]. We recall that $\mathbb{O}$ is a non-associative division algebra of rank 8 over $\mathbb{Q}$:

$$\begin{cases} \mathbb{Q} + \mathbb{Q}e_1 + \mathbb{Q}e_2 + \mathbb{Q}e_3 + \mathbb{Q}e_4 + \mathbb{Q}e_5 + \mathbb{Q}e_6 + \mathbb{Q}e_7 \\ e_i^2 = -1 \text{ all } i \\ e_i \cdot (e_{i+1} \cdot e_{i+3}) = (e_i \cdot e_{i+1}) \cdot e_{i+3} \text{ all } i \pmod 7. \end{cases} \tag{6.1}$$

The map $x = a_0 + \sum a_i e_i \mapsto \bar{x} = a_0 - \sum a_i e_i$ defines an anti-involution of $\mathbb{O}$, with fixed field $\mathbb{Q}$.

On $\mathbb{O}$, we have the trace

$$\begin{aligned} \operatorname{Tr} : \mathbb{O} &\to \mathbb{Q} \\ x &\mapsto x + \bar{x} = 2a_0 \end{aligned} \tag{6.2}$$

which is $\mathbb{Q}$-linear, and the norm

$$\begin{aligned} \mathbb{N} : \mathbb{O} &\to \mathbb{Q} \\ x &\mapsto x \cdot \bar{x} = \bar{x} \cdot x = a_0^2 + \sum a_i^2 \end{aligned} \tag{6.3}$$

which satisifies $\mathbb{N}(x \cdot y) = \mathbb{N}(x)\mathbb{N}(y)$. Although the multiplication is neither commutative nor associative, we have

$$\begin{aligned} \mathrm{Tr}(x \cdot y) &= \mathrm{Tr}(y \cdot x) \\ \mathrm{Tr}(x \cdot (y \cdot z)) &= \mathrm{Tr}((x \cdot y) \cdot z). \end{aligned} \tag{6.4}$$

We denote the latter rational number simply by $\mathrm{Tr}(xyz)$.

Let $\mathcal{R}$ [Co] be the $\mathbb{Z}$-lattice in $\mathbb{O}$ spanned by the e_i and the elements

$$\begin{cases} \frac{1}{2}(1 + e_1 + e_2 + e_4) \\ \frac{1}{2}(1 + e_1 + e_3 + e_7) \\ \frac{1}{2}(1 + e_1 + e_5 + e_6) \\ \frac{1}{2}(e_1 + e_2 + e_3 + e_5). \end{cases} \tag{6.5}$$

Then $\mathcal{R}$ is stable under octonionic multiplication, and $G = Aut(\mathcal{R})$ is the unique model over $\mathbb{Z}$ with good reduction at all primes [Gr]. We write $G(p)$ for the finite group $G(\mathbb{F}_p) = Aut(\mathcal{R}/p\mathcal{R})$.

The groups Γ_α stabilizing the cosets $g_\alpha K$ (2.2) have orders dividing $2^6 3^3 7$ [Se2]. When $K \subseteq G(\hat{\mathbb{Z}})$ they are all subgroups of $G(\mathbb{Z})$. The group $G(\mathbb{Z})$ has order $2^6 3^3 7 = 12096$, and is isomorphic to $G(2)$ under reduction modulo 2 [A].

The irreducible representations W of G considered in Section 1 of this chapter, can all be constructed from the irreducible representation V on octonions with trace 0. Namely, the 14-dimensional adjoint representation $\mathfrak{g}$ of G is the kernel of the map

$$\begin{aligned} \wedge^2 V &\to V \\ v \wedge w &\mapsto v \cdot w - w \cdot v \end{aligned} \tag{6.6}$$

and for $k_1, k_2 \geq 0$ there is an irreducible representation $W = W(k_1, k_2)$ of G defined over $\mathbb{Q}$ which occurs as the G-submodule of highest weight in $V^{\otimes^{k_1}} \otimes \mathfrak{g}^{\otimes^{k_2}}$. W has dimension

$$\frac{(k_1+1)(k_2+1)(k_1+k_2+2)(k_1+2k_2+3)(k_1+3k_2+4)(2k_1+3k_2+5)}{120}. \tag{6.7}$$

We fix the inner product $<,>$ on W by taking $< v, w >= Tr(\bar{v}w)$ on V, using the second exterior power of this product on $\mathfrak{g} \subset \wedge^2 V$, and then taking the tensor product of the previously defined inner products on $W \subseteq V^{\otimes^{k_1}} \otimes \mathfrak{g}^{\otimes^{k_2}}$.

Let $K_p = G(\mathbb{Z}_p)$. We make the Satake isomorphism (4.4)

$$\mathcal{H}_{K_p} \cong R(\hat{G}) \otimes \mathbb{Q}$$

completely explicit. Let $T \subset B \subset G$ be a maximal split torus in a Borel subgroup, all defined over $\mathbb{Z}_p$. Let $\{\alpha_1, \alpha_2\}$ be the corresponding root basis for the character group of T, where α_1 long and α_2 short, and let $\check{\omega}_1$ and $\check{\omega}_2$ be the dual basis for the co-characters. We define

$$\begin{cases} t_1 = char(K_p \check{\omega}_1(p) K_p) \\ t_2 = char(K_p \check{\omega}_2(p) K_p) \end{cases} \tag{6.8}$$

in $\mathcal{H}_{K_p}$. These have degrees (the number of single K_p-cosets)

$$\begin{cases} d(t_1) = p\frac{p^6-1}{p-1} \\ d(t_2) = p^5\frac{p^6-1}{p-1} \end{cases} \tag{6.9}$$

The elements $\check{\omega}_1$ and $\check{\omega}_2$ are the fundamental weights of the dual group $\hat{G}(\mathbb{C}) = G_2(\mathbb{C})$. Since $\check{\omega}_1$ is short and $\check{\omega}_2$ is long, $\check{\omega}_1$ corresponds to the 7-dimensional representation $\hat{V}$ and $\check{\omega}_2$ corresponds to the 14-dimensional adjoint representation $\hat{\mathfrak{g}}$. Let χ_1 and χ_2 denote the characters of these representations, so $R(\hat{G}) = \mathbb{Z}[\chi_1, \chi_2]$.

In the inverse of the Satake isomorphism:

$$\begin{cases} \chi_1 \text{ maps to } (t_1 + 1)/p^3 \\ \chi_2 \text{ maps to } (t_2 + t_1 + p^4 + 1)/p^5 \end{cases} \tag{6.10}$$

Hence $\mathcal{H}_{K_p} = \mathbb{Q}[t_1, t_2]$. If F in $A \otimes \mathbb{R}$ is an eigenvector for $\mathcal{H}_{K_p}$ with

$$\begin{cases} t_1 | F = \lambda_1 F \\ t_2 | F = \lambda_2 F \end{cases} \tag{6.11}$$

then the semi-simple class $s = s_p(F)$ in $\hat{G}(\mathbb{C})$ has the following characteristic polynomial on $\hat{V}$:

$$\det(1 - sT|\hat{V}) = 1 - a_1 T + a_2 T^2 - a_3 T^3 + a_4 T^4 - a_5 T^5 + a_6 T^6 - T^7 \tag{6.12}$$

where

$$\begin{aligned} a_1 =& (\lambda_1 + 1)/p^3 \\ a_2 =& (\lambda_2 + (p^2+1)\lambda_1 + (p^4 + p^2 + 1))/p^5 \\ a_3 =& a_1^2 + a_1 - a_2 \end{aligned}$$

and $a_i = a_{7-i}$ for $i = 1, \ldots, 6$.

The Iwahori Hecke algebra $\mathcal{H}_{I_p}$ has generators T_0, T_1, T_2 and relations

$$\begin{aligned} (T_i - p)(T_i + 1) = 0 \qquad i = 0, 1, 2 \\ T_0 T_2 = T_2 T_0 \\ T_0 T_1 T_0 = T_1 T_0 T_1 \\ (T_1 T_2)^3 = (T_2 T_1)^3. \end{aligned} \tag{6.13}$$

The subalgebra $\mathcal{H}(K_p // I_p) \cong \mathcal{H}(G(p)//B(p))$ has generators T_1 and T_2.

7. Examples of modular forms on $G = Aut(\mathbb{O})$.

We now do some explicit examples, where K is a subgroup of finite index in $G(\hat{\mathbb{Z}})$. Let p be a rational prime, and let $B(p) \subset G(p)$ be a Borel subgroup. We let

$$K(p) \subseteq K_0(p) \subseteq G(\hat{\mathbb{Z}}) \tag{7.1}$$

be the subgroups reducing to $1 \pmod p$ and to $B(p) \pmod p$ respectively. Then $K(p)$ is a normal subgroup of $G(\hat{\mathbb{Z}})$, and the local component of $K_0(p)$ at p is an Iwahori subgroup. In particular, the spaces

$$A(St_p) \subseteq A(\epsilon_p) \subseteq A(W, K_0(p)) \tag{7.2}$$

are defined.

Consider first the case when $K = K(2)$ and W is arbitrary. Since $G(\hat{\mathbb{Q}}) = G(\mathbb{Q})G(\hat{\mathbb{Z}})$, and $G(\mathbb{Z}) = G(2)$ [Gr],

$$G(\hat{\mathbb{Q}}) = G(\mathbb{Q}) \times K(2) \tag{7.3}$$

so we have one double coset, with $\Gamma = 1$. Hence by (2.3)

$$A(W, K(2)) \cong W \tag{7.4}$$

as $\mathbb{Q}$-vector spaces. This is actually an isomorphism of $G(2) = G(\mathbb{Z})$-modules. By (2.3) we obtain isomorphisms

$$\begin{cases} A(W, K_0(2)) \cong W^{B(2)} \\ A(W, G(\hat{\mathbb{Z}})) \cong W^{G(2)} \end{cases} \tag{7.5}$$

of $\mathbb{Q}$-vector spaces. The former is an isomorphism of $\mathcal{H}(G(\hat{\mathbb{Z}})//K_0(2)) \cong \mathcal{H}(G(2)//B(2))$-modules.

Consider the special case when $W = W(1,1)$ has dimension $64 = 2^6$. The restriction of W to $G(\hat{\mathbb{Z}})$ is isomorphic to the Steinberg representation st_2 of $G(2)$. Hence

$$A(\epsilon_2) = A(W, K_0(2)) \tag{7.6}$$

is one-dimensional.

Proposition 7.7. *There exists a unique automorphic representation π of $G(\mathbb{A})$ with $\pi_\infty \cong W(1,1) \otimes \mathbb{C}$, $\pi_2 \cong St_2$, the Steinberg representation, and π_p unramified for all $p \neq 2$.*

Proof. The $K_0(2)$-fixed vectors in such a π contribute a line to the space $A(St_2) \otimes \mathbb{C}$, so we must show that $A(\epsilon_2) = A(St_2)$.

The elements T_1 and T_2 generate the Hecke algebra $\mathcal{H}(G(2)//B(2))$. Since $T_1 = T_2 = -1$ on $A(\epsilon_2)$, and $T_0T_1T_0 = T_1T_0T_1$, we have $-T_0^2 = T_0$ on $A(\epsilon_2)$, so T_0 must act as -1 also.

Finally, consider the case when $W = W(0,0) = \mathbb{Q}$ is the trivial representation of G and $K = K_0(p)$. From (7.2) we have an isomorphism of $\mathbb{Q}$-vector spaces

$$\begin{cases} A(\mathbb{Q}, K_0(p)) \cong (\mathrm{Ind}_{B(p)}^{G(p)} \mathbb{Q})^{G(\mathbb{Z})} \\ A(\mathbb{Q}, K_0(p))(\epsilon_p) \cong (st_p)^{G(\mathbb{Z})} \end{cases} \tag{7.8}$$

Here st_p is the Steinberg representation of $G(p)$. Its dimension is p^6, and appears with multiplicity one in the induced representation from $B(p)$.

Let $d(p) = \dim(st_p)^{G(\mathbb{Z})}$. For small primes p, we have the following table (computed by D. Pollack and J. Lansky):

(7.9)

p	2	3	5	7	11	13	17	19
$d(p)$	0	0	1	13	142	416	1980	3931

For $p \neq 2, 3, 7$, K. Magaard has shown that

$$d(p) = (p^6 \pm 56p^3 + 315p^2 + ap + b)/12096 \tag{7.10}$$

where the sign $\pm$ is chosen so that $p \equiv \pm 1 \pmod 3$, and the coefficients a and b depend on the conguence of p modulo 24:

(7.11)

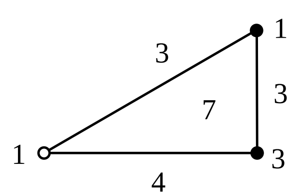
$p \equiv$	1 (24)	5 (24)	7 (24)	11 (24)	13 (24)	17 (24)	19 (24)	23 (24)
a	1932	−420	420	−1932	1932	−420	420	−1932
b	9792	−2304	3744	3744	3744	3744	−2304	9792

Proposition 7.12. *There is a unique automorphic representation π of $G(\mathbb{A})$ with $\pi_\infty \cong \mathbb{C}$, $\pi_5 \cong St_5$, and π_p unramified for $p \neq 5$.*

Proof. We must show that

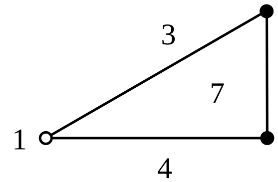
$$A(\mathbb{Q}, K_0(5))(St_5)$$

is one-dimensional. To do this, we will compute the dimensions of the spaces $A(\mathbb{Q}, K(i))$, where $K(i) = \prod_{p \neq 5} G(\mathbb{Z}_p) \times I_5(i)$ and $I_5(i)$ is an arbitrary parahoric at prime 5. The parahorics are indexed by the facets of a 30-60-90 triangle (a maximal simplex in the building, the white vertex is hyperspecial), and we will find the dimensions:



This gives $\dim A(St_5) = 7 - (3 + 3 + 4) + (1 + 1 + 3) - 1 = 1$ by Casselman's formula (5.5).

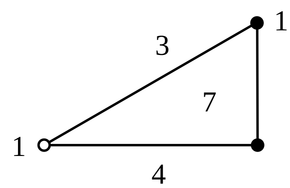
The dimensions



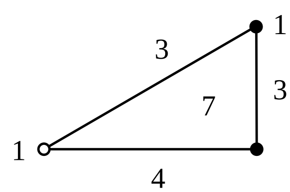
are easily computed by the above considerations, as the corresponding parahorics $I_5(i)$ are contained in $G(\mathbb{Z}_5)$. Now let $I_5(i)$ be the maximal parahoric whose reduction $\pmod 5$ is $SL_3(5)$. Using the mass formula [Gr], we find:

$$\sum_{G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/K(i)} \frac{1}{\#\Gamma_\alpha} = \frac{1}{2^5 3}.$$

Since every finite subgroup of $K(i)$ has order dividing $2^5 3$, we find a single double coset, with stabilizer of order $2^5 3$, reducing to the normalizer of a maximal split torus in $SL_3(5)$. Hence $\dim A(\mathbb{Q}, K(i)) = 1$, so we have obtained the dimensions:



Since the normalizer of a split maximal torus in $SL_3(5)$ has precisely 3 orbits on $\mathbb{P}^2(5)$, we obtain the dimensions:



It remains to compute the dimension for $K(i)$ with local component $I_5(i)$ maximal, whose reduction $\pmod 5$ is split $SO_4(5)$. Here the mass formula yields

$$\sum_{G(\mathbb{Q})\backslash G(\hat{\mathbb{Q}})/K(i)} \frac{1}{\#\Gamma_\alpha} = \frac{31}{2^6 3^2}.$$

Hence the number of double cosets is either 2 or 3. Since the order of a finite subgroup of $K(i)$ divides $2^6 3^2$, and we can not write the mass as a sum of two such terms, the dimension is 3 as claimed.

II Motives

In this chapter, we present a conjecture on motives of rank 7 over $\mathbb{Q}$ associated to automorphic forms on the anisotropic form $G = Aut(\mathbb{O})$ of G_2. Since we hope to construct these motives as the orthogonal complement of a Hodge class in a motive of rank 8, we give local criteria which allow one to show that a subgroup Γ of SO_8 is contained in either $Spin_7$ or $G_2 = Spin_7 \cap SO_7$. Finally, we discuss the simply connected form of E_7 of rank 3 over $\mathbb{Q}$, its 56-dimensional representation, and the dual pair $G \times G' = Aut(\mathbb{O}) \times PGSp_6$ in the associated adjoint group.

1. A conjecture on G_2-motives.

Let $G = Aut(\mathcal{R})$ be the form of G over $\mathbb{Z}$ constructed in (6.2) in Chapter I, and fix a finite, non-empty set S of primes. For p in S, let $K_p \subset G(\mathbb{Z}_p)$ be an Iwahori subgroup. For p not in S, let $K_p = G(\mathbb{Z}_p)$.

Fix an irreducible representation W of G, and put $K = \prod_p K_p$. Let $A = A(W, K)$ be the associated space of automorphic forms, and

$$A_S = A(St_S) \subseteq A \tag{1.1}$$

the subspace which is localized at the Steinberg representation St_p, for all primes p in S. Then A_S is a finite-dimensional inner product space over $\mathbb{Q}$, with

$$\dim A_S \approx \frac{\dim W}{12096} \prod_{p \in S} (p^5 - 1)(p - 1) \tag{1.2}$$

by (5.2) of Section 1.

The spherical Hecke algebra $\hat{\otimes}_{p \notin S} \mathcal{H}(G(\mathbb{Q}_p)//G(\mathbb{Z}_p))$ acts on A_S, via self-adjoint commuting operators. Let F be a simultaneous eigenvector, defined over a totally real number-field E. Then F corresponds to an irreducible automorphic representation of $G(\mathbb{A})$

$$\pi = \hat{\otimes} \pi_v \tag{1.3}$$

with $\pi_\infty \cong W \otimes \mathbb{C}$, $\pi_p \cong St_p$ for all p in S and π_p spherical for all primes p not in S.

Conjecture 1.4. *Associated to the eigenvector F in A_S (or to the automorphic representation π), there is a motive M of rank 7 and weight 0 over $\mathbb{Q}$ with coeffcients in E. The motive M enjoys the following local properties.*

(1) *Assume $W = W(k_1, k_2)$. The Hodge components $M^{p,q}$ of $M_B \otimes \mathbb{C}$ have rank 1 over $E \otimes \mathbb{C}$ for those (p, q) which satisfy $p + q = 0$ and*

$$\begin{aligned} p = & 3 + k_1 + 2k_2 \\ & 2 + k_1 + k_2 \\ & 1 + k_2 \\ & 0 \\ & -(1 + k_2) \\ & -(2 + k_1 + k_2) \\ & -(3 + k_1 + 2k_2). \end{aligned}$$

Otherwise $M^{p,q} = 0$. The real Frobenius F_∞ acts as -1 on $M^{0,0}$.

(2) *Assume $p \notin S$, so π_p is spherical with Satake parameter s_p. Let λ be a finite prime of E not dividing p. Then the λ-adic representation M_λ of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ is unramified at p. If F_p is a Frobenius element, then*

$$\det(1 - F_p T | M_\lambda) = \det(1 - s_p T | \hat{V}_{\mathbb{C}})$$

has coefficients in E.

