Let $V$ be a Euclidean vector space, that is, a vector space over $\mathbb{R}$ with a scalar product $(x,y)$. Then $V$ is a normed space with the norm $\|x\|^2 = (x,x)$. We shall need the following continuity of the dot product.

Exercise. Let $x,y \in V$ and $(x_n)$ a sequence in $V$ converging to $x$. Then

$$\lim_n (x_n, y) = (x, y).$$

Hint: Use Cauchy Schwarz inequality.

Now assume that $V$ is a Hilbert space, i.e. a separable and complete Euclidean space. Let $e_1, e_2, \ldots$ its orthonormal basis, see the previous lecture. In particular, the subspace $U$ spanned by $e_1, e_2, \ldots$ is a dense subset.

Lemma 0.1. Bessel’s inequality. For every $v \in V$, and every $n \in \mathbb{N}$,

$$(v, e_1)^2 + \ldots + (e_n, v)^2 \leq \|v\|^2.$$

Proof. Let

$$v_n = (v, e_1)e_1 + \ldots + (v, e_n)e_n.$$

Then, for every $i \leq n$, $(v - v_n, e_i) = (v, e_i) - (v_n, e_i) = 0.$

Since $v_n$ is a linear combination of $e_i$ for $i \leq n$, it follows that $v_n$ and $v - v_n$ are perpendicular. By the Pythagorean equality,

$$\|v_n\|^2 \leq \|v\|^2 + \|v - v_n\|^2 = \|v\|^2.$$

The lemma follows since $\|v_n\|^2 = (v, e_1)^2 + \ldots + (v, e_n)^2$. □

Now we can prove the main result in the theory of (infinite dimensional) Hilbert spaces.

Theorem 0.2. (Riesz-Fischer) Let $V$ be a Hilbert space, and $e_1, e_2, \ldots$ its orthonormal basis. Then

1. Fourier series. For every $v \in V$,

$$v = (v, e_1)e_1 + (v, e_2)e_2 + \ldots$$

i.e. the series is absolutely convergent and it converges to $v$.

2. Parseval’s identity. For every $v \in V$,

$$\|v\|^2 = (v, e_1)^2 + (v, e_2)^2 + \ldots$$
(3) If \((x_1, x_2, \ldots)\) is a sequence of real numbers such that \(x_1^2 + x_2^2 + \ldots < \infty\) then the series
\[
x_1e_1 + x_2e_2 + \ldots
\]
is absolutely convergent and it converges to an element in \(V\).

**Proof.** (1) Let \(v_n = (v, e_1)e_1 + \ldots + (v, e_n)e_n\), for \(n \in \mathbb{N}\). We need to show that this sequence converges to \(v\). By the Bessel’s inequality, the series \(\sum_{n=0}^{\infty} (v, e_n)^2\) is convergent. Thus, for every \(\epsilon > 0\) there exists \(N\) such that
\[
\sum_{n>N} (v, e_n)^2 < \epsilon.
\]
If \(m > n > N\) then
\[
||v_m - v_n||^2 = (v, e_{n+1})^2 + \ldots + (v, e_m)^2 < \epsilon.
\]
This shows that the sequence \((v_n)\) is Cauchy. Since \(V\) is complete, it has a limit \(\lim_n v_n = w \in V\). It remains to show that \(v = w\). Observe that, using the exercise,
\[
(w, e_i) = (\lim_n v_n, e_i) = \lim_n(v_n, e_i) = (v, e_i).
\]
Hence \(w - v\) is perpendicular to all \(e_i\) and to the linear span \(U\) of \(e_i\). But this space is dense, hence \(w - v = 0\), as we proved in the last lecture.

(2) follows form
\[
||v||^2 = \lim_n ||v_n||^2,
\]
since \(v = \lim_n v_n\), and \(||v_n||^2 = (v, e_1)^2 + \ldots + (v, e_n)^2\).

(3) Let \(v_n = x_1e_1 + \ldots + x_n e_n\). Then \((v_n)\) is Cauchy by the same argument as in (1) thus the series is converging to an element in \(V\) since \(V\) is complete.

\(\square\)

**Corollary 0.3.** Any Hilbert space \(V\) is isomorphic to \(\ell^2(\mathbb{N})\).

**Proof.** Indeed, by the map
\[
v \mapsto ((v, e_1), (v, e_2), \ldots),
\]
for every \(v \in V\), is a norm-preserving isomorphism from \(V\) onto \(\ell^2(\mathbb{N})\). \(\square\)

This is a great result, for it gives a classification of Hilbert spaces. There is only one, up to an isomorphism.