MATH 5210, LECTURE 8 - RIESZ-FISCHER THEOREM APRIL 03

Let V be a Euclidean vector space, that is, a vector space over $\mathbb{R}$ with a scalar product (x, y). Then V is a normed space with the norm $||x||^2 = (x, x)$. We shall need the following continuity of the dot product.

Exercise. Let $x, y \in V$ and (x_n) a sequence in V converging to x. Then

$$\lim_n (x_n, y) = (x, y).$$

Hint: Use Cauchy Schwarz inequality.

Solution. (x_n) converging to x means $\lim_n ||x_n - x|| = 0$.

$$|(x_n, y) - (x, y)| = |(x_n - x, y)| \leq ||x_n - x|| \cdot ||y||$$

hence (x_n, y) converges to (x, y).

Now assume that V is a Hilbert space, i.e. a separable and complete Euclidean space. Let $e_1, e_2, \ldots$ its orthonormal basis, see the previous lecture. In particular, the subspace U spanned by $e_1, e_2, \ldots$ is a dense subset.

Lemma 0.1. *Bessel's inequalty. For every $v \in V$, and every $n \in \mathbb{N}$,*

$$(v, e_1)^2 + \ldots + (e_n, v)^2 \leq ||v||^2.$$

Proof. Let

$$v_n = (v, e_1)e_1 + \ldots + (v, e_n)e_n.$$

Then, for every $i \leq n$,

$$(v - v_n, e_i) = (v, e_i) - (v_n, e_i) = 0.$$

Since v_n is a linear combination of e_i for $i \leq n$, it follows that v_n and $v - v_n$ are perpendicular. By the Pythagorean equality,

$$||v_n||^2 \leq ||v_n||^2 + ||v - v_n||^2 = ||v||^2.$$

The lemma follows since $||v_n||^2 = (v, e_1)^2 + \ldots + (v, e_n)^2$. □

Now we can prove the main result in the theory of (infinite dimensional) Hilbert spaces.

Theorem 0.2. *(Riesz-Fischer) Let V be a Hilbert space, and $e_1, e_2, \ldots$ its orthonormal basis. Then*

(1) *Fourier series. For every $v \in V$,*
$$v = (v, e_1)e_1 + (v, e_2)e_2 + \ldots$$
i.e. the series is absolutely convergent and it converges to v.

(2) *Parsevals' identity. For every $v \in V$,*
$$||v||^2 = (v, e_1)^2 + (v, e_2)^2 + \ldots$$

(3) *If $(x_1, x_2, \ldots)$ is a sequence of real numbers such that $x_1^2 + x_2^2 + \ldots < \infty$ then the series*
$$x_1e_1 + x_2e_2 + \ldots$$
is absolutely convergent and it converges to an element in V.

Proof. (1) Let $v_n = (v, e_1)e_1 + \ldots + (v, e_n)e_n$, for $n \in \mathbb{N}$. We need to show that this sequence converges to v. By the Bessel's inequality, the series $\sum_{n=0}^{\infty}(v, e_n)^2$ is convergent. Thus, for every $\epsilon > 0$ there exists N such that
$$\sum_{n>N}(v, e_n)^2 < \epsilon.$$
If $m > n > N$ then
$$||v_m - v_n||^2 = (v, e_{n+1})^2 + \ldots + (v, e_m)^2 < \epsilon.$$
This shows that the sequence (v_n) is Cauchy. Since V is complete, it has a limit $\lim_n v_n = w \in V$. It remains to show that $v = w$. Observe that, using the exercise,
$$(w, e_i) = (\lim_n v_n, e_i) = \lim_n (v_n, e_i) = (v, e_i).$$
Hence $w - v$ is perpendicular to all e_i and to the linear span U of e_i. But this space is dense, hence $w - v = 0$, as we proved in the last lecture.

(2) follows form
$$||v||^2 = \lim_n ||v_n||^2,$$
since $v = \lim_n v_n$, and $||v_n||^2 = (v, e_1)^2 + \ldots + (v, e_n)^2$.

(3) Let $v_n = x_1e_1 + \ldots + x_ne_n$. Then (v_n) is Cauchy by the same argument as in (1) thus the series is converging to an element in V since V is complete.

□

Corollary 0.3. *Any Hilbert space V is isomorphic to $\ell^2(\mathbb{N})$.*

Proof. Indeed, by the map
$$v \mapsto ((v, e_1), (v, e_2), \ldots),$$
for every $v \in V$, is a norm-preserving isomorphism from V onto $\ell^2(\mathbb{N})$. □

This is great result, for it gives a classification of Hilbert spaces. There is only one, up to an isomorphism.