(3) *Assume $p \in S$, so $\pi_p \cong St_p$. Let λ be a finite prime of E not dividing p. Then the λ-adic representation M_λ of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ is tamely ramified at p, and isomorphic to $S^6H^1(X, E_\lambda)(3)$, where X is a Tate elliptic curve over $\mathbb{Q}_p$.*

In particular, part 2) of the conjecture implies that the local components of π are all tempered. This need not be true for arbitrary eigenvectors F in $A \otimes \mathbb{C}$, but should be true for eigenvectors in the subspace $A(St_p)$ for any p in S. Also, we expect the spectrum of the spherical Hecke algebra to be multiplicity free on $A_S \otimes \mathbb{R}$.

If the motive of Conj. 1.4 exists, its L-function at finite primes is given by the Euler product

$$L(M, s) = \prod_{p \in S} (1 - p^{-3-s})^{-1} \prod_{p \notin S} \det(1 - s_p \cdot p^{-s} | \hat{\mathbb{V}})^{-1} \tag{1.5}$$

which converges for $Re(s) > 1$. This is a Dirichlet series $\sum_{n \geq 1} a_n n^{-s}$ with coefficients in the subfield E of $\mathbb{C}$. If [De; pg 329]

$$L_\infty(M, s) = \Gamma_{\mathbb{C}}(s + (3 + k_1 + 2k_2))\Gamma_{\mathbb{C}}(s + (2 + k_1 + k_2))\Gamma_{\mathbb{C}}(s + (1 + k_2))\Gamma_{\mathbb{R}}(s + 1) \tag{1.6}$$

is the Archimedean L-factor, then the product $\Lambda(M, s) = L_\infty(M, s)L(M, s)$ should have an analytic continuation to the entire s-plane, and satisfy the functional equation

$$\Lambda(M, s) = N^{\frac{1}{2} - s}\Lambda(M, 1 - s) \tag{1.7}$$

with exponential factor $N = \prod_{p \in S} p^6$.

The fact that $F_\infty = -1$ on $M^{0,0}$ implies that $s = 0$ and $s = 1$ are critical for $L(M, s)$, in the sense of Deligne [De; pg 318]. If M is realized as the complement of a Hodge class in the orthogonal motive M' of rank 8, as suggested in the introduction, then

$$\begin{cases} L(M', s) = \zeta(s)L(M, s) \\ L(M, 1) = Res_{s=1}L(M', s)ds. \end{cases} \tag{1.8}$$

2. Subgroups of O_8.

As described in the introduction, we hope to construct $M \subset M' \subseteq H_c^6(X, \mathcal{F})(3)$, where M' is an orthogonal motive of rank 8. It is therefore useful to have criteria which allow to conclude that the motivic Galois group of M' is a proper subgroup of the orthogonal group $O(M') = O_8$.

Let W be a non-degenerate quadratic space of dimension 8 over $\mathbb{C}$, and let s be a semi-simple conjugacy class in $O(W) = O_8$. Let

$$f(T) = \det(1 - sT|W) = \sum_{k=0}^{8} (-1)^k \operatorname{Tr}(s| \wedge^k W)T^k \tag{2.1}$$

be the characteristic polynomial of s on W.

Proposition 2.2.

1) If s lives in the normal subgroup SO_8 of elements with $\det(s) = 1$, then $f(T)$ has the form

$$f(T) = 1 - AT + BT^2 - CT^3 + DT^4 - CT^5 + BT^6 - AT^7 + T^8$$

where the coefficients (A, B, C, D) are (arbitrary) elements in $\mathbb{C}^4$.

2) If s lives in the subgroup $SO_7 \subset SO_8$ fixing a non-isotropic line, or in the subgroup $SO_3 \times SO_5 \subset SO_8$ stabilizing an orthogonal decomposition $W = W_3 \oplus W_5$, then the coefficients (A, B, C, D) of $f(T)$ are (arbitrary) elements in the hyperplane

$$2A - 2B + 2C - D - 2 = 0.$$

3) If s lies in the subgroup $Spin_7 \subset SO_8$ embedded by the spin representation, or in the subgroup $SL_2 \times Sp_4/\Delta < \pm 1 > \cong Spin_3 \times Spin_5/\Delta < \pm 1 > \subset SO_8$ embedded by the tensor product of the two spin representations, then the coefficients (A, B, C, D) of $f(T)$ are (arbitrary) elements in the hypersurface

$$A^2(D + 2B + 1) = C^2 + 2AC + A^4.$$

4) If s lies in the subgroup $G_2 = Spin_7 \cap SO_7 \subset SO_8$ of $Spin_7$ fixing a non-isotropic line, or in the subgroup $PGL_3 \subset SO_8$ embedded by the adjoint representation, then the coefficients A and B of $f(T)$ are (arbitrary) elements of $\mathbb{C}$, and the coefficients C and D are given by

$$\begin{cases} C = A^2 - A \\ D = 2(A^2 - B - 1). \end{cases}$$

Proof. 1) This is well-known. 2) If s lies in SO_7, then $f(T) = (T-1)g(T)$, so $f(1) = 1$. This gives the linear relation on coefficients. In fact, $g(T) = (T-1)h(T)$, although this gives no new relations. The same holds for s in $SO_3 \times SO_5$, as $s = s_1 \times s_2$, where s_1 fixes a vector in the 3-dimensional representation W_3, and s_2 fixes a vector in the 5-dimensional representation W_5.

3) In these two cases, the smallest degree invariant lies in $\wedge^4 W$, where there is a unique fixed line. The polynomial relation is computed from the representation rings of the two groups.

4) These equations are simply a combination of 2) and 3). They state that the roots of $f(T)$ have the form $\{1, 1, \alpha, \beta, \gamma, \alpha^{-1}, \beta^{-1}, \gamma^{-1}\}$ with $\alpha\beta\gamma = 1$. This is also true for semi-simple elements in PGL_3.

We say that a subgroup $\Gamma \subseteq SO_8$ is locally contained in SO_7 if the coefficients of the characteristic polynomials of all elements s in Γ satisfy the equation in 2), Prop. 2.2. Similarly, we say that $\Gamma \subseteq SO_8$ is locally contained in $Spin_7$ if the coefficients of the characteristic polynomials of all elements s in Γ satisfy the equation in 3), Prop 2.2. Note that the group $SO_3 \times SO_5$ is locally contained in SO_7, even though it does not globally fix a line. Similarly, the group $Spin_3 \times Spin_5/\Delta < \pm 1 >$ is locally contained in $Spin_7$. If we introduce regular unipotent elements, we can eliminate these examples.

Proposition 2.3. *Let $\Gamma \subseteq SO_8$ be a subgroup which acts semi-simply on W. Assume that*

1. *Γ contains a regular unipotent element u.*
2. *Γ is locally contained in SO_7.*

Then Γ is contained in SO_7.

Similarly, assume that

1. *Γ contains a regular unipotent element u.*
2. *Γ is locally contained in $Spin_7$.*

Then Γ is contained in $Spin_7$.

Proof. We the fact that the connected, reductive subgroups of SO_8 containing a regular unipotent u form a chain:

$$\begin{array}{ccccc} & Spin_7 & & & \\ SO_8 & & G_2 & PGL_2 \\ & SO_7 & & & \end{array}$$

where $G_2 = SO_7 \cap Spin_7$, and PGL_2 is the pricipal subgroup of SO_8 determined by u.

Let $C \subseteq SO_8$ be the Zariski closure of Γ. Then the connected component C^0 is reductive, as it has a faithful semi-simple representation W. It also contains a regular unipotent element, so is one of the groups in the chain. If Γ is locally contained in SO_7, then so is C, hence

$$C^0 = SO_7,\ G_2 \text{ or } PGL_2.$$

The normalizers of these in SO_8 are $C^0 \times < \pm 1 >$, but -1 is not locally contained in SO_7. Hence C is connected, and Γ is contained in SO_7.

The same argument works when Γ is locally contained in $Spin_7$, but now C can be $G_2 \times < \pm 1 >$ or $PGL_2 \times < \pm 1 >$. Since these are both contained in $Spin_7$, Γ is contained in $Spin_7$.

Corollary 2.4. *Let $\Gamma \subseteq SO_8$ be a subgroup which acts semi-simply on W. Assume that*

1. *Γ contains a regular unipotent element u.*
2. *Γ is locally contained in G_2: the coefficients of the characteristic polynomial of $s \in \Gamma$ satisfy the equations in Prop. 2.2, part 4).*

Then Γ is contained in G_2.

Proof. This is a combination of the two results of Prop. 2.3, as $G_2 = Spin_7 \cap SO_7$ in SO_8. The proof shows that the Zariski closure of Γ in O_8 is either G_2, or the principal PGL_2 in G_2 determined by u.

3. A form of E_7.

As mentioned in the introduction, there is a (unique) form H of the split adjoint group of type E_7, which has rank 3 over $\mathbb{Q}$. This group is split over $\mathbb{Q}_p$ for all primes p, and acts on the exceptional tube domain over $\mathbb{R}$; it can be constructed from the Cayley division algebra $\mathbb{O}$ [Fr]. We sketch such a construction of the simply connected double cover H_{sc} over $\mathbb{Q}$, which lies in the exact sequence of algebraic groups

$$1 \to \mu_2 \to H_{sc} \to H \to 1. \tag{3.1}$$

The constuction is based on the existence of a faithful representation W of dimenesion 56 over $\mathbb{Q}$. We note that the only other simply-connected group of type E_7 admitting a 56-dimensional representation over $\mathbb{Q}$ is the split form.

Let $J_{\mathbb{O}}$ be the exceptional 27-dimensional Jordan algebra of all 3×3 Hermitian symmetric matrices over $\mathbb{O}$:

$$A = \begin{pmatrix} a & z & \bar{y} \\ \bar{z} & b & x \\ y & \bar{x} & c \end{pmatrix} \tag{3.2}$$

where a, b, c lie in $\mathbb{Q}$ and x, y, z lie in $\mathbb{O}$. The Jordan multiplication is given by the formula

$$A \circ B = \frac{1}{2}(AB + BA). \tag{3.3}$$

There is a cubic form $\det : J_{\mathbb{O}} \to \mathbb{Q}$, defined by

$$\det(A) = abc + \mathrm{Tr}(xyz) - a\mathbb{N}(x) - b\mathbb{N}(y) - c\mathbb{N}(z). \tag{3.4}$$

Let (A, B, C) be the unique symmetric trilinear form such that [EG]

$$(A, A, A) = 6\det(A). \tag{3.5}$$

Let M be the reductive algebraic group over $\mathbb{Q}$ of invertible linear mappings $m : J_{\mathbb{O}} \to J_{\mathbb{O}}$ which satisfy

$$\det(m(A)) = \lambda(m)\det(A) \tag{3.6}$$

for a similitude $\lambda(m)$ in $\mathbb{Q}^\times$. The center of M is $\mathbb{G}_{\mathrm{m}}$, acting by scalar matrices, and the kernel of the morphism $\lambda : M \to \mathbb{G}_{\mathrm{m}}$ is a simply connected group of type E_6 and rank 2 over $\mathbb{Q}$ [CS]. On the center, $\lambda(a) = a^3$.

Let N be a unipotent abelian group over $\mathbb{Q}$, isomorphic to $J_{\mathbb{O}}$. In H_{sc} we have a maximal parabolic subgroup

$$P_{sc} = MN \tag{3.7}$$

where the conjugation action of M on N is given by

$$mAm^{-1} = \lambda^{-1}(m)\, m(A). \tag{3.8}$$

Note that this action has a kernel μ_2 (the center of H_{sc}).

We now define a representation of P_{sc} on the 56-dimensional module

$$W = \mathbb{Q} \oplus J_{\mathbb{O}} \oplus J_{\mathbb{O}}^* \oplus \mathbb{Q}^*, \tag{3.9}$$

where $J_{\mathbb{O}}^* = Hom(J_{\mathbb{O}}, \mathbb{Q})$, and $\mathbb{Q}^* = Hom(\mathbb{Q}, \mathbb{Q}) \cong \mathbb{Q}$. The subgroup M acts on W by

$$m(y, Y, Y^*, y^*) = (\lambda(m)y, m(Y), m^*(Y^*), \lambda^{-1}(m)y^*), \tag{3.10}$$

where m^* is defined as follows; if $< X, Y^* >$ is the pairing $J_{\mathbb{O}} \times J_{\mathbb{O}}^* \to \mathbb{Q}$, we have $< mX, m^*Y^* >=< X, Y^* >$ for all $X \in J_{\mathbb{O}}$ and $Y^* \in J_{\mathbb{O}}^*$. One can give complicated formulas for the action of N on W [Ki; pg 143], but since we are in characterisitic 0, it suffices to define the action of $Lie(N) \cong J_{\mathbb{O}}$. This acts by

$$X(y, Y, Y^*, y^*) = (0, yX, X \times Y, < X, Y^* >). \tag{3.11}$$

where $X \times Y$ is the element of $J_{\mathbb{O}}^*$ mapping Z to (X, Y, Z).

Using (3.8), one can check that the formulas (3.10) and (3.11) define an action of P_{sc} on W, which preserves the natural symplectic form

$$\{(x, X, X^*, x^*), (y, Y, Y^*, y^*)\} = (xy^* - yx^*) + (< X, Y^* > - < Y, X^* >). \tag{3.12}$$

The group $H_{sc} \subset Sp(W)$ is generated by P_{sc} and an element w of order 4, giving a simple reflection in the Weyl group which normalizes M (a Levi factor of P_{sc}).

To define w in $Sp(W)$, we need to choose a polarization I of $J_{\mathbb{O}}$ with $\det(I) = 1$ and $I > 0$ in $J_{\mathbb{O}} \otimes \mathbb{R}$ [EG; Ch. 2]. Since M acts transitively on polarizations over $\mathbb{Q}$, there is no loss of generality in taking I to be the identity matrix in $J_{\mathbb{O}}$. This gives a positive-definite bilinear form on $J_{\mathbb{O}}$, defined by

$$< A, B >= -(A, B, I) + (A, I, I)(B, I, I)/4. \tag{3.13}$$

If I is the identity matrix, then $< A, B >= \mathrm{Tr}(AB) = \mathrm{Tr}(A \circ B) = \mathrm{Tr}(BA)$. This form defines an identification $J_{\mathbb{O}} \cong J_{\mathbb{O}}^*$. With this identification, we define w, which depends on I and satisfies $w^2 = -1$, by

$$w(y, Y, Y^*, y^*) = (-y^*, -Y^*, Y, y). \tag{3.14}$$

Then $H_{sc} =< P_{sc}, w >$ in $Sp(\mathbb{W})$. The element w acts by inversion on the center $\mathbb{G}_{\mathrm{m}}$ of M, and w^2 is an involution in the center of M, which generates the center μ_2 of H_{sc}. In the quotient $H = H_{sc}/\mu_2$, the image $\bar{w}$ of w has order 2, and gives the Cartan involution of $Lie(H)$ over $\mathbb{R}$.

The adjoint group H has the maximal parabolic

$$P = P_{sc}/\mu_2 = MN \tag{3.15}$$

with isomorphic Levi factor ($M/\mu_2 \cong M$), but with a different conjugation action of M on $N \cong J_{\mathbb{O}}$:

$$mAm^{-1} = m(A). \tag{3.16}$$

Note that this action is faithful.

We can use the Coxeter order $\mathcal{R} \subset \mathbb{O}$ to give a model for H_{sc} over $\mathbb{Z}$ with good reduction at all primes p [Gr]. Let $J_{\mathcal{R}} \subset J_{\mathbb{O}}$ be the lattice consisitng of all elements with a, b, c in $\mathbb{Z}$, and x, y, z in $\mathcal{R}$. Then

$$W_{\mathbb{Z}} = \mathbb{Z} \oplus J_{\mathcal{R}} \oplus J_{\mathcal{R}}^* \oplus \mathbb{Z}^* \tag{3.17}$$

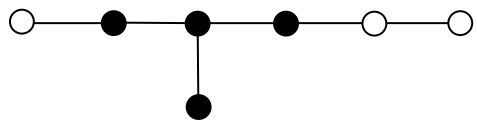
is a lattice in W, where $J_{\mathcal{R}}^* = Hom(J_{\mathcal{R}}, \mathbb{Z}) \cong J_{\mathcal{R}}$ [EG], and $\mathbb{Z}^* = Hom(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$. The stabilizer of $W_{\mathbb{Z}}$ gives a desired model with $H_{sc}(\mathbb{Z}) \subset Sp_{56}(\mathbb{Z})$. Then $Lie(H_{sc}/\mathbb{Z}) \subset_2 Lie(H/\mathbb{Z})$ as lattices in $Lie(H)$. If we normalize the Killing form so that the determinat of $Lie(H_{sc}/\mathbb{Z})$ is 2, then $Lie(H/\mathbb{Z})$ is the dual lattice.

4. Dual pair $G \times G'$.

We now give two constructions of the dual pair $G \times G' = Aut(\mathbb{O}) \times PGSp_6$ in H over $\mathbb{Q}$. The first uses the relative root system, and the second our construction of H as a group generated by P and $\bar{w}$.

Over $\mathbb{Q}$, the simple group H has index [Ti; pgs 59-60]

(4.1)



and relative root system Φ of type C_3. Let S be a maximal split torus of dimension 3 in H. The derived group of the centralizer $C(S)$ (the semi-simple anisotropic kernel) is isomorphic to the group [Gr]

$$\begin{aligned} &Spin_8(\mathbb{O}) = \\ &\{(\gamma_1, \gamma_2, \gamma_3) \in GL(\mathbb{O})^3 \mid \mathbb{N}(\gamma_i x) = \mathbb{N}(x),\ \mathrm{Tr}(\gamma_1 x \cdot \gamma_2 y \cdot \gamma_3 z) = \mathrm{Tr}(xyz)\}. \end{aligned} \tag{4.2}$$

The group has 3 orthogonal representations V_1, V_2, V_3 of dimension 8, via the action of $\gamma_1, \gamma_2, \gamma_3$ on $\mathbb{O}$.

The 6 long root spaces $Lie(H)_\alpha$ ($\alpha = \pm 2e_i$) for S have dimension 1, and trivial action of $Spin_8(\mathbb{O})$. The 12 short root spaces $Lie(H)_\alpha$ ($\alpha = \pm e_i \pm e_j$) for S have dimension 8, and $Spin_8(\mathbb{O})$ acts on $Lie(H)_{\pm e_i \pm e_j}$ by the representation V_k, ($i \neq j \neq k$).

From this viewpoint G is the subgroup of all triples (γ, γ, γ) in $Spin_8(\mathbb{O})$ with γ in $Aut(\mathbb{O})$. Since $\gamma(1) = 1$, the restriction of each V_i to G is isomorphic to $\mathbb{Q} \oplus V$. Hence $Lie(H)^G_\alpha$ has dimension 1 for all $\alpha \in \Phi$, and the centralizer of G in H is a split group of type C_3. Since the roots Φ give a basis for the character group of S, G' is of adjoint type, and $G' = PGSp_6$. Conversely, the centralizer of G' is contained in $Spin_8(\mathbb{O})$, and fixes a vector in each 8-dimensional representation V_i. This shows that the centralizer of G' is contained in G, hence equal to G.

