MATH 6370, LECTURE 7
APRIL 01

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Let F be a Galois extension of $\mathbb{Q}$ of degree n and let A be the ring of integers in F. Recall the Dedekind zeta function

$$\zeta_F(s) = \sum_{I \subseteq A} \frac{1}{N(I)^s} = \prod_{P \subset A} \frac{1}{\left(1 - \frac{1}{N(P)^s}\right)}.$$

Assume now that F is Galois with the Galois group G. Let p be a prime. We have a factorization $Ap = P_1^e \cdot \ldots \cdot P_g^e$, let P be any of these primes. Then A/P is a degree f extension of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ where $n = efg$. Then $N(P_1) = \ldots = N(P_g) = p^f$. If we group together the primes P dividing p in the factorization of $\zeta_F(s)$, then we get

$$\zeta_F(s) = \prod_p \frac{1}{\left(1 - \frac{1}{p^{fs}}\right)^g}.$$

Let $(r, \mathbb{C}[G])$ be the regular representation of G.

Proposition 0.1. *The Dedekind zeta function $\zeta_F(s)$ is equal to the Artin L-function attached to the regular representation.*

Proof. We need to establish identities

$$\frac{1}{\left(1 - \frac{1}{p^{fs}}\right)^g} = L_p(r, s)$$

for all primes p. Assume that p is unramified, $Ap = P_1 \cdot \ldots \cdot P_g$, let P be any of these primes. Let Fr_P be the Frobenius element where $P \subseteq A$ a prime dividing p. We need to show that

$$\left(1 - \frac{1}{p^{fs}}\right)^g = \det\left(1 - \frac{r(\mathrm{Fr}_P)}{p^s}\right).$$

To simplify notation, write $\sigma = \mathrm{Fr}_P$, and $\lambda = p^s$. Since $r(\mathrm{Fr}_P)$ is an $n \times n$ matrix, we have

$$\lambda^n \det\left(1 - \frac{r(\sigma)}{\lambda}\right) = \det(\lambda - r(\sigma)).$$

Thus we need to compute the characteristic polynomial of $r(\sigma)$. Let $H \subseteq G$ be the cyclic group generated by σ. The order of H is f. (Of course, $H = D_P$ the decomposition group, though this is not important in this moment.) Write G as a union of H-cosets

$$G = \sigma_1 H \cup \ldots \cup \sigma_g H.$$

Then

$$\mathbb{C}[G] = \mathbb{C}[\sigma_1 H] \oplus \ldots \oplus \mathbb{C}[\sigma_g H] \cong \mathbb{C}[H]^g,$$

and $r(\sigma)$ preserves each summand. The space $\mathbb{C}[H]$ has basis of delta functions $\delta_\sigma, \dots, \delta_{\sigma^f}$ and $r(\sigma)$ permutes them cyclically. Hence $r(\sigma)$ is represented by a block-diagonal matrix with g blocks, and each block is an $f \times f$ cycle permutation matrix. The characteristic polynomial of this permutation matrix is $\lambda^f - 1$ (check this), thus

$$\det(\lambda - r(\sigma)) = (\lambda^f - 1)^g$$

and

$$\det\left(1 - \frac{r(\sigma)}{\lambda^n}\right) = \left(1 - \frac{1}{\lambda^f}\right)^g$$

as desired.

Exercise. Work out the local factors at ramified primes. □

It is well known that every irreducible representation (ρ, V) appears as a direct summand of the regular representation with multiplicity $n_\rho = \dim V$. Thus from the multiplicative property of Artin's L-functions and the above theorem it follows that

$$\zeta_F(s) = \prod_\rho L(\rho, s)^{n_\rho}$$

where the product is taken over all isomorphism classes of irreducible representations of G. We know that $\zeta_F(s)$ has a simple pole at $s = 1$, and so does the Riemann zeta function which appears on the right, corresponding to the trivial representation. So it is conjectured that Artin L-functions for irreducible non-trivial ρ have analytic continuations to entire functions on $\mathbb{C}$. This is known for abelian G, by Tate's thesis, and for solvable G by Langlands. In general this conjecture is wide open.

Home-work problem. Let ℓ be a prime, and $\omega = e^{2\pi i/\ell}$. Let $F = \mathbb{Q}(\omega)$. Let $E = F(2^{\frac{1}{\ell}})$. In words, E is the splitting field of $x^\ell - 2$. Let G be the Galois group of E. Show that

$$G \cong \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_\ell) \right\}$$

Hint: every $\sigma \in G$ is determined by its action on ω and $2^{\frac{1}{\ell}}$. Use this matrix representation of G to determine conjugacy classes in G. Prove the Čebotarev density theorem for G. Hint: first factor p in F, use the distribution statement for F and that the set of primes p that split completely in E has density $1/|G|$.