MATH 6370, LECTURE 3
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Fix a natural number m. For every integer $0 \leq a < m$ consider the arithmetic sequence $a, a+m, a+2m, \ldots$. This sequence can contain primes only if a and m are relatively prime. There are $\phi(m)$ such sequences where $\phi(m)$ is the Euler's function. The theorem of Dircihlet says that primes are equally distributed amongst these sequence. More precisely, the set of primes contained in any sequence has Dirichlet's density $1/\phi(m)$. We shall prove this theorem in the case $m = \ell$ a prime.

This theorem is proved using the cyclotomic extension $\mathbb{Q}(\omega)$ where $\omega = e^{2\pi i/\ell}$ is the ℓ-th root of one. Recall that ω is root of $\Phi_\ell = (x^\ell - 1)/(x-1)$, and the ring of integers is $A \cong \mathbb{Z}[x]/(\Phi_\ell)$. Hence a prime p splits completely in A iff the polynomial Φ_ℓ splits completely mod p, concretely, there exists a primitive root of order ℓ in $\mathbb{F}_p^\times$. Since $\mathbb{F}_p^\times$ is a cyclic group of order $p-1$, it contains a primitive root of order ℓ if and only if ℓ divides $p-1$ i.e. p belongs to the sequence $1, 1+\ell, 1+2\ell, \ldots$. But we already know that the set of primes that split completely in $\mathbb{Q}(\omega)$ has density $1/\ell - 1$. This proves the density in one case. We shall now prove the general case. (In a later lecture we shall relate to splitting of primes in $\mathbb{Q}(\omega)$.)

Let $\chi : (\mathbb{Z}/\ell\mathbb{Z})^\times \to \mathbb{C}^\times$ be a non-trivial character. For convenience, we extend χ to all integers by $\chi(n) = 0$ if ℓ divides n. Then ℓ is a periodic function on $\mathbb{Z}$ satisfying $\chi(mn) = \chi(n)\chi(m)$ (check it). Let

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_n \frac{a_n}{n^s}$$

be the Dirichlet's L-function attached to χ. This series is clearly absolutely convergent for $\Re(s) > 1$.

Exercise: Prove that we have a factorization

$$L(\chi, s) = \prod_{p \neq \ell} \frac{1}{1 - \frac{\chi(p)}{p^s}}.$$

Exercise: Prove that $\sum_{i=1}^{n} a_i = O(1)$ (bounded).

It follows that $L(\chi, s)$ has a holomorphic continuation on half-plane $\Re(s) > 0$, if $\chi \neq 1$. A key point, for us, is the factorization

$$\zeta_{\mathbb{Q}(\omega)}(s) = \zeta_{\mathbb{Q}}(s) \prod_\chi L(\chi, s)$$

where the product is taken over all non-trivial characters χ of $(\mathbb{Z}/\ell\mathbb{Z})^\times$. (You should think $\zeta_{\mathbb{Q}}(s)$ as the L-function attached $\chi = 1$.) This is a special case of a more general factorization of the Dedekind zeta functions in terms of Artin L-functions. Proof of this general fact is quite easy, and we shall do it in a later lecture. The Dedekind zeta function has a pole of order 1, and so does the Riemann zeta function that appears on the right, thus it follows that

$$L(\chi, 1) \neq 0$$

for all non-trivial characters χ. This non-vanishing is the most difficult part in proofs found in literature.

Lemma 0.1. *Recall that X is the set of all prime numbers. Let χ be a non-trivial character of $(\mathbb{Z}/\ell\mathbb{Z})^\times$. Then*

$$\lim_{s\to 1^+} \frac{\sum_{p\in X} \frac{\chi(p)}{p^s}}{\sum_{p\in X} \frac{1}{p^s}} = 0.$$

Proof. Since $L(\chi, 1) \neq 0$,

$$\lim_{s\to 1^+} \frac{\log L(\chi, s)}{\log \zeta_{\mathbb{Q}}(s)} = 0.$$

In the previous lecture we proved

$$\log \zeta_{\mathbb{Q}}(s) = \sum_{p\in X} \frac{1}{p^s} + e(s)$$

where $e(s)$ is holomorphic at $s = 1$. Similarly, we have

$$\log L(\chi, s) = \sum_{p\in X} \frac{\chi(p)}{p^s} + e_\chi(s)$$

where $e_\chi(s)$ is also holomorphic at $s = 1$. The lemma is now an easy combination of the above facts.

□

Remark: Since χ takes complex values, we need to pick a branch of log specified by $\log(1) = 0$. Then locally, around 1, $\log(xy) = \log x + \log y$.