The following alternative construction of the closed subgroup $G \times G'$ of H uses our construction in the previous section. The group $G = Aut(\mathbb{O})$ is a subgroup of $M \subset P$, via tha action on the matrix entries x, y, z of A in $J_\mathbb{O}$. Let $U \subset N$ be the group fixed by this action, consisting of matrices with rational entries x, y, z. Then U has dimension 6 over $\mathbb{Q}$. We have an embedding of $L = GL_3$ into M given by

$$\det(g)^{-1}\ gAg^t. \tag{4.3}$$

where $\det(g)$ and g^t denote the determinant and the transpose of the 3×3 matrix g. Note that the restriction of λ to $L \cong GL_3$ is $\det(g)^{-1}$. The image stabilizes the subgroup U, and the semi-direct product

$$Q = LU \tag{4.4}$$

is a subgroup of P, commuting with G.

Assume that the polarization I defining w has rational entries, (for example, take I the identity matrix). Then the image $\bar{w}$ of w in H commutes with G and normalizes L. Moreover,

$$G' = PGSp_6 =< Q, \bar{w} > \tag{4.5}$$

in H. This gives the dual pair $G \times G'$ in H.

III Real correspondences

Let $H(\mathbb{R})$ be the adjoint algebraic group of type E_7 whose connected component is the group of conformal transformations of the exceptional symmetric domain. We have the dual pair

$$G(\mathbb{R}) \times G'(\mathbb{R}) = Aut(\mathbb{O} \otimes \mathbb{R}) \times PGSp_6(\mathbb{R}) \tag{0.1}$$

in $H(\mathbb{R})$ with $G(\mathbb{R})$ the compact form of G_2.

Let $\hat{\Pi}$ be the minimal representation of $H(\mathbb{R})$. In this chapter we show that

$$\Pi|_{G(\mathbb{R}) \times G'(\mathbb{R})} = \hat{\oplus}_\pi \pi \otimes \Theta(\pi) \tag{0.2}$$

where the sum is taken over all finite dimensional representations of $G(\mathbb{R})$ and $\Theta(\pi)$ is an irreducible representation whose restriction to $Sp_6(\mathbb{R})$ is a sum of a holomorphic and an anti-holomorphic discrete series representation. The lift $\pi \mapsto \Theta(\pi)$ is functorial for the inclusion of dual groups $G_2(\mathbb{C}) \to Spin_7(\mathbb{C})$ (note that G_2 is the stabilizer of a generic vector in the spin-module of $Spin_7$).

1. Minimal representation of $H_{sc}(\mathbb{R})$.

Let $H_{sc}(\mathbb{R})$ be the simply connected group of type $E_{7,3}$ over $\mathbb{R}$. Its real rank is 3 and the reduced root system is C_3. Let $e_i - e_j$, $(1 \le i < j \le 3)$ and $e_i + e_j$, $(1 \le i \le j \le 3)$ be the standard set of positive roots. The root spaces corresponding to $e_i + e_j$, $(1 \le i < j \le 3)$ are 8-dimensional and can be identified with $\mathbb{O} \otimes \mathbb{R}$. The root spaces corresponding to strongly othogonal $2e_i$ are one-dimensional, hence we have an embedding

$$SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \subset H_{sc}(\mathbb{R}). \tag{1.1}$$

Choose $K_{sc}(\mathbb{R})$, a maximal compact subgroup of $H_{sc}(\mathbb{R})$, such that

$$Z_1(\mathbb{R}) \times Z_2(\mathbb{R}) \times Z_3(\mathbb{R}) = K_{sc}(\mathbb{R}) \cap SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \tag{1.2}$$

is a compact maximal Cartan subgroup of $SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$. Let

$$Z(\mathbb{R}) = K_{sc}(\mathbb{R}) \cap SL_2(\mathbb{R}) \tag{1.3}$$

where $SL_2(\mathbb{R}) \subset SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ is diagonally embedded. Note that $Z(\mathbb{R}) \cong SO_2(\mathbb{R})$ is the center of $K_{sc}(\mathbb{R})$, and $< \pm 1 > \subset Z(\mathbb{R})$ is the center of $H_{sc}(\mathbb{R})$.

We henceforth denote by H_{sc}, K_{sc}, Z ... the complexifications of $H_{sc}(\mathbb{R})$, $K_{sc}(\mathbb{R})$ $Z(\mathbb{R})$... Let $\mathfrak{h}$ and $\mathfrak{k}$ be the Lie algebras of H_{sc} and K_{sc}. Then

$$\mathfrak{h} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+ \tag{1.4}$$

where $\mathfrak{p}^\pm$ are two fundamental 27-dimensional representations of the exceptional Lie algebra $\mathfrak{e}_6 = [\mathfrak{k}, \mathfrak{k}]$ of type E_6. Under the action of $Z_1 \times Z_2 \times Z_3$, $\mathfrak{p}^+$ decomposes as a sum of root spaces $\gamma_i + \gamma_j$, $(1 \leq i \leq j \leq 3)$.

Let e^-, z, e^+ be a standard basis of $sl(2) \subset sl(2) \oplus sl(2) \oplus sl(2) \subset \mathfrak{h}$ such that z spans the Lie algebra of Z, $e^- \in \mathfrak{p}^-$ and $e^+ \in \mathfrak{p}^+$.

Let λ^+ be the highest weight of the irreducible $\mathfrak{e}_6$-module $\mathfrak{p}^+$. Let $E(n)$ be the irreducible representation of $\mathfrak{e}_6$ with highest weight $n\lambda^+$. Since $\mathfrak{k} = \mathfrak{e}_6 \oplus \mathbb{C}z$, let $E(n,k)$ be the representation of $\mathfrak{k}$ such that the restriction to $\mathfrak{e}_6$ is isomorphic to $E(n)$ and z acts via the scalar k. Let Π^+ be the irreducible $(\mathfrak{h}, K_{sc})$-module corresponding to a holomorphic representation of $H_{sc}(\mathbb{R})$, with K_{sc}-types [Wl]:

$$\Pi^+|_{K_{sc}} = \oplus_{n \geq 0} E(n, 2n+12). \tag{1.5}$$

The annihilator of Π^+ in the enveloping algebra of $\mathfrak{h}$ is Joseph's ideal. In particular, the Gelfand-Kirillov dimension of Π^+ is the smallest amongst non-trivial modules. There is also an anti-holomorphic module Π^-, contragredient to Π^+. By (1.5) the center $< \pm 1 > \subset Z(\mathbb{R})$ acts trivially on Π^- and Π^+.

Let $\hat{\Pi}^-$ and $\hat{\Pi}^+$ denote the unitary completion of Π^- and Π^+. Since $H_{sc}(\mathbb{R})$ and $H(\mathbb{R})$ are related by the exact sequence

$$1 \to < \pm 1 > \to H_{sc}(\mathbb{R}) \to H(\mathbb{R}) \to \mathbb{R}^\times/(\mathbb{R}^\times)^2 \to 1, \tag{1.6}$$

it follows that there exist unique representation $\hat{\Pi}$ of $H(\mathbb{R})$ such that

$$\hat{\Pi}|_{H_{sc}(\mathbb{R})} = \hat{\Pi}^+ \oplus \hat{\Pi}^-. \tag{1.7}$$

It is precisely this representation that we call the minimal representation of $H(\mathbb{R})$.

2. Dual pairs.

We now describe several dual pairs in $\mathfrak{h}$ using Jordan algebras. Note that

$$x \circ y = \frac{1}{2}[[x, e^-], y] \tag{2.1}$$

gives a Jordan product on $\mathfrak{p}^+$. Then $(\mathfrak{p}^+, \circ)$ is isomorphic to the exceptional Jordan algebra of 3×3 hermitian matrices

$$\begin{pmatrix} d_1 & z_{1,2} & \bar{z}_{1,3} \\ \bar{z}_{1,2} & d_2 & z_{2,3} \\ z_{1,3} & \bar{z}_{2,3} & d_3 \end{pmatrix} \tag{2.2}$$

with coefficients in the octonion algebra over $\mathbb{C}$. Under this isomorphism, the 1-dimensional root spaces $2\gamma_i$, $(1 \leq i \leq 3)$ are given by diagonal matrices such that $d_j = 0$ if $j \neq i$, and the 8-dimensional root spaces $\gamma_i + \gamma_j$, $(i < j)$ are given by off-diagonal matrices such that $z_{k,l} = 0$ if $\{k, l\} \neq \{i, j\}$.

Let $\mathfrak{q}^+ \subset \mathfrak{p}^+$ be a Jordan subalgebra containing e^+. Let

$$\mathfrak{a} = C_{\mathfrak{k}}(\mathfrak{q}^+). \tag{2.3}$$

Assume, conversly, that $\mathfrak{q}^+$ is the set of all elements in $\mathfrak{p}^+$ annihilated by $\mathfrak{a}$. Let $\mathfrak{u} = C_{\mathfrak{k}}(\mathfrak{a})$, and $\mathfrak{q}^- \subset \mathfrak{p}^-$ such that $\mathfrak{q}^+ = [e^+, [e^+, \mathfrak{q}^-]]$. Then

$$\mathfrak{b} = \mathfrak{q}^- \oplus \mathfrak{u} \oplus \mathfrak{q}^+ \tag{2.4}$$

is the centralizer of $\mathfrak{a}$ in $\mathfrak{b}$. Obviously, the converse is also true, i.e. $\mathfrak{a} \times \mathfrak{b}$ is a dual reductive pair.

Some of the possible cases are:

(2.5)

$\dim \mathfrak{q}^+$	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{u}$
1	$\mathfrak{f}_4$	$sl(2)$	$u(1)$
3	$\mathfrak{d}_4$	$sl(2) \oplus sl(2) \oplus sl(2)$	$u(1) \oplus u(1) \oplus u(1)$
6	$\mathfrak{g}_2$	$sp(6)$	$u(3)$

where the subalgebra $\mathfrak{q}^+$ is given respectively by 3×3 scalar, diagonal and symmetric matrices with coefficients in $\mathbb{C}$.

3. Correspondences.

In this section we restrict the representation Π^+ to the dual pairs $A(\mathbb{R}) \times B(\mathbb{R})$ given by (2.5), with $A(\mathbb{R})$ compact.

We start with $A(\mathbb{R}) = F_4(\mathbb{R})$. The 27-dimensional module $\mathfrak{p}^+$ decomposes 1+26, under the action of $F_4(\mathbb{R})$. Let λ the highest weight of the 26-dimensional summand, and let $F(n)$ be the irreducible representation of $F_4(\mathbb{R})$ with highest weight $n\lambda$. By Thm. 6.1 in [HPS]

$$E(n) = \oplus_{m \leq n} F(m). \tag{3.1}$$

Proposition 3.2. *Consider the dual pair $F_4(\mathbb{R}) \times SL_2(\mathbb{R})$. Then*

$$\Pi^+|_{F_4(\mathbb{R}) \times sl(2)} = \oplus_{n \geq 0} F(n) \otimes d(2n + 12),$$

where $d(n)$ is the irreducible $(sl(2), Z)$-module corresponding to the holomorphic discrete series of $SL_2(\mathbb{R})$ with the minimal Z-type n.

Proof. By (3.1) one can write

$$\Pi^+|_{F_4(\mathbb{R}) \times sl(2)} = \oplus_{n \geq 0} F(n) \otimes V_n.$$

where V_n are certain $(sl(2), Z)$-modules. Since $F(n)$ appears in $E(k, 2k + 12)$ only for $k = n, n + 1, \ldots$, Z-types of V_n are $2n + 12, 2n + 14, \ldots$ and are one-dimensional. The proposition is proved.

Next, consider the case $A(\mathbb{R}) = D_4(\mathbb{R})$. Then $D_4(\mathbb{R})$-invariant subspaces of $\mathfrak{p}^+$ are precisely the root spaces. The three 8-dimensional root spaces are 3 different fundamental 8-dimensional representations of $D_4(\mathbb{R})$. Let λ_i, $(1 \leq i \leq 3)$ be the highest weight of the fundamental representation given by the root space $\gamma_j + \gamma_k$, where $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$. Let $D(a_1, a_2, a_3)$ be the irreducible representation of $D_4(\mathbb{R})$ with a highest weight $a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3$.

Proposition 3.3. *Consider the dual pair $D_4(\mathbb{R}) \times SL_2(\mathbb{R})^3$. Then*

$\Pi^+|_{D_4(\mathbb{R})\times sl(2)^3} = \oplus_{a_1,a_2,a_3\geq 0} D(a_1,a_2,a_3) \otimes d(a_2+a_3+4) \otimes d(a_1+a_3+4) \otimes d(a_1+a_2+4),$

where $d(n)$ is the irreducible $(sl(2), Z)$-module corresponding to the holomorphic discrete series of $SL_2(\mathbb{R})$ with the minimal Z-type n.

Proof. We want to decompose $E(n, 2n)$ with respect to the action of $D_4 \times Z_1 \times Z_2 \times Z_3$. Write

$$E(n, 2n) = \Gamma(\mathcal{L}^{\otimes n}),$$

where $\mathcal{L}$ is a line bundle on the flag variety. Since $E(1, 2) \cong \mathfrak{p}^+$, we know how to decompose $E(1, 2)$. Let $v_i \in \Gamma(\mathcal{L})$ be highest weight vectors of the three 8-dimensional representations with the highest weights λ_i. Let $w_i \in \Gamma(\mathcal{L})$ be non-zero vectors with weights $2\gamma_i$. Let a_i and r_i $(i = 1, 2, 3)$, be non-negative integers such that

$$a_1 + a_2 + a_3 + r_1 + r_2 + r_3 = n.$$

Then

$$v_1^{a_1} v_2^{a_2} v_3^{a_3} w_1^{r_1} w_2^{r_2} w_3^{r_3} \in \mathcal{L}^{\otimes n}$$

generates a D_4-module isomorphic to $D(a_1, a_2, a_3)$, with a $Z_1 \times Z_2 \times Z_3$-type

$$(a_2 + a_3 + 2r_1, a_1 + a_2 + 2r_2, a_1 + a_2 + 2r_3).$$

Note that different choices of a_i and r_i produce non-isomorphic modules. We claim that these modules give a complete decomposition of $E(n, 2n)$. Indeed, let

$$x = n + 1 - (a_1 + a_2 + a_3).$$

Since $E(n) = \oplus_{m\leq n} F(m)$ it follows from Prop. 4.7 that the multiplicity of $D(a_1, a_2, a_3)$ in $E(n, 2n)$ is $x(x + 1)/2$. Since

$$\frac{x(x+1)}{2} = \#\{(r_1, r_2, r_3) | r_1 + r_2 + r_3 = x - 1\}$$

the claim follows.

Write

$$\Pi^+|_{D_4(\mathbb{R})\times sl(2)+sl(2)+sl(2)} = \oplus_{a_1,a_2,a_3\geq 0} D(a_1, a_2, a_3) \otimes V_{a_1,a_2,a_3}.$$

Taking into account the additional shift by 12, it follows that $Z_1 \times Z_2 \times Z_3$-types of V_{a_1,a_2,a_3} are

$$(a_2 + a_3 + 2r_1 + 4, a_1 + a_3 + 2r_2 + 4, a_2 + a_3 + 2r_3 + 4),$$

and they are one-dimensional. The proposition is proved.

Finally, we consider the case $A(\mathbb{R}) = G_2(\mathbb{R})$. Let ω_1 and ω_2 be the fundamental weights for G_2, such that ω_1 is the highest weight of the 7-dimensional representation. Let $W(k_1, k_2)$ be the irreducible representation of $G_2(\mathbb{R})$ with highest weight $k_1\omega_1 + k_2\omega_2$.

Note that $B(\mathbb{R}) = Sp_6(\mathbb{R})$, and the reduced root system of $H(\mathbb{R})$ restricts to a root system of $Sp_6(\mathbb{R})$. Also, let $U(3, \mathbb{R})$ be a maximal compact subgroup of $Sp_6(\mathbb{R})$ given by

$$U(3, \mathbb{R}) = K_{sc}(\mathbb{R}) \cap Sp_6(\mathbb{R}). \tag{3.4}$$

We identify irreducible representations of $U(3, \mathbb{R})$ with their highest weights $l_1\gamma_1 + l_2\gamma_2 + l_3\gamma_3$, $(l_1 \geq l_2 \geq l_3)$ with respect to the maximal Cartan subgroup $Z_1(\mathbb{R}) \times Z_2(\mathbb{R}) \times Z_2(\mathbb{R})$.

Theorem 3.5. *Consider the dual pair $G_2(\mathbb{R}) \times Sp_6(\mathbb{R})$. Then*

$$\Pi^+|_{G_2(\mathbb{R})\times sp(6)} = \oplus_{k_1,k_2\geq 0} W(k_1,k_2)\otimes d(k_1,k_2),$$

where $d(k_1,k_2)$ is the irreducible $(sp(6),U(3))$-module corresponding to the holomorphic discrete series representation of $Sp_6(\mathbb{R})$ with infinitesimal character

$$(k_1+2k_2+3)e_1+(k_1+k_2+2)e_2+(k_2+1)e_3$$

and the minimal $U(3)$-type

$$(k_1+2k_2+4)\gamma_1+(k_1+k_2+4)\gamma_2+(k_2+4)\gamma_3.$$

Proof. Write

$$\Pi^+|_{G_2(\mathbb{R})\times sp(6)} = \oplus_{k_1,k_2\geq 0} W(k_1,k_2)\otimes V_{k_1,k_2}.$$

By [HPS] the infinitesimal character of V_{k_1,k_2} is $(k_1+2k_2+3)e_1+(k_1+k_2+2)e_2+(k_2+1)e_3$. It is Z-admissible, with positive Z-types. Hence it is a direct sum of finitely many unitary lowest weight modules.

Lemma 3.6. *In addition to $d(k_1,k_2)$ one has the following unitary lowest weight modules with infinitesimal character $(k_1+2k_2+3)e_1+(k_1+k_2+2)e_2+(k_2+1)e_3$:*

(1) *$k_2=0$ and $k_1\neq 0$. There is a module with the minimal Z-type $2k_1+10$.*
(2) *$k_1=k_2=0$. There are three modules. Their minimal Z-types are $0,6$ and 10.*

Note that the minimal Z-type of $d(k_1,k_2)$ is $2k_1+4k_2+12$, hence it is strictly bigger then the minimal Z-type of any other modules with the same infinitesimal character.

Proof. This follows from the classification of unitary lowest weight modules [EHW].

By Prop. 4.8, $W(k_1,0)$ does not appear in $D(a_1,a_2,a_3)$ unless $k_1\leq a_1+a_2+a_3$. It follows from Prop. 4.2 that the minimal Z-type of $V_{k_1,0}$ is greater then $2k_1+12$. It follows from the lemma that in all cases V_{k_1,k_2} is a finite multiple of $d(k_1,k_2)$. This implies that the minimal Z-type of V_{k_1,k_2} is $2k_1+4k_2+12$, so by Prop. 3.3, if $W(k_1,k_2)$ is contained in $D(a_1,a_2,a_3)$, then

$$a_1+a_2+a_3\geq k_1+2k_2.$$

Hence, the multiplicity of

$$(k_2+4)\gamma_1+(k_1+k_2+4)\gamma_2+(k_1+2k_2+4)\gamma_3,$$

the lowest weight of the minimal $U(3)$-type of $d(k_1,k_2)$ in V_{k_1,k_2}, is equal to the multiplicity of $W(k_1,k_2)$ in $D(k_1+k_2,k_2,0)$. And this is one by Prop. 4.8. Hence the minimal $U(3)$-type appears with multiplicity one and this implies that $V_{k_1,k_2}\cong d(k_1,k_2)$.

