Let $F$ be a Galois extension of $\mathbb{Q}$ of degree $n$ and let $A$ be the ring of integers in $F$. Recall the Dedekind zeta function

$$
\zeta_F(s) = \sum_{I \subseteq A} \frac{1}{N(I)^s} = \prod_{P \subseteq A} \left(1 - \frac{1}{N(P)^s}\right).
$$

Assume now that $F$ is Galois with the Galois group $G$. Let $p$ be a prime. We have a factorization $A_p = P_1^{e_1} \cdots P_g^{e_g}$, let $P$ be any of these primes. Then $A/P$ is a degree $f$ extension of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ where $n = efg$. Then $N(P_1) = \cdots = N(P_g) = p^f$. If we group together the primes $P$ dividing $p$ in the factorization of $\zeta_F(s)$, then we get

$$
\zeta_F(s) = \prod_p \left(1 - \frac{1}{p^fs}\right)^g.
$$

Let $(r, \mathbb{C}[G])$ be the regular representation of $G$.

**Proposition 0.1.** The Dedekind zeta function $\zeta_F(s)$ is equal to the Artin $L$-function attached to the regular representation.

**Proof.** We need to establish identities

$$
\left(1 - \frac{1}{p^fs}\right)^g = L_p(r, s)
$$

for all primes $p$. Assume that $p$ is unramified, $A_p = P_1^{e_1} \cdots P_g^{e_g}$, let $P$ be any of these primes. Let $\text{Fr}_P$ be the Frobenius element where $P \subseteq A$ a prime dividing $p$. We need to show that

$$
\left(1 - \frac{1}{p^fs}\right)^g = \det \left(1 - r(\text{Fr}_P)/p^s\right).
$$

To simplify notation, write $\sigma = \text{Fr}_P$, and $\lambda = p^s$. Since $r(\text{Fr}_P)$ is an $n \times n$ matrix, we have

$$
\lambda^n \det \left(1 - \frac{r(\sigma)}{\lambda}\right) = \det(\lambda - r(\sigma)).
$$

Thus we need to compute the characteristic polynomial of $r(\sigma)$. Let $H \subseteq G$ be the cyclic group generated by $\sigma$. The order of $H$ is $f$. (Of course, $H = D_P$ the decomposition group, though this is not important in this moment.) Write $G$ as a union of $H$-cosets

$$
G = \sigma_1 H \cup \ldots \cup \sigma_g H.
$$

Then

$$
\mathbb{C}[G] = \mathbb{C}[\sigma_1 H] \oplus \ldots \oplus \mathbb{C}[\sigma_g H] \cong \mathbb{C}[H]^g,
$$
and \( r(\sigma) \) preserves each summand. The space \( \mathbb{C}[H] \) has basis of delta functions \( \delta_{\sigma}, \ldots, \delta_{\sigma^f} \) and \( r(\sigma) \) permutes them cyclically. Hence \( r(\sigma) \) is represented by a block-diagonal matrix with \( g \) blocks, and each block is an \( f \times f \) cycle permutation matrix. The characteristic polynomial of this permutation matrix is \( \lambda^f - 1 \) (check this), thus

\[
\det(\lambda - r(\sigma)) = (\lambda^f - 1)^g
\]

and

\[
\det \left( 1 - \frac{r(\sigma)}{\lambda^n} \right) = \left( 1 - \frac{1}{\lambda^f} \right)^g
\]
as desired.

Exercise. Work out the local factors at ramified primes.

It is well known that every irreducible representation \((\rho, V)\) appears as a direct summand of the regular representation with multiplicity \( n_\rho = \dim V \). Thus from the multiplicative property of Artin’s \( L \)-functions and the above theorem it follows that

\[
\zeta_F(s) = \prod_\rho L(\rho, s)^{n_\rho}
\]

where the product is taken over all isomorphism classes of irreducible representations of \( G \). We know that \( \zeta_F(s) \) has a simple pole at \( s = 1 \), and so does the Riemann zeta function which appears on the right, corresponding to the trivial representation. So it is conjectured that Artin \( L \)-functions for irreducible non-trivial \( \rho \) have analytic continuations to entire functions on \( \mathbb{C} \). This is known for abelian \( G \), by Tate’s thesis, and for solvable \( G \) by Langlands. In general this conjecture is wide open.

Home-work problem. Let \( \ell \) be a prime, and \( \omega = e^{2\pi i/\ell} \). Let \( F = \mathbb{Q}(\omega) \). Let \( E = F(2^{1/\ell}) \). In words, \( E \) is the splitting field of \( x^\ell - 2 \). Let \( G \) be the Galois group of \( E \). Show that

\[
G \cong \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(F\ell) \right\}
\]

Hint: every \( \sigma \in G \) is determined by its action on \( \omega \) and \( 2^{1/\ell} \). Use this matrix representation of \( G \) to determine conjugacy classes in \( G \). Prove the Čebovarev density theorem for \( G \). Hint: first factor \( p \) in \( F \), use the distribution statement for \( F \) and that the set of primes \( p \) that split completely in \( E \) has density \( 1/|G| \).