MATH 5210, LECTURE 9 - CLASSICAL FOURIER EXPANSION I APRIL 06

Let V be a Hilbert space. Let $u_1, u_2, \ldots$ be a sequence of non-zero vectors mutually perpendicular, such that their linear span U is a dense subspace of V. In other words, $e_i = u_i/||u_i||$ is an orthonormal basis of V, however, it is sometimes more convenient to work in this more general setting, with an orthogonal basis not necessarily normal. We have shown that, for every $v \in V$,

$$v = \frac{(v, u_1)}{(u_1, u_1)} u_1 + \frac{(v, u_2)}{(u_2, u_2)} u_2 + \ldots$$

We shall now use this general result to obtain Fourier decomposition of periodic functions. More precisely, function $f : \mathbb{R} \to \mathbb{R}$ is periodic if $f(t+1) = f(t)$ for all $t \in \mathbb{R}$. In particular such f is determined by restriction to the interval

$$X = \left(-\frac{1}{2}, \frac{1}{2}\right].$$

On X we developed Lebesgue measure μ such that $\mu(X) = 1$, and $L^2(X)$ the space of square integrable functions on X is a Hilbert space with respect to the dot product

$$(f, g) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)g(t)\ dt.$$

As is well known from calculus, trigonometric functions $\cos(2\pi nt)$, $\sin(2\pi mt)$, $n, m \in \mathbb{Z}$, $m \neq 0$, are mutually perpendicular elements in $L^2(X)$:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2\pi nt)\sin(2\pi mt)\ dt = 0,$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2\pi nt)\cos(2\pi mt)\ dt = 0 \text{ if } n \neq m,$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin(2\pi nt)\sin(2\pi mt)\ dt = 0 \text{ if } n \neq m.$$

We also have, for $n \neq 0$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2\pi nt)^2\ dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin(2\pi nt)^2\ dt = \frac{1}{2}.$$

Let U be the subspace of $L^2(X)$ spanned by the trigonometric functions $\cos(2\pi nt)$, $\sin(2\pi mt)$. We shall show that U is dense $L^2(X)$. With that assumption, the trigonometric functions are an orthogonal basis of $L^2(X)$, hence $f \in L^2(X)$ can be Fourier expanded, using the formula on the top of the page,

$$f(t) = a_0 + \sum_{n\neq 0} a_n \cos(2\pi nt) + \sum_{m\neq 0} b_n \sin(2\pi mt)$$

where

$$a_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)dt,$$

$$a_n = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)\cos(2\pi nt)\ dt \text{ if } n \neq 0,$$

$$b_m = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)\sin(2\pi mt)\ dt \text{ if } m \neq 0.$$

It remains to show that U, the space of trigonometric functions, is dense. Let

$$\mathbb{T} = \{z = x + iy \in \mathbb{C} \mid x^2 + y^2 = 1\}$$

the unit circle in the plane, which we think of as the set of complex numbers. Consider the exponential function $\exp : \mathbb{R} \to \mathbb{T}$

$$\exp(t) = \cos(2\pi t) + i\sin(2\pi t).$$

The restriction of exp to X gives a bijection of X and $\mathbb{T}$. Using this bijection we can transfer the Lebesgue measure from X to $\mathbb{T}$. In this way we get a rotation invariant measure on $\mathbb{T}$, and we have an isomorphism

$$L^2(\mathbb{T}) = L^2(X)$$

that maps a measurable function f, representing an element in $L^2(\mathbb{T})$ to the periodic measurable function $f \circ \exp$, the composite of exp and f. For example the function $f(x+iy) = x$ maps to $\cos(2\pi t)$ and $f(x+iy) = y$ to $\sin(2\pi t)$. More generally, as f runs through all polynomials $p(x, y)$ in two variables, we get all trigonometric functions on X. Thus in the isomorphism $L^2(\mathbb{T}) = L^2(X)$ the space U corresponds to polynomial functions on $\mathbb{T}$. Let $C(\mathbb{T})$ be the space of continuous functions on $\mathbb{T}$, equipped with the sup norm. By the Stone-Weierstrass theorem polynomial functions are dense in $C(\mathbb{T})$. But $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$, hence U is dense in $L^2(X)$.

The Fourier expansion is an equality between two elements in $L^2(X)$, so it is not an equality of functions, strictly speaking. In the next lecture we shall show that we have an equality of functions, under some assumption on f.