Let $\bar{d}(k_1,k_2)$ be the contragradient of $d(k_1,k_2)$. It corresponds to an anti-holomorphic representation of $Sp_6(\mathbb{R})$. Since

$$1\to<\pm 1>\to Sp_6(\mathbb{R})\to PGSp_6(\mathbb{R})\to \mathbb{R}^\times/(\mathbb{R}^\times)^2\to 1, \tag{3.7}$$

there exists unique irreducible discrete series representation $\mathcal{D}(k_1,k_2)$ of $PGSp_6(\mathbb{R})$, which is the unitary completion of

$$d(k_1,k_2)\oplus\bar{d}(k_1,k_2). \tag{3.8}$$

This observation gives a proof of the following.

Corollary 3.9. *Let $\hat{\Pi}$ be the minimal representation of $H(\mathbb{R})$. Then*

$$\hat{\Pi}|_{G(\mathbb{R})\times PGSp_6(\mathbb{R})} = \hat{\oplus}_{k_1,k_2\geq 0} W(k_1,k_2)\otimes \mathcal{D}(k_1,k_2).$$

The lift

$$W(k_1,k_2)\to \mathcal{D}(k_1,k_2) \tag{3.10}$$

is functorial for the inclusion of dual groups $G_2(\mathbb{C})\to Spin_7(\mathbb{C})$. This is shown in [HPS].

4. Branching formulas.

In this section we work out the branching laws used in the previous section. Let $(a_1,\dots,a_n)$ be the standard coordinates [Bu] for the root system of type B_n, with

$$a_1\geq \ldots \geq a_n\geq 0 \tag{4.1}$$

a dominant Weyl chamber. Also, let $(b_1,\dots,b_n)$ be the standard coordinates for the root system of type D_n, with

$$b_1\geq \ldots\geq b_{n-1}\geq |b_n| \tag{4.2}$$

a dominant Weyl chamber. Recall that in both cases a dominant weight represents a highest weight of a finite dimensional representation if the coefficients are in $\frac{1}{2}\mathbb{Z}$ but their differences are in $\mathbb{Z}$.

4.3 Branching $B_n\downarrow D_n$. *Let $\pi(\lambda)$ be an irreducible representation of B_n with the highest weight $\lambda=(a_1,\dots,a_n)$. Let $\pi(\mu)$ be an irreducible representation of D_n with the highest weight $\mu=(b_1,\dots,b_n)$. Then the multiplicity of $\pi(\mu)$ in $\pi(\lambda)$ is 0 or 1. It is 1 if and only if $a_i-b_i\in\mathbb{Z}$ and*

$$a_1\geq b_1\geq a_2\geq b_2\geq \ldots\geq a_n\geq |b_n|.$$

4.4 Branching $D_n\downarrow B_{n-1}$. *Let $\pi(\lambda)$ be an irreducible representation of D_n with the highest weight $\lambda=(a_1,\dots,a_n)$. Let $\pi(\mu)$ be an irreducible representation of B_{n-1} with the highest weight $\mu=(b_1,\dots,b_{n-1})$. Then the multiplicity of $\pi(\mu)$ in $\pi(\lambda)$ is 0 or 1. It is 1 if and only if $a_i-b_i\in\mathbb{Z}$ and*

$$a_1\geq b_1\geq a_2\geq b_2\geq\ldots\geq b_{n-1}\geq |a_n|.$$

Let λ be the highest weight of the 26-dimensional representation of F_4. Let $F(n)$ be the irreducible representation with the highest weight $n\lambda$.

4.5 Branching $F_4\downarrow B_4$. *The restriction of $F(n)$ to B_4 decomposes with multiplicities 0 or 1. It is 1 only for $\pi(\mu)$ with*

$$\mu=(y+x,x,x,x)$$

and $2x+y\leq n$.

The branching laws 4.3 and 4.4 are well known, and 4.5 is in [Le; Thm. 8]. This reference also contains proofs of 4.3 and 4.4.

Recall that D_4 has three 8-dimensional representations. Let λ_i, $i=1,2,3$, be their highest weights. Let $D(a_1,a_2,a_3)$ be the irreducible representation with highest weight $a_1\lambda_1+a_2\lambda_2+a_3\lambda_3$. In terms of (4.2) this highest weight is

$$(a_1+\frac{a_2+a_3}{2},\frac{a_2+a_3}{2},\frac{a_2+a_3}{2},\frac{a_2-a_3}{2}). \tag{4.6}$$

4.7 Branching $F_4 \downarrow D_4$**.** *The representation* $F(n)$ *decomposes as a sum of* $D(a_1, a_2, a_3)$ *with multiplicities*

$$(n+1) - (a_1 + a_2 + a_3).$$

Proof. We know that the restriction of $F(n)$ to B_4 is a sum of representations with highest weights

$$(y + x, x, x, x),$$

$2x + y \leq n$. Restriciting further down to D_4 we get a sum of all representations with highest weights (u_1, u_2, u_3, u_4) such that

$$y + x \geq u_1 \geq x \geq u_2 \geq x \geq u_3 \geq x \geq |u_4|.$$

It follows that $u_2 = u_3 = x$ and the representation can be written as

$$D(u_1 - x, x - u_4, x + u_4),$$

by (4.6). Now

$$D(u_1 - x, x - u_4, x + u_4) = D(a_1, a_2, a_3)$$

implies that $2x = a_2 + a_3$ and $u_1 = a_1 + x$. Since $y + x \geq u_1$, we have $y \geq a_1$. Hence the multiplicity of $D(a_1, a_2, a_3)$ in $F(n)$ is the number of integers y such that $y \geq a_1$ and $y + a_2 + a_3 \leq n$. Clearly, this number is $(n + 1) - (a_1 + a_2 + a_3)$.

Proposition 4.8.

(1) *The multiplicity of* $W(k_1, k_2)$ *in* $D(k_1 + k_2, k_2, 0)$ *is* 1.
(2) *The multiplicity of* $W(k_1, 0)$ *in* $D(a_1, a_2, a_3)$*, is* 0 *unless* $a_1 + a_2 + a_3 \geq k_1$.

Proof. Let μ_1, μ_2 and μ_3 be the fundamental weights for B_3 such that μ_1 is the highest weight of the standard 7-dimensional representation and μ_3 is the highest weight of the 8-dimensional spin-representation. Let $B(m_1, m_2, m_3)$ be the irreducible representation with highest weight $m_1\mu_1 + m_2\mu_2 + m_3\mu_3$. In terms of (4.1), this highest weight is

$$(m_1 + m_2 + \frac{m_3}{2}, m_2 + \frac{m_3}{2}, \frac{m_3}{2}).$$

Lemma 4.9.

(1) *The multiplicity of* $W(k_1, k_2)$ *in* $B(j, 0, k_2)$*,* $j \leq k_1 + k_2$*, is* 0 *or* 1*. It is* 1 *precisely when* $j = k_1 + k_2$.
(2) *The multiplicity of* $W(k_1, 0)$ *in* $B(m_1, m_2, m_3)$*, is* 0 *or* 1*. It is* 1 *precisely when*

$$m_1 + m_2 + m_3 \geq k_1 \geq m_1 + m_2.$$

Proof. These are two easy, special cases of the formula given by Mc Govern, [MG; Thm. 3.4].

By (4.6) the highest weight of $D(k_1+k_2,k_2,0)$ is

$$(k_1+k_2+\frac{k_2}{2},\frac{k_2}{2},\frac{k_2}{2},\frac{k_2}{2}).$$

The branching $D_4 \downarrow B_3$ implies that $D(k_1+k_2,k_2,0)$ decomposes as a sum of representations with highest weights

$$(j+\frac{k_2}{2},\frac{k_2}{2},\frac{k_2}{2})$$

with $j \le k_1+k_2$. These are $B(j,0,k_2)$ with $j \le k_1+k_2$. The first statement follows from the lemma.

Since the highest weight of $D(a_1,a_2,a_3)$ is given by (4.6), the restricition to B_3 consists of representations with highest weights (x,y,z) such that

$$a_1+\frac{a_2+a_3}{2} \ge x \ge \frac{a_2+a_3}{2} \ge y \ge \frac{a_2+a_3}{2} \ge z \ge \frac{a_2-a_3}{2}$$

It follows that $y=(a_2+a_3)/2$ and these are the representations:

$$B(x-\frac{a_2+a_3}{2},\frac{a_2+a_3}{2}-z,2z).$$

The lemma implies that $k_1 \le x+z$, and since $x+z \le a_1+a_2+a_3$, the second statement follows. The proposition is proved.

IV p-adic correspondences

Our goal in this chapter is to understand the restriction of the minimal representation Π of $H(\mathbb{Q}_p)$ to the closed subgroup $G(\mathbb{Q}_p)\times G'(\mathbb{Q}_p)$.

The minimal representation of a split, adjoint group $H(\mathbb{Q}_p)$ of type D_n or E_n is an unramified representation whose Satake parameter is

$$s_{\min}=\varphi\begin{pmatrix} p^{\frac{1}{2}} & 0 \\ 0 & p^{-\frac{1}{2}} \end{pmatrix}, \tag{0.1}$$

where φ is a map

$$\varphi: SL_2(\mathbb{C}) \to \hat{H}(\mathbb{C}) \tag{0.2}$$

corresponding to the subregular unipotent orbit in $\hat{H}(\mathbb{C})=H_{sc}(\mathbb{C})$. The representation Π restricts to the irreducible representation of $H_{sc}(\mathbb{Q}_p)$ constructed by Kazhdan and Savin in [KS]. On the space of Iwahori-fixed vectors in Π, the Iwahori-Hecke algebra of $H_{sc}(\mathbb{Q}_p)$ acts via the reflection representation [Lu].

1. Parameters.

In this section, we give a conjectural decription of the irreducible representations $\pi \otimes \pi'$ of $G(\mathbb{Q}_p) \times G'(\mathbb{Q}_p)$ which occur as quotients of Π. This description is given in terms of the Langlands-Deligne-Lusztig-Vogan parametrization of irreducible representations, using admissible homomorphisms from Weil-Deligne group of $\mathbb{Q}_p$ to the dual group. Even though this parametrization is still conjectural, in the next section we derive some implications which can be stated independently of the parametrization. We check some of this implications in Section 3.

We first review the parametrization, for G any semi-simple, split group of adjoint type over $\mathbb{Q}_p$. Let $\hat{G}$ be the Langlands dual group, so $\hat{G}(\mathbb{C})$ is semi-simple and simply-connected complex Lie group. The conjectural parameter of an irreducible, admissible, complex representation of $G(\mathbb{Q}_p)$. is a pair (φ, χ), where

$$\varphi : W' \to \hat{G}(\mathbb{C}) \tag{1.1}$$

is an admissible homomorphism of the Weil-Deligne group W' of $\mathbb{Q}_p$ [B1], and χ is an irreducible complex representation of a finite group B_φ associated to φ.

We recall that φ is a continuous homomorphism of the Weil group W taking Frobenius elements to a semi-simple class, together with a nilpotent element N in $\hat{\mathfrak{g}}$, the Lie algebra of $\hat{G}(\mathbb{C})$, with $Ad(w)(N) = ||w||\, N$. By the Jacobson-Morozov theorem, giving a parameter φ as in (1.1) is equivalent to giving a continuous, semi-simple representation

$$\eta : W \times SL_2(\mathbb{C}) \to \hat{G}(\mathbb{C}) \tag{1.2}$$

with

$$\begin{cases} \varphi(w) = \eta(w, \begin{pmatrix} ||w||^{1/2} & 0 \\ 0 & ||w||^{-1/2} \end{pmatrix}) \\ \exp(N) = \eta(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) \end{cases} \tag{1.3}$$

Associated to φ, we have the finite group

$$A_\varphi = \pi_0(Cent(\eta)) \tag{1.4}$$

where $Cent(\eta)$ is the algebraic subgroup of $\hat{G}$ which centralizes the image of η and π_0 donotes the corresponding group of connected components. If $Z(\hat{G})$ is the center of $\hat{G}$, the inclusion $Z(\hat{G}) \subseteq Cent(\varphi)$ induces a map (not necessarily injective)

$$Z(\hat{G}) \to A_\varphi. \tag{1.5}$$

The image is a normal subgroup, and we define B_φ as the quotient. Then an irreducible representation π of $G(\mathbb{Q}_p)$ should correspond to a pair (φ, χ), where χ is an irreducible representation of B_φ.

If $G = G_2(\mathbb{Q}_p)$, then $\hat{G}(\mathbb{C}) = G_2(\mathbb{C})$, and $A_\varphi = B_\varphi$. If $G' = PGSp_6(\mathbb{Q}_p)$, then $\hat{G}'(\mathbb{C}) = Spin_7(\mathbb{C})$ and $B_{\varphi'}$ is the quotient by the image of -1. The possible finite groups which arise are given by the following. Let μ_k denote the cyclic group of k^{th} roots of 1, and S_k the symmetric group on k letters.

Proposition 1.6.

(1) *If* $\varphi : W' \to G_2(\mathbb{C})$ *is a parameter for* $G = G_2$*, then the group* B_φ *is isomorphic to* μ_3*,* S_3*, or* μ_2^k*, with* $0 \leq k \leq 3$*.*

(2) *If* $\varphi' : W' \to Spin_7(\mathbb{C})$ *is a parameter for* $G' = PGSp_6$ *then the group* $B_{\varphi'}$ *is isomorphic to* μ_2^k*, with* $0 \leq k \leq 3$*. If* a *is the number of distinct orthogonal representations in the decomposition of the semi-simple* $W \times SL_2(\mathbb{C})$*-module* $\mathbb{V} = \mathbb{C}^7$*, and* b *is the dimension of the subspace of* $Hom(W, \pm 1)$ *spanned by the determinants of the orthogonal summands in* $\mathbb{V}$*, then* $k = a - b - 1$*.*

Proof.
1) Examples of subgroups $Im(\eta)$ of $G_2(\mathbb{C})$ with the given groups B_φ are given by the following table:

$Im(\eta)$	B_φ
G_2	1
SL_3	μ_3
SO_3	S_3
SO_4	μ_2
$T \rtimes <\pm 1>$	μ_2^2
$T_2 \rtimes <\pm 1>$	μ_2^3

Here $T \cong GL_1^2$ is a maximal torus, T_2 the subgroup of T killed by 2, and -1 the central element in the Weyl group of T. The latter case ($T_2 \rtimes <\pm 1> \cong \mu_2^3$) can only be the image of η when $p = 2$. This list exhausts the possible groups B_φ, as we will see in the proof of Proposition 1.10.
2) Examples of subgroups $Im(\eta')$ of $Spin_7(\mathbb{C})$ with the given groups B_φ are given by the following table:

$Im(\eta')$	$A_{\varphi'}$	$B_{\varphi'}$
$Spin_7$	$<\pm 1>$	1
$Spin_6 = SL_4$	μ_4	μ_2
$T' \rtimes <\pm 1>$	μ_2^3	μ_2^2
$T_2' \rtimes <\pm 1>$	μ_2^4	μ_2^3

Again, $T' \cong GL_1^3$ is a maximal torus, T_2' the subgroup of T' killed by 2, and -1 the central element in the Weyl group of T'. The last case ($T_2' \rtimes <\pm 1> \cong \mu_2^4$) can only be the image of η' when $p = 2$.

To see that these are the only possibilities for $B_{\varphi'}$, and to verify the formula for k, we note that the representation V gives a parameter for Sp_6:

$$\bar{\varphi}' : W' \to Spin_7(\mathbb{C}) \to SO_7(\mathbb{C}).$$

We can compute the component group $A_{\bar{\varphi}'}$ using the results in [GP; Cor. 7.7], and find that

$$A_{\bar{\varphi}'} = B_{\bar{\varphi}'} \cong \mu_2^{a-1}$$

where a is the number of distinct orhogonal summands in the semi-simple representation V. But we have an exact sequence [GP; pg 983]

$$\begin{aligned} 1 \to B_{\varphi'} \to B_{\bar{\varphi}'} &\to Hom(W, \pm 1) \\ x &\mapsto \det(\mathbb{V}^{x=-1}) \end{aligned}$$

so $k = a - b - 1$ as claimed.

Now let

$$f : \hat{G}(\mathbb{C}) = G_2(\mathbb{C}) \to \hat{G}'(\mathbb{C}) = Spin_7(\mathbb{C}) \tag{1.7}$$

be the inclusion, well defined up to conjugacy, that realizes G_2 as the fixer of a non-isotropic line in the 8-dimensional spin representation of $Spin_7$. If φ is a parameter for G, then

$$\varphi' = f \circ \varphi \tag{1.8}$$

is a parmeter for G'. Moreover, f induces a map $f_\varphi : A_\varphi \to A_{\varphi'}$. Since $A_\varphi = B_\varphi$ and $B_{\varphi'}$ is the quotient of $A_{\varphi'}$ by the image of -1, we get an induced map

$$f_\varphi : B_\varphi \to B_{\varphi'} \tag{1.9}$$

Proposition 1.10. *The map f_φ (1.9) is surjective, with kernel the Sylow 3-subgroup of B_φ (either 1 or μ_3).*

Proof. In the absence of the intelligent argument, we can prove this in a case by case manner, considering the connected component of the image of η in $G_2(\mathbb{C})$, which is a reductive subroup C. Considering possible normalizers of C, and their action on $\mathbb{V}$, we can compare B_φ with $B_{\varphi'}$ computed in Proposition 1.6.

For example, assume that C has rank 2, i.e. it contains the maximal torus T. Then C is determined by its root system which is contained in the root system of G_2. Hence the possibilities for C are G_2, SL_3, SO_4, $GL_{2,s}$, $GL_{2,l}$, and T, where SL_3 is spanned by long roots,

$$SO_4 = SL_{2,s} \times SL_{2,l}/\Delta < \pm 1 >$$

is a group spanned by a pair of perpendicular roots, one short and one long, and $GL_{2,s}$ and $GL_{2,l}$ are Levi factors of maximal parabolic subgroups of G_2. The corresponding groups B_φ and $B_{\varphi'}$ are

$Im(\eta)$	B_φ	$B_{\varphi'}$
G_2	1	1
SO_4	μ_2	μ_2
$N(SL_3)$	1	1
SL_3	μ_3	1
$N(GL_{2,s})$	μ_2	μ_2
$GL_{2,s}$	1	1
$N(GL_{2,l})$	μ_2	μ_2
$GL_{2,l}$	1	1
$N(T)$	1	1
$T \rtimes S_3$	μ_3	1
$T \rtimes (2,2)$	μ_2	μ_2
$T \rtimes \mu_6$	1	1
$T \rtimes \mu_3$	μ_3	1
$T \rtimes < \pm 1 >$	μ_2^2	μ_2^2
$T \rtimes (2)_s$	μ_2	μ_2
$T \rtimes (2)_s$	μ_2	μ_2
T	1	1

We leave the analysis when the connected component of the image of η has rank 1 or 0 to the reader to check. The group $B_\varphi = S_3$ arises only when the image is $SO_3 \subset SL_3 \subset G_2$. In this case $\mathbb{V} = 2\mathbb{C}^3 \oplus \mathbb{C}$ where $\mathbb{C}^3$ is the standard representation of SO_3. In particular, it is an orthogonal representation of W of determinant 1. Hence, $B_{\varphi'} = \mu_2$, and the map $S_3 \to \mu_2$ is the sign character.

If φ is a parameter for G and $\varphi' = f \circ \varphi$, by Proposition 1.10 we have a surjective map

$$f_\varphi : B_\varphi \to B_{\varphi'} \cong \mu_2^k. \tag{1.11}$$

If χ' is an irreducible representation of $B_{\varphi'}$ (i.e. a quadratic character), we obtain a quadratic character

$$\chi = \chi' \circ f_\varphi \text{ of } B_\varphi. \tag{1.12}$$

We can now state the conjecture on the restriction of Π.

Conjecture 1.13. *Let Π be the minimal representation of $H(\mathbb{Q}_p)$. The representation*

$$\pi \otimes \pi' = \pi(\varphi, \chi) \otimes \pi'(\varphi', \chi)$$

of $G(\mathbb{Q}_p) \times G'(\mathbb{Q}_p)$ is a quotient of Π if and only if

$$\begin{aligned} \varphi' &= f \circ \varphi \\ \chi &= \chi' \circ f_\varphi. \end{aligned}$$

In this case $Hom_{G \times G'}(\Pi, \pi \otimes \pi')$ has dimension 1.

2. Some consequences.

Let π be an irreducible representation of $G(\mathbb{Q}_p)$ and π' an irreducible representation of $G'(\mathbb{Q}_p)$. Define

$$\Theta(\pi) \tag{2.1}$$

to be the set of equivalence classes of irreducible representations σ' of $G'(\mathbb{Q}_p)$ such that $\pi \otimes \sigma'$ is a quotient of Π. Similarly, define

$$\Theta(\pi') \tag{2.2}$$

to be the set of equivalence classes of irreducible representations σ of $G(\mathbb{Q}_p)$ such that $\sigma \otimes \pi'$ is a quotient of Π. The conjecture 1.14 implies

Conjecture 2.3.

(1) *Card* $\Theta(\pi) \leq 1$, *with equality if the character* χ *of* B_φ *is quadratic.*
(2) *Card* $\Theta(\pi') \leq 1$, *with equality if the parameter* φ' *has image in the subgroup* $G_2(\mathbb{C})$ *of* $Spin_7(\mathbb{C})$.

We write

$$\pi \leftrightarrow \pi' \tag{2.4}$$

and say that π corresponds to π' if

$$\Theta(\pi(\varphi)) = \{\pi(\varphi')\} \text{ and } \Theta(\pi(\varphi)) = \{\pi(\varphi')\}. \tag{2.5}$$

In particular, $\pi \otimes \pi'$ is a quotient of Π.

Recall that for each semi-simple conjugacy class s in $G_2(\mathbb{C})$, there is an unramified representation $\pi(s)$ of $G(\mathbb{Q}_p)$ with Satake parameter s. Similarly, if s' is a semi-simple conjugacy class in $Spin_7(\mathbb{C})$, there is an unramified representation $\pi(s')$ of $G'(\mathbb{Q}_p)$ with Satake parameter s'.

Conjecture 2.5.

(1) *If* $s' = f(s)$, *then* $\pi(s) \leftrightarrow \pi(s')$. *In particular, the trivial representation* 1 *of* G *corresponds to the trivial representation* $1'$ *of* G'.
(2) *Card* $\Theta(\pi(s')) = 0$ *unless* $s' = f(s)$ *for some* s.
(3) *The Steinberg representation* St *of* G *corresponds to the Steinberg representation* St' *of* G'.

These predictions follow immediately from Conjecture 1.13. For (1) we note that the parameter of $\pi(s)$ is a homomorphism $\varphi : W' \to \hat{G}(\mathbb{C})$ with $N = 0$, φ trivial on the inertia subgroup and $\varphi(Frob_p) = s$. This has $B_\varphi = 1$, so $\chi = 1$. It follows that φ' is a the parameter of $\pi(s')$.

The Satake parameter of the trivial representation is $\check{\rho}(p)$ where $\check{\rho}$ is the co-character given by half the sum of positive co-roots. Since the image of the principal SL_2 in G_2 under the map $f : G_2 \to Spin_7$ is the principal SL_2 in $Spin_7$, we have $f \circ \check{\rho} = \check{\rho}'$. Hence the trivial representation 1 of G should correspond to the trivial representation $1'$ of G'. The same argument shows that the Steinberg representations should correspond, as those parameters factor through the principal SL_2.

The part (1) is true for tempered representations (recall that the representation $\pi(s)$ is tempered if s lies in a maximal compact subgroup). This is shown in [MS]. In the next section we show that $1 \leftrightarrow 1'$, and we verify (2). We also obtain a partial verification of (3). For example, we show that $\Theta(St') \subseteq \{St\}$.

3. Some calculations.

We first prove a statement slightly stronger then Conj. 2.5 (2):

Proposition 3.1. *Let $\pi(s')$ be an unramified representation of $G'(\mathbb{Q}_p)$. If $\pi(s')$ is a quotient of Π, then $s' = f(s)$ for some s in $G_2(\mathbb{C})$.*

We note that every unramified representation π' can be realized as a submodule

$$\pi' \subseteq \operatorname{Ind}_{\bar{Q}}^{G'} \bar{\sigma} \tag{3.2}$$

for some unramified representation $\bar{\sigma}$ of GL_3. Here $\bar{Q} = L\bar{U}$, $L \cong GL_3$, is the maximal parabolic subgroup of G', opposite to Q, defined in (3.20), Chapter II. Let

$$\bar{s} = |p| \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_3 \end{pmatrix} \quad \text{in} \quad \hat{GL}_3(\mathbb{C}) = GL_3(\mathbb{C}) \tag{3.3}$$

be the Satake parameter of $\bar{\sigma}$. Here $|\cdot|$ is a norm on $\mathbb{Q}_p$, such that $|p| = 1/p$. The factor $|p|$ enters through the normalization of the parabolic induction. The corresponding modular function $\rho_{\bar{U}}$ of GL_3 is

$$\rho_{\bar{U}} = |\det|. \tag{3.4}$$

So if π' is contained in $\operatorname{Ind}_{\bar{Q}}^{G'} \bar{\sigma}$, then the Satake parameter of π' is

$$s' = \bar{s}|p|^{-1} \quad \text{in} \quad GL_3(\mathbb{C}) \subset Spin_7(\mathbb{C}). \tag{3.5}$$

The 8-dimensional spin representation of $Spin_7$ restrict to $GL_3(\mathbb{C})$ as

$$\det \oplus \mathbb{C}^3 \oplus (\mathbb{C}^3)^* \oplus (\det)^*, \tag{3.6}$$

where $\mathbb{C}^3$ is the standard representatio of $GL_3(\mathbb{C})$, and $*$ denotes dual representations. Hence the parameter s' fixes a vector in the spin representation (so $s' = f(s)$ for some s in $G_2(\mathbb{C})$) if

$$z_1 z_2 z_3 = 1 \quad \text{or} \quad z_i = 1 \quad \text{for some} \quad i. \tag{3.7}$$

By the Frobenius reciprocity,

$$Hom_{G'}(\Pi, \operatorname{Ind}_{\bar{Q}}^{G'} \bar{\sigma}) = Hom_{GL_3}(\Pi_{\bar{U}}, \bar{\sigma}) \tag{3.8}$$

it suffices to determine which representations of GL_3 appear as a quotient of $\Pi_{\bar{U}}$, the maximal $\bar{U}$-invariant quotient of Π.

To describe $\Pi_{\bar{U}}$ we need some notation. Let Q_1 and Q_2 be the two non-conjugated maximal parabolic subgroups of GL_3 intersecting in the group of lower triangular 3×3 matrices, with Levi factors $GL_1 \times GL_2$ and $GL_2 \times GL_1$ respectively. Then

$$\rho_1'(g_1, g_2) = |g_1|^{-1} |\det g_2|^{1/2} \quad \text{and} \quad \rho_2'(g_2, g_1) = |\det g_2|^{-1/2} |g_1| \tag{3.9}$$

are their modular characters.

Maximal parabolic subgroups of $G(\mathbb{Q}_p)$ can be defined as stabilizers of non-trivial nil subalgebras of $\mathbb{O}_p = \mathbb{O} \otimes \mathbb{Q}_p$. A nil subalgebra is a subspace of $\mathbb{O}_p$ consisting of traceless elements with trivial multiplication (i.e. the product of any two elements is 0). The possible dimensions are 1 and 2. Fix $V_1 \subset V_2$, a pair of nil-subalgebras. Then P_1 and P_2, the stabilizers of V_1 and V_2, are two non-conjugated maximal parabolic subgroups of G, with $P_1 \cap P_2$ a Borel subgroup. In particular, P_r has a quotient $GL_r = GL(V_r)$. For P_2, this quotient is isomorphic to the Levi factor. The Levi factor of P_1 is isomorphic to $GL_2 = GL(V_3/V_1)$, where

$$V_3 = \{x \in \mathbb{O} \mid \bar{x} = -x, \quad \text{and} \quad xV_1 = 0\}. \tag{3.10}$$

The action of GL_2, the Levi factor of P_1, on V_1 is given by det. The respective modular characters are

$$\rho_1 = |\det|^{5/2} \quad \text{and} \quad \rho_2 = |\det|^{3/2}. \tag{3.11}$$

We fix the above identifications, in particular, $P_r \times Q_r$, $r = 1, 2$, has a quotient isomorphic to

$$GL_r \times GL_r. \tag{3.12}$$

Proposition 3.13. [MS; Thm. 5.3] *The $G \times GL_3$-module $\Pi_{\bar{U}}$ has a filtration*

$$0 = V_0 \subset V_1 \subset V_2 \subset V_3 = \Pi_{\bar{U}}$$

such that

(1)
$$V_1/V_0 \cong \mathrm{ind}_{P_2 \times Q_2}^{G \times GL_3} (C_c^\infty(GL_2)) \otimes |\det|^2$$

(2)
$$V_2/V_1 \cong \mathrm{ind}_{P_1 \times Q_1}^{G \times GL_3} (C_c^\infty(GL_1)) \otimes |\det|^2$$

(3)
$$V_3/V_2 = \Pi_{\bar{N}} \cong \Pi(M) \otimes |\det| \;\oplus\; 1 \otimes |\det|^2$$

Here $C_c^\infty(GL_i)$ denotes the space of locally constant, compactly supported functions on GL_r, $r = 1, 2$. In both cases, $P_r \times Q_r$ acts through the quotient isomorphic to $GL_r \times GL_r$. In (3), $\Pi(M)$ is the minimal representation of M (the center of M, which coincides with the center of GL_3, acts trivially on $\Pi(M)$).

Corollary 3.14. *The possible Satake parameters of unramified GL_3-quotients of V_i/V_{i-1} are:*

1. $|p|(z_1, z_2, 1)$ *if* $i = 1$.
2. $|p|(z_1, |p|, 1)$ *if* $i = 2$.
3. $|p|(z_1, z_2, z_3)$ *with* $z_1 z_2 z_3 = 1$, *or* $|p|(|p|^2, |p|, 1)$ *if* $i = 3$. *The latter is the parameter of* $|\det|^2$.

Proposition 3.1 follows from Corollary 3.14.

Let π be an irreducible representation of G and π' an irreducible representation of G'. We now explain how Proposition 3.13 can be used to obtain an upper bound on $\Theta(\pi')$ and a lower bound on $\Theta(\pi)$.

Assume that π' is a submodule of $\mathrm{Ind}_{\bar{Q}}^{G'}(\bar{\sigma})$, where $\bar{\sigma}$ is a representation of GL_3. If σ is in $\Theta(\pi')$, then by the Frobenius reciprocity

$$Hom_{G\times G'}(\Pi, \sigma \otimes \mathrm{Ind}_{\bar{Q}}^{G'}(\bar{\sigma})) \cong Hom_{G\times GL_3}(\Pi_{\bar{U}}, \sigma \otimes \bar{\sigma}), \tag{3.15}$$

$\sigma \otimes \bar{\sigma}$ is a quotient of $\Pi_{\bar{U}}$. Hence, if we can determine all representations σ of G such that $\sigma \otimes \bar{\sigma}$ is a quotient of $\Pi_{\bar{U}}$, then we have an upper bound on $\Theta(\pi')$. Conversly, if $\pi \otimes \bar{\sigma}$ is a a quotient of $\Pi_{\bar{U}}$, for some $\bar{\sigma}$, then $\Theta(\pi)$ is not empty, for it contains a subquotient of $\mathrm{Ind}_{\bar{Q}}^{G'}(\bar{\sigma})$ by (3.15).

To illustrate this principle, we prove

Proposition 3.16. $1 \leftrightarrow 1'$.

Proof. We first prove the following lemma.

Lemma 3.17.

1. $\Theta(1)$ *is not empty.*
2. $\Theta(1') \subseteq \{1\}$.

Proof. By Proposition 3.13, $1 \otimes |\det|^2$ is a quotient of $\Pi_{\bar{U}}$. Hence by (3.15) $\Theta(1)$ is not empty. On the other hand,

$$1' \subset \mathrm{Ind}_{\bar{Q}}^{G'} \bar{1}$$

where $\bar{1}$ is the trivial representation of GL_3. Its Satake parameter is

$$\begin{pmatrix} |p|^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & |p| \end{pmatrix} = |p| \begin{pmatrix} |p|^{-2} & 0 & 0 \\ 0 & |p|^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Comparing with Corollary 3.14, we see that $\sigma \otimes \bar{1}$ can be only a quotient of

$$V_1/V_0 = \mathrm{ind}_{P_2\times Q_2}^{G\times GL_3}(C_c^\infty(GL_2)) \otimes |\det|^2.$$

Since $C_c^\infty(GL_2)$ is the regular representation of GL_2, $\sigma \otimes |\det|^{-2}$ is a quotient of

$$\mathrm{Ind}_{P_2}^{G}(\tau) \otimes \mathrm{Ind}_{Q_2}^{GL_3}(\tau^*)$$

for some irreducible representation τ of GL_2. This implies that $\tau^* = |\det|^{-3}$. Hence $\tau = |\det|^3 = \rho_2^2$. Since 1 is unique quotient of

$$\mathrm{Ind}_{P_2}^{G}(\rho_2^2)$$

it follows that $\sigma \cong 1$. The lemma is proved.

Let $\bar{U}_2$ be the unipotent radical of $\bar{P}_2$, the maximal parabolic of G, opposite to P_2. Since $\Pi_{\bar{U}_2}$ is given by [MS; Thm. 7.6], we can prove a statement complementary to Lemma 3.17:

Lemma 3.18.

(1) $\Theta(1')$ *is not empty.*
(2) $\Theta(1) \subseteq \{1'\}$.

The two lemmas combined imply the proposition.

We finish this section with a discussion on Steinberg representations.

Proposition 3.19. $\Theta(St') \subseteq \{St\}$.

Proof. The representation St' is unique submodule of

$$\mathrm{Ind}_{\bar{Q}}^{G'}(\bar{S}t \otimes |\det|^2)$$

where $\bar{S}t$ is the Steinberg representation of GL_3. Again, if $\sigma \otimes St'$ is a quotient of Π, then by (3.15) $\sigma \otimes (\bar{S}t \otimes |\det|^2)$ is a quotient of $\Pi_{\bar{U}}$. However, it can not be a quotient of V_3/V_2, because the central character of $(\bar{S}t \otimes |\det|^2)$ is $|\cdot|^6$, and the central character of $\Pi_M \otimes |\det|^2$ is $|\cdot|^3$. Also, it can not be a quotient of V_2/V_1, because the Steinberg representation is generic. Hence $\sigma \otimes (\bar{S}t \otimes |\det|^2)$ must be a quotient of V_1/V_0. So as in the proof of Lemma 3.17, $\sigma \otimes \bar{S}t$ must be a quotient of

$$\mathrm{Ind}_{P_2}^{G}(\tau) \otimes \mathrm{Ind}_{Q_2}^{GL_3}(\tau^*)$$

for an irreducible representation τ of GL_2. It follows that τ^* is the Steinberg representation of GL_2. Hence τ is also the Steinberg representation. Since St is unique quotent of

$$\mathrm{Ind}_{P_2}^{G}(\tau),$$

the proposition follows.

Again, in the same fashion using $\Pi_{\bar{U}_2}$, we can prove that if π' is a *unitarizable* Θ-lift of St then

$$\pi' \cong St'. \tag{3.20}$$

As we shall see in the next chapter, the Θ-lift of a form in $A(St_S)$, (S not empty) will be cuspidal, hence all its local components must be unitarizable. So (3.20) implies that the lift is Steinberg at all places in S.

V Global correspondences

Let H be the adjoint goup of type E_7, defined and of rank three over $\mathbb{Q}$. Kim [Ki] has constructed a square-integrable modular form on the exceptional hermitian domain which gives an automorphic realization

$$\theta : \otimes_p \Pi_p \to L^2(H(\mathbb{Q})\backslash H(\mathbb{A})) \tag{0.1}$$

where Π_p is the minimal representation of $H(\mathbb{Q}_p)$, and Π_∞ is the space of K-finte vectors in the irreducible unitary representation $\hat{\Pi}$ of $H(\mathbb{R})$, studied in Chapter III.

In view of the local results, it is of great interest to study the lift of automorphic forms from G to G' via the kernel constructed by Kim. Let $\theta = \theta(\otimes_p f_p)$ for some $\otimes_p f_p \in \otimes_p \Pi_p$. Let π be an automorphic representation of $G = Aut(\mathbb{O})$. Let α be a form in π. Define β, a holomorphic form on G' by

$$\beta(g') = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \theta(gg')\alpha(g)dg. \tag{0.2}$$

The integral converges, because we are integrating two smooth functions over a compact set.

Let π' be a cuspidal automorphic representation of G'. We say that π' is a Θ lift of π if

$$\int_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \alpha'(g')\beta(g')dg' \tag{0.3}$$

is not zero for some α and α' in π'. The integral (0.3) converges because α' is rapidly decreasing at cusps, and β is of moderate growth. Hence (0.2) and (0.3) define a linear functional on

$$\Pi \otimes \pi \otimes \pi'. \tag{0.4}$$

If this functional is not trivial, then local components of π and π' are related by the local correspondences studied in Chapters III and IV.

The existence of π' depends on the affirmative answer to these two questions:

(1) Under which conditions is the form β cuspidal?
(2) Under which conditions is the form β non-zero?

To answer these questions we study the Fourier expansion of β along the unipotent radical U of the Siegel parabolic subgroup of $G' = PGSp_6$. Note that the Fourier coefficients are parametrized by ternary quadratic forms. The holomorphic form β is cuspidal, if the Fourier coefficients corresponding to degenerate forms are zero [An; pg 78]. We show that β is cuspidal if a local component of π is generic, i.e. it admits a non-zero Whittaker functional. For example, if π corresponds to a modular form in $A(\epsilon_p)$, (5.3) in Chapter I, then the local component at the place p has a non-zero Whittaker functional. This gives a fairly satisfactory answer to (1). On the other hand, $L(\mathbb{Q})$-conjugacy classes (L is the Levi factor of the Siegel parabolic) of non-degenerate Fourier coefficients are parametrized by quaternion algebras. Let D be a quaternion subalgebra of $\mathbb{O}$ (the class of Fourier coefficients is zero if $D \otimes \mathbb{R}$ is not definite). Let C be the centralizer in G of D. It is isomorphic to the group of norm one elements in D. Then we show that the Fourier coefficient of β in the conjugacy class parametrized by D is non-zero if and only if the integral of the form α over C is non-zero.

This result has a striking similarity to the following well known classical result [Shn] and [Wa]: Let $PD^\times$ be the automorphism group of D. Then one has a dual pair

$$PD^\times \times SL_2 \subset Sp_6,$$

and one can use the Weil representation of $\tilde{Sp}_6$ to lift automorphic forms from $PD^\times$ to $\tilde{SL}_2$. The conjugacy classes of non-degenerate Fourier coefficients on $\tilde{SL}_2$ are parametrized by quadratic algebras. Let K be a quadratic subalgebra of D. Fix an automorphic form on $PD^\times$. The Fourier coefficients (the conjugacy class parametrized by K) of the lift are non-zero if and only if the integral of the form over the centralizer in $PD^\times$ of K is non-zero. Again, the class of Fourier coefficients is zero if K can not be embedded into D.

We finish this chapter by showing that the two modular forms for G constructed in Chapter I lift non-trivially to G'.

1. Kim's form.

In this section we recall few results from [Ki]. Let $\mathbb{O}_\infty = \mathbb{O} \otimes \mathbb{R}$ and let $J^+_{\mathbb{O}_\infty}$ be the cone of positive definite matrices in $J_{\mathbb{O}_\infty}$. The exceptional symmetric domain $\mathcal{D}^+$ is the set

$$\mathcal{D}^+ = \{Z = X + iY \mid X \text{ in } J_{\mathbb{O}_\infty} \text{ and } Y \text{ in } J^+_{\mathbb{O}_\infty}\}. \tag{1.1}$$

The group of holomorphic transformation of $\mathcal{D}^+$ is isomorphic to $H_{sc}(\mathbb{R})/ < \pm 1 >$, the connected component of $H(\mathbb{R})$.

Let f^0_∞ be a non-zero vector in the one-dimensional minimal K-type in Π^+_∞, and let f^0_p (p is a finite prime) be the spherical vector in Π_p. Kim has constructed a modular form F on $\mathcal{D}^+$ of weight 4 and level 0, which corresponds to $\theta(\otimes_p f^0_p)$. Moreover, he has obtained a Fourier series decomposition

$$F(Z) = 1 + 240 \sum_{T \in J_{\mathcal{R}}} a_T \; e^{2\pi i \; \mathrm{Tr}(T \circ Z)} \tag{1.2}$$

where $J_{\mathcal{R}}$ is the set of A in $J_{\mathbb{O}}$ with entries in $\mathcal{R}$. A coefficient a_T is zero unless the rank of T is 1, and it is in the closure of the cone $J^+_{\mathbb{O}_\infty}$. In this case,

$$a_T = \sum_{d \mid c(T)} d^3, \tag{1.3}$$

where $c(T)$ is the largest integer such that $c(T)^{-1}T$ is in $J_{\mathcal{R}}$. In particular, $a_T > 0$.

2. Local theory.

In this section all objects are over $\mathbb{Q}_p$. As in Chapter II, Section 3, let $P = MN$ be the maximal parabolic subgroup of H such that M is the group of isogenies of the determinant form on the exceptional Jordan algebra $J_{\mathbb{O}_p}$, and $N \cong J_{\mathbb{O}_p}$ as an M-module. Let λ be the isogeny character of M defined by (3.5) in Chapter II.

Let ψ be a non-trivial character of $\mathbb{Q}_p$. Let A be an element in $J_{\mathbb{O}_p}$. Define a character ψ_A of N by

$$\psi_A(B) = \psi(\mathrm{Tr}(A \circ B)) \tag{2.1}$$

where $B \in J_{\mathbb{O}_p} \cong N$. A non-zero element A in $J_{\mathbb{O}_p}$ has rank 1 if

$$A^2 = \mathrm{Tr}(A)A. \tag{2.2}$$

Let Ω be the set of elements of rank 1. We wil need the following result [MS; Thm. 1.1]:

Proposition 2.3. *The minimal representation* Π *of* H *fits into the sequence of* P*-modules*

$$C_c^\infty(\Omega) \subset \Pi \subset C^\infty(\Omega),$$

where $C^\infty(\Omega)$ *denotes the space of locally constant functions on* Ω *and* $C_c^\infty(\Omega)$ *is the subspace of compactly supported functions. Moroever,*

(1) *if* $f \in C^\infty(\Omega)$, *then*

$$\begin{cases} \Pi(n)f(X) = \psi_X(n)f(X), & n \in N \cong J_{\mathbb{O}_p} \\ \Pi(m)f(X) = |\lambda(m)|^2 f((\tilde{m}(X)), & m \in M, \end{cases}$$

where $\tilde{m}(X)$ *is defined by*

$$\mathrm{Tr}(\tilde{m}(X) \circ Y) = \mathrm{Tr}(X \circ m(Y)),$$

for all Y *in* $J_{\mathbb{O}_p}$.

(2)

$$\Pi / C_c^\infty(\Omega) = \Pi_N,$$

where Π_N *is the maximal* N*-invariant quotient of* Π.

Let (π, E) be a smooth N-module. Define E_{N,ψ_A} to be the quotient of E by the subspace $E(N, \psi_A)$ spanned by the elements $\{\pi(n)v - \psi_A(n)v \mid n \in N, v \in E\}$. Since the functor $E \rightsquigarrow E(N, \psi_A)$ is exact [BZ], Prop. 2.3 implies the following:

Corollary 2.4. *Let* A *be a non-zero element in* $J_{\mathbb{O}}$. *Then*

$$\dim \Pi_{N,\psi_A} \leq 1.$$

It is 1 if and only if the rank of A *is 1. In this case, the dual of* Π_{N,ψ_A} *is spanned by "evaluation at* A*".*

3. Fourier coefficients.

Fix ψ, a non-trivial character of $\mathbb{A}/\mathbb{Q}$, with conductor $\prod_p \mathbb{Z}_p$. Let $A \in J_{\mathbb{Q}}$. As in the previous section we define a character ψ_A of $N(\mathbb{A})$.

For a function $\theta = \theta(\otimes_p f_p)$, define the Fourier coefficient θ_A by

$$\theta_A(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \theta(ng)\psi_A(n)dn. \tag{3.1}$$

Corollary 2.3 implies that $\theta_A(g) = 0$ if A has rank > 1.

We consider the Fourier expansion of β defined by (0.2) along U, the unipotent radical of the Siegel parabolic of G'. We identify $U(\mathbb{Q})$ with the set of 3×3 symmetric matrices with coefficients in $\mathbb{Q}$. Let B be an element in $U(\mathbb{Q})$. Then

$$\beta_B(1) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \theta(ug)\alpha(g)\psi_B(u)dgdu. \tag{3.2}$$

Write $\theta(g) = \sum_{rkA \leq 1} \theta_A(g)$ and substitute into the formula for $\beta_B(1)$. Then

$$\beta_B(1) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \sum_{\psi_A|_{U(\mathbb{A})} = \psi_B^{-1}} \theta_A(g)\alpha(g)dg. \tag{3.3}$$

Lemma 3.4. *The group $G(\mathbb{Q})$ acts transitively on the set of all rank-one elements A in $J_{\mathbb{O}} \cong N(\mathbb{Q})$ such that*

$$\psi_A|_{U(\mathbb{A})} = \psi_B^{-1}.$$

Proof. After conjugating B with an element in $L(\mathbb{Q}) \cong GL_3(\mathbb{Q})$, we can assume that B is given by a diagonal matrix:

$$B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} d & \bar{z} & y \\ z & e & \bar{x} \\ \bar{y} & x & f \end{pmatrix}$$

such that $\psi_A|_U = \psi_B^{-1}$. Then

$$\begin{array}{lll} d = -a, & e = -b, & f = -c \\ \overline{x} = -x, & \overline{y} = -y, & \overline{z} = -z. \end{array}$$

Since A is rank-one, we have $A^2 = \mathrm{Tr}(A)A$ and this implies

$$\begin{array}{lll} x^2 = -bc, & y^2 = -ca, & z^2 = -ab \\ yz = -ax, & zx = -by, & xy = -cz. \end{array}$$

We have four cases:

(1) $a = b = c = 0$. Then $x^2 = y^2 = z^2 = 0$. Since $\mathbb{O}$ is a division algebra, this implies that $x = y = z = 0$. Hence $A = 0$ is the only possibility.
(2) $a \neq 0, b = c = 0$. Then again $x^2 = y^2 = z^2 = 0$, and $A = -B$ is the only possibility.
(3) $ab \neq 0$ but $c = 0$. Then $x^2 = y^2 = 0$. Therefore $x = y = 0$. Since $z^2 = -ab$, $K = \mathbb{Q}(z)$ is a quadratic subalgebra of $\mathbb{O}$. By a theorem of Jacobson [J3], $G(\mathbb{Q})$ acts transitively on the set of traceless z such that $z^2 = -ab$. The stabilizer of a point is isomorphic to $SU(K^{\perp}) = SU_3^K$. This form of SL_3 is compact over $\mathbb{R}$, quasi-split for all finite primes, and split by K.
(4) $abc \neq 0$. In this case x, y and z are standard generators of a quaternion subalgebra D of $\mathbb{O}$. By a theorem of Jacobson [J3], $G(\mathbb{Q})$ acts transitively on the set of triples (x, y, z). The stabilizer of a point is isomorphic to the group of norm one elements in D.

Finally, note that A will exist only if B is a semi-definite matrix. The lemma is proved.

Assume that $B = 0$. The lemma implies that

$$\beta_0(1) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \theta_0(g)\alpha(g)dg. \tag{3.5}$$

Since $\theta_0(1) = \theta_0(m)$ for any $m \in [M(\mathbb{A}), M(\mathbb{A})]$ (the Fourier coefficient a_0 is constant in the Kim's formula), and $G(\mathbb{A}) \subset [M(\mathbb{A}), M(\mathbb{A})]$,

$$\beta_0(1) = \theta_0(1) \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \alpha(g)dg. \tag{3.6}$$

Remark: Ginzburg, Rallis, and Soudry have constructed in [GRS1] an automorphic representation of the split group $E_{7,7}(\mathbb{A})$, whose local components are the minimal representations. Moreover, they have shown that the constant term along the unipotent radical of the $E_{6,6}$-maximal parabolic is a sum of two automorphic representations of $[M(\mathbb{A}), M(\mathbb{A})] = E_{6,6}(\mathbb{A})$: the trivial representation, and a representation whose local components are the minimal representations. In the case of $E_{7,3}(\mathbb{A})$, the latter summand can not appear, because the local component $E_{6,2}(\mathbb{R})$ of $[M(\mathbb{A}), M(\mathbb{A})]$ has no minimal representation.

Now assume that $B \neq 0$. Fix A', a representative of the orbit. Since A' and nA' ($n \in \mathbb{Z}$) are in the same $L(\mathbb{Q})$-orbit, we shall assume that the entries of A' lie in $\mathcal{R}$, the maximal order of $\mathbb{O}$. Let $\mathcal{A}$ be the composition algebra generated by the entries of A'. Let

$$C(\mathbb{Q}) = C_{G(\mathbb{Q})}(\mathcal{A}). \tag{3.7}$$

Then

$$\beta_B(1) = \int_{C(\mathbb{Q})\backslash G(\mathbb{A})} \theta_{A'}(g)\alpha(g)dg. \tag{3.8}$$

Recall that $\theta = \theta(\otimes_p f_p)$ and we fix f_∞. The map

$$\theta \mapsto \theta_{A'}(1) \tag{3.9}$$

defines a linear functional on $\mathbb{C}f_\infty \otimes (\otimes_{p\neq\infty}\Pi_p)$ which by the local uniqueness (Cor. 2.3) must be a product of local functionals, i.e. evaluations at A'. Therefore, there exists a non-zero constant c such that

$$\theta_{A'}(1) = c \prod_{p\neq\infty} f_p(A'). \tag{3.10}$$

We have to say a word or two about the above "infinite" product. Let $f_p^0 \in \Pi_p$ be the spherical vector. Note that $f_p^0(A') \neq 0$. This follows from the Kim's formula, which says that $\theta_{A'}(1) \neq 0$ if $f_p = f_p^0$ for all p, and the coefficients of A' are in $\mathcal{R}$. We normalize f_p^0 so that $f_p^0(A') = 1$, hence the above product is always just a finite product.

Let $g = g_\infty g_f$ be an element in $G(\mathbb{A})$, where g_∞ is in $G(\mathbb{R})$ and g_f in $G(\hat{\mathbb{Q}})$. Arguing as before,

$$\theta_{A'}(g) = c(g_\infty) \prod_{p\neq\infty} f_p(g_p^{-1}(A')), \tag{3.11}$$

where $c(g_\infty)$ is a constant depending on g_∞. It follows that $\theta_{A'}(g)$ is $C(\hat{\mathbb{Q}})$-left invariant. Since it is also $C(\mathbb{Q})$-invariant, and $C(\mathbb{Q})C(\hat{\mathbb{Q}})$ is dense in $C(\mathbb{A})$ by the weak approximation [Kn], it follows that $\theta_{A'}(g)$ is $C(\mathbb{A})$-left invariant. Hence, from (3.8)

$$\beta_B(1) = \int_{C(\mathbb{A})\backslash G(\mathbb{A})} \theta_{A'}(g) P_\alpha^C(g)dg, \tag{3.12}$$

where

$$P_\alpha^C(g) = \int_{C(\mathbb{Q})\backslash C(\mathbb{A})} \alpha(vg)dv. \tag{3.13}$$

4. Non-vanishing and cuspidality.

We first give a criterion for non-vanishing of non-degenerate Fourier coefficients. Every 3×3 symmetric is $L(\mathbb{Q})$-conjugated to a diagonal matrix. Let

$$B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \quad \text{in} \quad U(\mathbb{Q}) \tag{4.1}$$

be of rank three, i.e. $abc \neq 0$. Let D be a quaternion algebra spanned by 1 and traceless x, y, and z, subject to the following relations:

$$\begin{aligned} x^2 &= -bc, & y^2 &= -ca, & z^2 &= -ab \\ yz &= -ax, & zx &= -by, & xy &= -cz. \end{aligned} \tag{4.2}$$

Replacing B with another diagonal matrix in the same conjugacy class amounts to rescaling and permuting the generators x, y and z. In particular, generic $L(\mathbb{Q})$-orbits are parametrized by quaternion algebras.

Henceforth, we assume that D is contained in $\mathbb{O}$ (this is possible iff B is a definite matrix). In particular, x, y and z are traceless octonions. Then

$$A' = -\begin{pmatrix} a & z & \bar{y} \\ \bar{z} & b & x \\ y & \bar{x} & c \end{pmatrix} \tag{4.3}$$

is a rank one matrix in $J_{\mathbb{O}}$ such that $\psi_{A'}|_U = \psi_B^{-1}$. Again, there is no harm in assuming that the entries of A' are in $\mathcal{R}$, the maximal order in $\mathbb{O}$.

Let π be an automorphic representation of G. Fix a form α in π. Let S be a set of finite places such that if $p \notin S$, then

$$\begin{cases} \alpha \ \text{ is } \ G(\mathbb{Z}_p) - \text{invariant} \\ D_p \text{ is the algebra of } \ 2\times 2 \ \text{ matrices} \\ a, b, c \ \text{ are not divisible by } \ p. \end{cases} \tag{4.4}$$

Proposition 4.5. *Let C be the centralizer of D in G. Then the following are equivalent:*

(1) *$P_\alpha^C \neq 0$, and*
(2) *$\beta_B(1) \neq 0$ for a choice of $\theta = \theta(\otimes_p f_p)$, with $f_p = f_p^0$ for all $p \notin S$.*

In particular, if $P_\alpha^C \neq 0$, then β is unramified for all $p \notin S$.

Proof. Obviously, if $P_\alpha^C = 0$ then $\beta_B(1) = 0$ for any choice of θ, by (3.12). Assume that $P_\alpha^C \neq 0$, and let $\theta^0 = \theta(\otimes_p f_p)$, such that

(1) $f_\infty = f_\infty^0$, a vector in the minimal "K-type" of Π_∞^+.
(2) If $p \notin S$, then $f_p = f_p^0$ ($H(\mathbb{Z}_p)$-invariant vector).

Lemma 4.6. *Let $\mathbb{Q}_S = \prod_{p\in S} \mathbb{Q}_p$. If $g \in G(\mathbb{A})$, we write $g = g_\infty g_S g^S$ where $g_\infty \in G(\mathbb{R})$, $g_S \in G(\mathbb{Q}_S)$ and $g^S \in \prod_{p\notin S} G(\mathbb{Q}_p)$. Then there exists a non-zero constant c, such that for every $g \in G(\mathbb{A})$*

$$\theta_{A'}^0(g) = c\ f_S(g_S^{-1}(A')) \prod_{p\notin S} \chi_p(g_p)$$

where $f_S = \otimes_{p\in S} f_p$, and χ_p is the characteristic function of $C(\mathbb{Z}_p)\backslash G(\mathbb{Z}_p)$ where $C(\mathbb{Z}_p) = C(\mathbb{Q}_p) \cap G(\mathbb{Z}_p)$. Note that, since D_p is split, $C(\mathbb{Q}_p)$ is isomorphic to $SL_2(\mathbb{Q}_p)$.

Proof. By (3.11) we have

$$\theta_{A'}^0(g) = c \prod_{p\neq\infty} f_p(g_p^{-1}(A')).$$

In this case, however, the constant c does not depend on g_∞ because f_∞^0 is $G(\mathbb{R})$-invariant.

Let $g_p \in G(\mathbb{Q}_p)$ such that $f_p^0(g_p^{-1}(A')) \neq 0$. Let x', y', z' be the off-diagonal terms of $g_p^{-1}A'$. Since f_p^0 is $N(\mathbb{Z}_p)$-invariant, it follows from Prop. 2.3 (1) that f_p^0 is supported in $J_{\mathcal{R}_p} = J_{\mathcal{R}} \otimes \mathbb{Z}_p$, hence $x', y', z' \in \mathcal{R}_p$.

Consider $\mathcal{R}/p\mathcal{R}$, the octonion algebra over $\mathbb{Z}/p\mathbb{Z}$. The projections of (x, y, z), the off-diagonal terms of A', and (x', y', z') onto $\mathcal{R}/p\mathcal{R}$ are $G(p)$-conjugated by the theorem of Jacobson. It follows from Hensel's lemma that (x, y, z) and (x', y', z') are $G(\mathbb{Z}_p)$-conjugated. Therefore, the function

$$g_p \mapsto f_p^0(g_p^{-1}(A'))$$

is supported in $C(\mathbb{Z}_p)\backslash G(\mathbb{Z}_p) \subset C(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)$. Since f_p^0 is $G(\mathbb{Z}_p)$-invariant,

$$f_p^0(g_p^{-1}(A')) = f_p^0(A') = 1,$$

for g_p in $G(\mathbb{Z}_p)$. The lemma follows.

Let Y be a finite collection of elements n_i in $N(\mathbb{R})$, together with a collection of numbers c_i. Define

$$\theta^Y(g) = \sum_i c_i \theta^0(gn_i).$$

Obviously,

$$\theta^Y_{A'}(g) = \sum_i c_i \psi_{A'}(g_\infty n_i g_\infty^{-1})\theta^0_{A'}(g^\infty).$$

Furthermore, recall that $\psi_x(y) = \psi(\mathrm{Tr}(x \circ y)) = \psi_y(x)$, so

$$\theta^Y_{A'}(g) = \sum_i c_i \psi_{n_i}(g_\infty^{-1}(A'))\theta^0_{A'}(g^\infty),$$

and

$$\theta^Y_{A'}(g) = c\sum_i c_i \psi_{n_i}(g_\infty^{-1}(A')) f_S(g_S^{-1}(A')) \prod_{p\notin S} \chi_p(g_p)$$

by Lemma 4.6.

In general, θ^Y is not a "K-finite" vector. Still, it is a smooth function on $H(\mathbb{Q})\backslash H(\mathbb{A})$, and we can use it to define a function β on G' using the formula (0.2). By (3.12)

$$\beta_B(1) = \int_{C(\mathbb{A})\backslash G(\mathbb{A})} \theta^Y_{A'}(g) P^C_\alpha(g) dg.$$

Substituting the expresion for $\theta^Y_{A'}$, and using that P^C_α is $G(\mathbb{Z}_p)$-invariant for every $p \notin S$,

$$\beta_B(1) = c\int_{C(\mathbb{R}\times\mathbb{Q}_S)\backslash G(\mathbb{R}\times\mathbb{Q}_S)} \sum_i c_i \psi_{n_i}(g_\infty^{-1}(A')) f_S(g_S^{-1}(A')) P_\alpha(g_\infty g_S) dg_\infty dg_S \prod_{p\notin S} \mu_p,$$

where

$$\mu_p = \int_{SL_2(\mathbb{Q}_p)\backslash G_2(\mathbb{Q}_p)} \chi_p dg_p = \frac{\#G_2(p)}{\#SL_2(p)p^{11}} = (1-p^{-6}).$$

The infinite product is $\zeta_S^{-1}(6)$, and therefore non-zero. Let p be a prime in S. By Prop. 2.3, f_p can be any compactly supported, locally constant function on the set of rank one matrices in $J_{\mathbb{O}_p}$, hence the integral over $C(\mathbb{Q}_S)\backslash G(\mathbb{Q}_S)$ will be non-zero for a suitable choice of f_S. Since $S(\mathbb{R})\backslash G(\mathbb{R})$, the $G(\mathbb{R})$-orbit of A', is compact, by the Stone-Weierstrass theorem $\{\sum_i c_i\psi_{n_i}\}$ is a dense family of continuous functions. Hence the integral over $C(\mathbb{R})\backslash G(\mathbb{R})$ can be arranged to be non-zero, too.

Since

$$\theta^Y = \sum_{n\in\mathbb{Z}} \theta^Y_n$$

in locally uniform convergence of smooth functions on $H(\mathbb{Q})\backslash H(\mathbb{A})$ where θ_n^Y belongs to the n-th "K-type" of Π_∞, at least one of these summands has to produce a non-zero β, with a non-trivial Fourier coefficient at B. The proposition is proved.

Next, let

$$B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{in} \quad U(\mathbb{Q}) \tag{4.7}$$

be of rank two, i.e. $ab \neq 0$. Let $K = \mathbb{Q}(z)$ be a quaternion subalgebra of $\mathbb{O}$, such that z is traceless and $z^2 = -ab$.

Proposition 4.8. *Let C be the centralizer of K in G. Then the following are equivalent:*

(1) *$P_\alpha^C \neq 0$, and*
(2) *$\beta_B(1) \neq 0$ for a choice of θ.*

Proof. Analogous to the proof of Prop. 4.5. One also needs non-vanishing of a special value of the zeta-function ζ_χ, where χ is the character of $\mathbb{A}^\times$ corresponding to K via the class-field theory. This is provided in [EG].

Corollary 4.9. *Let $\pi = \hat{\otimes}_v \pi_v \subset L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ be an irreducible automorphic representation. Assume that:*

(1) *$\pi_\infty \cong W(k_1, k_2) \otimes \mathbb{C}$, with $k_2 \neq 0$, or there exists a finite prime p such that π_p has a Whittaker functional.*
(2) *There exists $\alpha \in \pi$ and D, a quaternion subalgebra of $\mathbb{O}$, such that $P_\alpha^C \neq 0$.*

Then $\Theta(\pi)$ contains a non-trivial cusp form on G', unramified at all places $p \notin S$, where S is given by (4.4).

Proof. The second condition and Prop. 4.5 imply that β is non-zero, and unramified outside S.

To show cuspidality we need to show that $\beta_B = 0$ for any B of rank less then or equal to 2. Assume that the rank is 0 or 1. The first condition implies that the π is not isomorphic to the trivial form on G. Hence the period over G is 0. This implies that β_B is also 0, by (3.6) and (3.12).

To show vanishing for B of rank 2, we have to show that SU_3-periods vanish, by Prop 4.8. Assume not. Then each local component of π has a non-trivial SU_3-invaraint functional. This means that π_∞ has an $SU_3(\mathbb{R})$-fixed vector, and by the Frobenius reciprocity, $(\pi_p)^*$, the contragredient of π_p, is a quotient of $C_c^\infty(G(\mathbb{Q}_p)/SU_3^K(\mathbb{Q}_p))$. By the branching law $G_2 \downarrow A_2$ [Sa2], the representation $W(k_1, k_2)$ has an $SU_3(\mathbb{R})$-fixed vector iff $k_2 = 0$. Also, a generic p-adic representation can not be a quotient of $C_c^\infty(G(\mathbb{Q}_p)/SU_3^K(\mathbb{Q}_p))$, in view of the following:

Lemma 4.10. *Let F be a p-adic field, and $K \subset \mathbb{O} \otimes F$ a quadratic subalgebra. Let $SU(K^\perp) = SU_3^K$ be the centralizer of K in $G = Aut(\mathbb{O} \otimes F)$. Then the G-module*

$$C_c^\infty(G/SU_3^K)$$

does not have a Whittaker functional.

Proof. Let $B \subset G$ be a Borel subgroup. The space $C_c^\infty(G/SU_3^K)$ has a B-invaraint filtration with succesive quotients $C_c^\infty(\mathcal{O})$, where $\mathcal{O}$ runs over the finite set of B-orbits on G/SU_3^K. We need to show that each of the subquotients does not have a Whittaker functional, so our task is to compute B-orbits on G/SU_3^K, or equivalently, SU_3^K-orbits on G/B.

Let $M_2(F)$ be the algebra of 2×2-matrices over F, with involution

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = -\begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

Then $\mathbb{O} \otimes F$ is isomorphic to the algebra $M_2(F) \oplus M_2(F)$ with multiplication

$$(a, b)(a', b') = (aa' + \bar{b}'b, b'a + b\bar{a}').$$

Using this realization of $\mathbb{O} \otimes F$, it is easy to compute SU_3^K-orbits on G/B. Assume, for example, that $K \cong F \oplus F$, the subalgebra of diagonal matrices in $M_2(F)$. Then $SU_3^K = SL_3(F)$. Let V be the space of traceless elements in $\mathbb{O} \otimes F$, then

$$V = V_3 \oplus V_3^* \oplus K^0.$$

under the action of $SL_3(F)$. Here V_3 is the standard representation of $SL_3(F)$, and $K^0 = K \cap V$. The set G/B can be identified with partial flags

$$V_1 \subset V_2 \subset V$$

consisting of 1 and 2-dimensional spaces with trivial octonion multiplication. We note that V_2 is always contained in the 3-dimensional space

$$V_1\Delta = \{x \in V \mid x \cdot V_1 = 0\}.$$

We have three diferent cases.

(1) The group $SL_3(F)$ acts transitively on partial flags $V_1 \subset V_2$ such that $V_1 \subset V_3$. Indeed, $SL_3(F)$ acts transtively on lines in V_3, and the stabilizer of a line V_1 is a maximal parabolic subgroup. Its Levi factor $GL_2(F)$ acts transitively on on the 2-dimensional space

$$V_1\Delta/V_1.$$

By the same argument, $SL_3(F)$ acts transitively on partial flags such that $V_1 \subset V_3^*$.

(2) The group $SL_3(F)$ acts with two orbits on partial flags $V_1 \subset V_2$ such that $V_1 \subset V_3 \oplus V_3^*$, but not contained in V_3 or V_3^*. Indeed, in this case

$$V_1 = F(x + y) \subset V_3 \oplus V_3^*$$

where $x^2 = xy = y^2 = 0$, and we have two orbits, depending whether

$$V_2 = Fx + Fy = V_1\Delta \cap (V_3 \oplus V_3^*)$$

or not.

(3) The group $SL_3(F)$ acts transitively on partial flags $V_1 \subset V_2$ such that V_1 is not contained in $V_3 \oplus V_3^*$. Indeed, $SL_3(F)$ acts transitively on such V_1, and the stabilizer is $SL_2(F)$, which acts transitively on on on the 2-dimensional space

$$V_1\Delta/V_1.$$

Next, let B be a Borel subgroup fixing a partial flag $V_1 \subset V_2$. Its short simple root group acts by $x \mapsto x + b$ on V_2/V_1 and and its long simple root group by $x \mapsto x + b$ on $V_1\Delta/V_2$. Now, it is a simple matter to check that in each case $B' = B \cap SL_3(F)$ will contain 1-dimensional subgroup acting by $x \mapsto x + b$ on V_2/V_1 or $V_1\Delta/V_2$. Hence, $C_c^\infty(B/B')$ does not have a Whittaker functional, and this implies the lemma.

5. Examples.

We prove the non-vanishing of the Θ-lifts of the automorphic representations π constructed in Prop. 7.7 and Prop. 7.12 of Chapter I.

Let D be a definite quaternion algebra over $\mathbb{Q}$. Then the $\mathbb{Q}$ algebra $D \oplus Dv$ with multiplication

$$(a + bv)(a' + b'v) = (aa' - \bar{b}'b) + (b'a + b\bar{a}')v \tag{5.1}$$

is a definite octonion algebra, hence it is isomorphic to the Cayley's octonion algebra $\mathbb{O}$. Thus D embeds in $\mathbb{O} = D \oplus D^\perp$, and if we fix an element v in $D^\perp$ with $v^2 = -1$, we get an isomorphism of the subgroup C of $G = Aut(\mathbb{O})$ fixing D with $D^\times_{\mathbb{N}=1}$, the group of norm-one elements in D:

$$d(a + bv) = a + (db)v, \qquad d\bar{d} = 1. \tag{5.2}$$

The subgroup $SO_4(D) = \{(d, d') \in D^\times \times D^\times \mid \mathbb{N}d = \mathbb{N}d'\}/\Delta\mathbb{Q}^\times$ of G acts on $\mathbb{O}$ as follows:

$$(d, d')(a + bv) = d's(d')^{-1} + (db(d')^{-1})v. \tag{5.3}$$

In particular, it stabilizes $D \subset \mathbb{O}$, and contains C as the subgroup $\{(d, 1)\}$. The group $SO_4(D)$ is the centralizer in G of the involution $i(a + bv) = a - bv$ of $\mathbb{O}$.

Now suppose D is the algebra of Hamilton's quaternions, ramified at 2 and ∞. We will show the following.

Proposition 5.4. *There is an embedding of D into $\mathbb{O}$ such that*

$$C(\mathbb{A}) = C(\mathbb{Q}) \times C(\mathbb{R}) \times (C(\hat{\mathbb{Q}}) \cap K(2))$$

where $K(2)$ is the subgroup of elements in $G(\hat{\mathbb{Z}})$ congruent to $1 \pmod 2$.

Proof. First some facts about Hamilton quaternions, independent of octonions. It has the maximal order R defined by Hurwitz:

$$R = \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}k \oplus \mathbb{Z}(\frac{1 + i + j + k}{2})$$

with unit group

$$R^\times =< \pm 1, \pm i, \pm j, \pm k, \frac{\pm 1 \pm i \pm j \pm k}{2} >$$

of order 24. This order gives a model for C over $\mathbb{Z}$, with $C(\mathbb{Z}) = R^\times$ and bad reduction at 2. For p odd, $C(\mathbb{Z}_p)$ is isomorphic to $SL_2(\mathbb{Z}_p)$.

For $p = 2$, let $R_2 = R \otimes \mathbb{Z}_2$ be a maximal order in $D_2 = D \otimes \mathbb{Z}_2$. Then

$$C(\mathbb{Z}_2) = (R_2^\times)_{\mathbb{N}=1} = C(\mathbb{Q}_2) = (D_2^\times)_{\mathbb{N}=1}.$$

The element $\varpi = (1+i)$ is a uniformizing element in R_2, and we have a filtration by normal subgroups:

$$(R_2^\times)_{\mathbb{N}=1} \supset_3 (1+\varpi R_2)_{\mathbb{N}=1} \supset_{(2,2)} (1+\varpi^2 R_2)_{\mathbb{N}=1} \supset_2 (1+\varpi^3 R_2)_{\mathbb{N}=1} \supset_{(2,2)} \cdots$$

where $\supset_{(2,2)}$, for example, denotes that the quotient is isomorphic to the Klein four-group.

We have

$$R^\times \cap (1+\varpi R_2)_{\mathbb{N}=1} =< \pm 1, \pm i, \pm j, \pm k >$$
$$R^\times \cap (1+\varpi^2 R_2)_{\mathbb{N}=1} =< \pm 1 >$$
$$R^\times \cap (1+\varpi^3 R_2)_{\mathbb{N}=1} = 1$$

Since the global units lie in distinct cosets of $(R_2^\times)_{\mathbb{N}=1}/(1+\varpi^3 R_2)_{\mathbb{N}=1}$, and the latter group has order 24, we obtain a direct product decomposition:

$$C(\mathbb{Z}_2) = R^\times \times (1+\varpi^3 R_2)_{\mathbb{N}=1}.$$

The mass formula (of Eichler) gives

$$C(\mathbb{A}) = C(\mathbb{Q})(C(\mathbb{R}) \times C(\hat{\mathbb{Z}}))$$

with intersection $C(\mathbb{Z})$. Hence we have a direct product

$$C(\mathbb{A}) = C(\mathbb{Q}) \times ((C(\mathbb{R}) \times (1+\varpi^3 R_2)_{\mathbb{N}=1} \times \prod_{p \neq 2} C(\mathbb{Z}_p)).$$

To finish the proof of the proposition, we need to find an embedding of D in $\mathbb{O}$ such that

$$C(\hat{\mathbb{Q}}) \cap K(2) = (1+\varpi^3 R_2)_{\mathbb{N}=1} \times \prod_{p \neq 2} C(\mathbb{Z}_p)).$$

Let $\mathcal{R}$ be the ring of integral octonions constructed by Coxeter, so $K(2)$ is the subgroup of $Aut(\mathcal{R} \otimes \hat{\mathbb{Z}})$ which acts trivially on $\mathcal{R}/2\mathcal{R}$. We embed the Hurwitz R into the Coxeter $\mathcal{R}$ via $i \mapsto e_1$, $j \mapsto e_2$, $k \mapsto e_4$. If we take $v = e_3$ then

$$\mathbb{O} = D \oplus D^\perp \text{ and}$$
$$\mathcal{R} \supset_{(2,2)} R \oplus R^\perp.$$

The general element of $\mathcal{R}$ has the form

$$\frac{1}{2}(a + be_3)$$

with a, b in ϖR, $\varpi = (1 + i) = (1 + e_1)$. If we write

$$a = \varpi\alpha$$
$$b = \varpi\beta$$

then for the above to lie in $\mathbb{O}$, we need the congruence

$$\alpha \equiv \bar{\beta} \pmod{\varpi R}.$$

It is now a simple matter to check that $(1 + \varpi R_2)_{\mathbb{N}=1} \times \prod_{p \neq 2} C(\mathbb{Z}_p)$ is the subgroup of $G(\hat{\mathbb{Z}})$ fixing D. Similarly, $(1 + \varpi^3 R_2)_{\mathbb{N}=1} \times \prod_{p \neq 2} C(\mathbb{Z}_p)$ is the subgroup fixing D and acting trivially on $\mathcal{R}/2\mathcal{R}$. The propositon is proved.

Proposition 5.5. *Let π be the automorphic representation of G given in Prop. 7.7 in Chapter I. (π_∞ is isomorphic to the irreducible 64-dimensional representation of $G(\mathbb{R})$ of highest weight ρ, π_2 is the Steinberg representation of $G(\mathbb{Q}_2)$ and π_p is unramified for all $p \neq 2$). Then π lifts to a cusp form π' on G', with:*

1. *π'_∞ is a holomorphic discrete series representation with infinitesimal character $(6, 4, 2)$.*
2. *π'_p, for $p \neq 2$ is unramified with Satake parameter in $G_2(\mathbb{C}) \subset Spin_7(\mathbb{C})$.*
3. *π'_2 is the Steinberg representation.*

Proof. We will show that the period over C given by Prop. 5.4 is non-zero for a $K(2)$-fixed function in π. Using the direct product decomposition

$$G(\mathbb{Q})\backslash G(\mathbb{A}) = G(\mathbb{R}) \times K(2),$$

one identifies the space of $K(2)$-fixed vectors in π with the space of matrix coefficients of W on $G(\mathbb{R})$. By Prop. 5.4 we have to show that there exists a matrix coefficient of W such that its integral over $C(\mathbb{R})$ is not zero.

Every finite-dimensional representation of $G(\mathbb{R})$ has a $C(\mathbb{R}) \cong SU_2(\mathbb{R})$-fixed vector. Indeed, we have a chain of groups

$$SU_2 \subset SU_3 \subset G$$

where SU_3 is the stabilizer in G of a quadratic subalgebra of D, for example $\mathbb{Q}(i)$. Since it is true that every representation of $SU_3(\mathbb{R})$ has $SU_2(\mathbb{R})$-fixed vectors, the same is true for $G(\mathbb{R})$.

Let w be a $C(\mathbb{R})$-fixed vector in W. Obviously, the matrix coefficient $< w, gw >$ has non-vanishing integral over $C(\mathbb{R})$. Hence the Fourier coefficient at

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is not zero. The proposition follows from Cor. 4.9. The information about the local components of π' follows from Cor. 3.9 of Chapter III and Prop. 3.1 of Chapter IV.

We now assume that D is the definite quaternion algebra ramified at 5 and ∞.

Proposition 5.6. *There is an embedding of D into $\mathbb{O}$ and a Borel subgroup $B(5) \subset G(5)$ such that*

$$C(\mathbb{A}) = C(\mathbb{Q}) \times C(\mathbb{R}) \times (C(\hat{\mathbb{Q}}) \cap K_0(5))$$

where $K_0(5)$ is the subgroup of elements in $G(\hat{\mathbb{Z}})$ reducing to $B(5) \pmod 2$.

Proof. Let R be a maximal order in D, which is unique up to $D^\times$-conjugacy. We have $R^\times = \mu_6$. The order R gives a model of C over $\mathbb{Z}$, with bad reduction at 5. The units lie in the 6 distinct cosets for the subgroup $(1 + \varpi R)$, where $\mathrm{Tr}(\varpi) = 0$ and $\mathbb{N}(\varpi) = 5$. Hence we obtain a direct product

$$C(\mathbb{A}) = C(\mathbb{Q}) \times (C(\mathbb{R}) \times (1 + \varpi R_5)_{\mathbb{N}=1} \times \prod_{p \neq 5} C(\mathbb{Z}_p)).$$

Let $\mathcal{R}$ be the Coxeter's order in $\mathbb{O}$. There is an embedding $R \to \mathcal{R}$, which is unique up to conjugacy by $Aut(\mathcal{R}) = G(\mathbb{Z})$. Let $R^\perp$ be the orthogonal complement of R in $\mathcal{R}$; again we find that

$$R \oplus R^\perp \subset_{(5,5)} \mathcal{R},$$

and that the subgroup of $G(\hat{\mathbb{Z}})$ fixing $R \otimes \hat{\mathbb{Z}}$ is precisely $(1 + \varpi R_5)_{\mathbb{N}=1} \times \prod_{p \neq 5} C(\mathbb{Z}_p)$.

Since $(1 + \varpi R_5)_{\mathbb{N}=1}$ is a 5-group, its reduction in $G(5)$ is contained in the unipotent radical of some Borel subgroup $B(5)$. Hence $C(\hat{\mathbb{Q}}) \cap K_0(5) = (1 + \varpi R_5)_{\mathbb{N}=1} \times \prod_{p \neq 5} C(\mathbb{Z}_p)$, which completes the proof.

In Section 7 of Chapter I, we showed that the space

$$\mathcal{S} = G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\mathbb{R}) \times K_0(5) \tag{5.7}$$

had 7 elements. We also showed that there was a non-zero function $f : \mathcal{S} \to \mathbb{Q}$, unique up to scaling, in the Steinberg subspace: $T_i f = -f$ for the three generators T_0, T_1, T_2 of the Iwahori-Hecke algebra at 5.

The function f determines a 1-dimensional subspace $< F >$ in the automorphic representation π of Prop. 7.12. The period of F over $C(\mathbb{Q}) \backslash C(\mathbb{A})$ is non-zero if and only if $f(e) \neq 0$, where e is the identity double coset of $\mathcal{S}$ (e is the image of $C(\mathbb{A})$, by the Prop. 5.6). A computer calculation, performed by D. Pollack and J. Lansky, showed that $f(e) \neq 0$. Hence we obtain

Proposition 5.8. *Let π be the automorphic representation of G given in Prop. 7.12 in Chapter I. ($\pi_\infty \cong 1$, π_5 is the Steinberg representation of $G(\mathbb{Q}_5)$ and π_p is unramified for all $p \neq 5$). Then π lifts to a cusp form π' on G', with*

1. *π'_∞ is a holomorphic discrete series representation with infinitesimal character $(3, 2, 1)$.*
2. *π'_p, for $p \neq 5$ is unramified with Satake parameter in $G_2(\mathbb{C}) \subset Spin_7(\mathbb{C})$.*
3. *π'_5 is the Steinberg representation.*

The automorphic representation π' corresponds to a classical holomorphic form of weight 4 and level 5.

VI Periods

As we indicated in the introduction, the complement of the motive M in M' should be given by the classes of Hilbert modular 3-folds. If so, the forms on G' coming from G should be characterized as those having non-zero periods over the cycles given by the Hilbert modular 3-folds. We give an affirmative answer to this question in the local setting.

1. A see-saw dual pair in E_n.

We let F be a p-adic field, E a separable cubic extension of F, and H a split adjoint group over F of type E_6, E_7 or E_8. Let $H' = H \rtimes \Gamma$ be the semi-direct product of H with its group of outer automorphisms Γ. Note that $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$ for E_6, and is trivial otherwise. The see-saw pair in H' is

$$\begin{array}{cc} D_4^E & G' \\ & \\ G_2 & C'_E \end{array} \tag{1.1}$$

where all groups are quasi-split over F, D_4^E is a twist of the simply connected group of type D_4 by the homomorphism $Gal(\bar{F}/F) \to S_3$ (the outer automorphisms of D_4) corresponding to the cubic etale algebra E, and the subgroups G' and C'_E are tabulated below.

(1.2)

Type of H	G'	C'_E
E_6	$PGL_3 \rtimes \Gamma$	$(R_{E/F}\mathbb{G}_{\mathrm{m}}/\mathbb{G}_{\mathrm{m}}) \rtimes \Gamma$
E_7	$PGSp_6$	$R_{E/F}SL_2/\mu_2$
E_8	F_4	D_4^E

The center of G' is trivial, and the pair $G_2 \times G'$ is a maximal subgroup of H', whereas the center of C'_E is

$$K_4^E = (R_{E/F}\mu_2)/\mu_2 = D_4^E \cap C'_E, \tag{1.3}$$

a twist of the Klein group K_4, and the subgroup $D_4^E \times C'_E/\Delta K_4^E$ is not maximal, for it is contained in the centralizer in H' of any non-trivial element of K_4^E.

In the following section, we will characterize the irreducible representations π' of $G'(F)$ which appear as quotients of the minimal representation of $H'(F)$; these are the representations with a non-zero, C'_E-invariant linear functional, for some E.

Let $\mathrm{Tr} : E \to F$ be the trace form. Since E is separable, the pairing $E \times E \to F$ given by $(v, w) = \mathrm{Tr}(vw)$ is non-degenerate. Let $\{v_1, v_2, v_3\}$ be a basis for E over F, and consider the F-linear embedding $E \to J(F)$

$$\alpha \mapsto A_\alpha = (\mathrm{Tr}(\alpha \cdot v_i v_j)), \tag{1.4}$$

where $J(F)$ is the 6-dimensional space of symmetric 3×3 matrices over F. The matrix

$$e = A_1 = (\mathrm{Tr}(v_i v_j)) \tag{1.5}$$

has $\det(e) \neq 0$, by the non-degeneracy of the pairing.

Now let $D \subseteq \mathbb{O} \otimes F$ be an F-subalgebra, on which the norm form is non-degenerate and represents 0. Let $J(D)$ be the F-vector space of 3×3 Hermitian symmetric matrices over D. We then have a chain of subspaces

$$Fe \subset E \subset J(F) \subset J(D) \subseteq J_{\mathbb{O}} \otimes F. \tag{1.6}$$

Let L' be the algebraic group of all invertible linear maps on $J(D)$ which preserve the determinant form $\det : J(D) \to F$. These groups are tabulated below [EG]:

(1.7)

D	$J(D)$	L'
$F + F$	$M_3(F)$	$(SL_3 \times SL_3/\Delta\mu_3) \rtimes \Gamma$
$M_2(F)$	$\wedge^2 F^6$	SL_6/μ_2
$\mathbb{O} \otimes F$	$J_{\mathbb{O}} \otimes F$	E_6

where we have identified $J(D)$ with a more familiar L'-module. Note that the center of L' is μ_3.

Let $\mathfrak{l}$ be the Lie algebra of L', and V_3 the standard 3-dimensional representation of $SL(3)$. In [Sa2] it is shown that the direct sum

$$\mathfrak{h} = (sl(3) \oplus \mathfrak{l}) \ \oplus \ (V_3 \otimes J(D)) \ \oplus \ (V_3 \otimes J(D))^* \tag{1.8}$$

has the structure of a simple, split Lie algebra of type E_n, $(n = 6, 7, 8)$, with a $\mathbb{Z}/3\mathbb{Z}$-gradation given by the action of the center of the subgroup

$$SL_3 \times L'/\Delta\mu_3 \tag{1.9}$$

of H'.

It can be checked that the centralizer of e in $L' \subset H'$ is G', and the centralizer of G' in H' has Lie algebra

$$\mathfrak{g}_2 = sl(3) \ \oplus \ (V_3 \otimes Fe) \ \oplus \ (V_3 \otimes Fe)^* \tag{1.10}$$

of type G_2.

Let $E_0 \subset E$ be the kernel of the trace map $\mathrm{Tr} : E \to F$. The centralizer of $E \subset J(D)$ in L' is C'_E, and the centralizer of C'_E in H' has Lie algebra

$$\mathfrak{d}_4^E = (sl(3) \oplus E_0) \ \oplus \ (V_3 \otimes E) \ \oplus \ (V_3 \otimes E)^* \tag{1.11}$$

of type D_4, split by E [Ru]. This gives a construction of the see-saw pair.

2. Periods.

The minimal representation Π, defined in Chapter IV, can be easily extended to H'; more precisely, the polarization used in [KS; pg 212] is Γ-invariant, and by taking the structural coefficients to be Γ-invariant, the representation extends to H' in an obvious way.

Consider the decomposition (1.8) of $\mathfrak{h}$. Let $\mathfrak{t}$ be the Cartan subalgebra of $sl(3)$, consisting of diagonal traceless matrices. Let

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathfrak{t} \subset sl(3) \subset \mathfrak{h}. \tag{2.1}$$

Define

$$\mathfrak{h}(k) = \{x \in \mathfrak{h} \mid [h, x] = kx\}. \tag{2.2}$$

Since the eigenvalues of h on the standard 3-dimensional representation V_3 of $sl(3)$ are $-1, 0, 1$, the decomposition (1.8) implies that $\mathfrak{h}(k) \neq 0$ for $k = -2, -1, 0, 1, 2$. For example,

$$\mathfrak{h}(0) = J(D)^* \oplus \mathfrak{t} \oplus \mathfrak{h} \oplus J(D). \tag{2.3}$$

Write $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{n}$ where

$$\begin{cases} \mathfrak{m} = \mathfrak{h}(0) \\ \mathfrak{n} = \mathfrak{h}(1) \oplus \mathfrak{h}(2) \\ \bar{\mathfrak{n}} = \mathfrak{h}(-1) \oplus \mathfrak{h}(-2). \end{cases} \tag{2.4}$$

Then $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ is a maximal parabolic subalgebra. The nilpotent radical $\mathfrak{n}$ is a two-step nilpotent (Heisenberg) Lie algebra, with the center $\mathfrak{z} = \mathfrak{h}(2)$. Analogous statements are true for the opposite algebra $\bar{\mathfrak{p}} = \mathfrak{m} \oplus \bar{\mathfrak{n}}$. Under the action of $\mathfrak{t} \oplus \mathfrak{l} \subset \mathfrak{m}$, we have direct sum decompositions

$$\begin{cases} \mathfrak{n}/\mathfrak{z} \cong \mathfrak{h}(1) = F \oplus J(D) \oplus J(D)^* \oplus F^* \\ \bar{\mathfrak{n}}/\bar{\mathfrak{z}} \cong \mathfrak{h}(-1) = F^* \oplus J(D)^* \oplus J(D) \oplus F. \end{cases} \tag{2.5}$$

Let $P' = M'N$ be the "Heisenberg" maximal parabolic subgroup of H', with Lie algebra $\mathfrak{p}$. Let Z be the center of N. The quotient of N by Z is commutative and $N/Z \cong \mathfrak{n}/\mathfrak{z}$ as M-modules. Let $\bar{P}' = M'\bar{N}$ be the parabolic subgroup opposite to P', and $\bar{Z}$ be the center of $\bar{N}$. The Killing form on $\mathfrak{h}$, the Lie algebra of H, defines a non-degenerate pairing $<,>$ between N/Z and $\bar{N}/\bar{Z}$. In terms of the identifications (2.5) this pairing is

$$< (x, u, u^*, x^*), (y^*, v^*, v, y) >= xy^* + < u, v^* > + < v, u^* > + yx^*. \tag{2.6}$$

Let Ω be the smallest non-trivial M'-orbit in N/Z. It is simply the orbit of a highest weight vector.

Proposition 2.7. *($p \neq 2$ if $G = E_8$) Let Π be the minimal representation of H'. Let $\bar{Z}$ be the center of $\bar{N}$ as above. Let $\Pi_{\bar{Z}}$ and $\Pi_{\bar{N}}$ be the maximal $\bar{Z}$-invariant and $\bar{N}$-invariant quotients of Π. Then*

$$0 \to C_c^\infty(\Omega) \to \Pi_{\bar{Z}} \to \Pi_{\bar{N}} \to 0$$

where $C_c^\infty(\Omega)$ denotes the space of locally constant, compactly supported functions on Ω. The action of $\bar{P}'$ on $\mathcal{C}_c^\infty(\Omega)$ is given by

$$\Pi(\bar{n})f(x) = \psi(< x, \bar{n} >)f(x), \qquad \bar{n} \in \bar{N}$$
$$\Pi(m)f(x) = |\det(m)|^{\frac{s}{d}} f(m^{-1}xm), \qquad m \in M',$$

where ψ is a non-trivial addititive character of F, det *is the determinant of the representation of M' on $\bar{N}/\bar{Z}$, d is the dimension of N/Z. The values are given by the following table.*

G	s	d
E_6	4	20
E_7	6	32
E_8	10	56

Proof. This is a simplified version of [MS; Thm 6.1].

We are now ready to prove the main result of this section. Let e be an element in $J(D)$, such that $(e, e, e) = 6$, and let $G' \subset L' \subset H'$ be the centralizer of e in L'.

Proposition 2.8. *Let π' be a representation of G' with a non-zero C'_E-invariant functional. Then $(\pi')^*$ (the contragredient of π') is a quotient of* Π.

Proof. If we compare the construction (1.10) of the dual pair $G_2 \times G'$ with the definition of P', we find that

$$G_2 \cap P' = P_2 = GL_2\, U_2$$

is the "Heisenberg" parabolic of G_2. Note that $Z \subset U_2$, and in terms of the identification (2.5),

$$U_2/Z \cong F \oplus Fe \oplus Fe^* \oplus F^* \subset F \oplus J(D) \oplus J(D)^* \oplus F^* \cong N/Z,$$

where e^* is a G'-fixed element of $J(D)^*$, normalized by $< e^*, e >= 3$.

The action of GL_2, the Levi factor of P_2, on U_2/Z is isomorphic to $S^3(F^2) \otimes \det^{-1}$, and this was studied in [Wr]. Generic GL_2-orbits correspond to cubic separable extensions of F as follows. A point (a, b, c, d) in U_2/Z defines a binary cubic form

$$ax^3 + bx^2y + cxy^2 + dy^3.$$

The corresponding GL_2-orbit is generic if and only if the form has three different solutions in $\mathbb{P}(\bar{\mathbb{Q}}_p)$. Assume that $a = 6$. This just means that $(6, 0)$ is not a solution of the cubic form. Then

$$E = F[x]/(x^3 + bx^2 + cx + d)$$

is the cubic separable algebra.

The pairing $<,>$ restricits to a non-degenerate pairing between U_2/Z and $\bar{U}_2/\bar{Z}$. Therefore, the point (a, b, c, d) and the additive character ψ define a character ψ_E of $\bar{U}_2$.

Lemma 2.9. *Let $\Pi_{\bar{U}_2,\psi_E}$ be the maximal quotient of Π such that $\bar{U}_2$ acts as the character ψ_E on it. Then*

$$\Pi_{\bar{U}_2,\psi_E} = C_c^\infty(G'/C'_E).$$

Proof. Prop. 2.8 and (2.6) imply that

$$\Pi_{\bar{U}_2,\psi_E} = C_c^\infty(\Omega)_{\bar{U}_2,\psi_E} = C_c^\infty(\Omega_E)$$

where

$$\Omega_E = \{(6,u,u^*,d) \in \Omega \mid \ < u, e^* >= b \text{ and } < e, u^* >= c\}.$$

We need to show that G' acts transitively on Ω_E. Let $\mathfrak{q} \subset \mathfrak{m}$ be a maximal parabolic subalgebra given by

$$\mathfrak{q} = (\mathfrak{t} \oplus \mathfrak{h}) \oplus J.$$

Let Q' be the corresponding subgroup of M', and let $U \cong J$ be the unipotent radical of Q'. The group Q' preserves the partial flag in N/Z

$$F \oplus J(D) \oplus J(D)^* \oplus F^* \supset J(D) \oplus J(D)^* \oplus F^* \supset J(D)^* \oplus F^* \supset F^*.$$

By [MS; Lemma 7.5], Q' has 4 orbits on Ω, given by the position of a point in Ω, relative to the flag. Hence, Ω_E is in the Q'-orbit of $(6,0,0,0)$. Moreover, by (2.5) the Levi factor of Q' preserves the line through $(6,0,0,0)$, hence Ω_E is contained in the U-orbit of $(6,0,0,0)$.

This action of U is given by (3.11) in Chapter II. In particular, if z is an element in $U \cong J$, then

$$z(6,0,0,0) = (1, 6z, 3z \times z, (z,z,z)).$$

So, if $(6,u,u^*,d)$ is an element in Ω_E, then

$$\begin{cases} u = 6z \\ u^* = 3z \times z \\ d = (z,z,z), \end{cases}$$

and, since $e^* = \frac{1}{2} e \times e$,

$$\begin{cases} b = 3(z,e,e) \\ c = 3(z,z,e) \\ d = (z,z,z). \end{cases}$$

Hence, the characteristic polynomial of z is

$$\begin{aligned} 6\det(\lambda - z) &= (\lambda - z, \lambda - z, \lambda - z) \\ &= (e,e,e)\lambda^3 - 3(z,e,e)\lambda^2 + 3(z,z,e)\lambda - (z,z,z) \\ &= 6\lambda^3 - b\lambda^2 + c\lambda - d. \end{aligned}$$

The group G' acts transitively on the set of such elements. In the first case, L' acts transitively on 3×3 matrices with determinant 1, so we can assume that e is the identity matrix. Hence Ω_E is the set of all 3×3 matrices with the characteristic polynomial (in the usual sense)

$$6\lambda^3 - b\lambda^2 + c\lambda - d,$$

i.e. it is a regular, semi-simple conjugacy class of 3×3 matrices. This implies the lemma in the first case. The general case can be reduced to this [J2; Thm. 10, pg 389]. The lemma is proved.

The proposition follows from Lemma 2.9 and the Frobenius reciprocity.

Corollary 2.10. *Let E be a cubic separable algebra over F. Let $\pi(s')$ be an unramified representation of $PGSp_6(F)$ with a non-trivial C'_E-functional. Then $s' = f(s)$ for some semisimple conjugacy class s in $G_2(\mathbb{C})$. If, in addition, $\pi(s')$ is tempered, then the converse is also true.*

Proof. Note that every unramified representation of $PGSp_6$ is self-contragredient. So, if $\pi(s')$ has a non-trivial C'_E-invariant functional, $\pi(s')$ is a quotient of Π by Prop. 2.8. Hence $s' = f(s)$ for some s in $G_2(\mathbb{C})$ by Prop. 3.1 of Chapter IV.

Now assume that $\pi(s')$ is tempered and $s' = f(s)$. Then $\pi(s) \otimes \pi(s')$ is a quotient of Π by [MS; Thm. 5.4]. Since $\pi(s)$ is a fully induced representation [Ke], it follows that $\pi(s)_{\bar{U}_2,\psi_E} \neq 0$. By Lemma 2.9

$$\Pi_{\bar{U}_2,\psi_E} = C_c^\infty(G'/C'_E),$$

so the second statement follows from the Frobenius reciprocity.

Acknowledgments

We would like to thank K. Magaard and J.-P. Serre for many stimulating and helpful discussions during the course of this work.

The second author was generously supported by an NSF Postdoctoral Fellowship and a Sloan Research Fellowship.

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