MATH 5210, LECTURE 2 - COMPLETENESS OF L^1
MARCH 20

Let V be a vector space over $\mathbb{R}$ and $||\cdot||$ a norm on V. Then $d(x,y) = ||x-y||$ is a metric on V. Let $\sum_{i=1}^{\infty} v_i$ be a series, where $v_i \in V$. The series is absolutely convergent if

$$\sum_{i=1}^{\infty} ||v_i|| < \infty$$

Recall that V is complete if every Cauchy sequence in V is convergent. As we proved in class, instead of working with sequences, in order to prove that V is complete, it suffices to prove that absolutely convergent series are convergent.

Let $X = [0,1]^k \subset \mathbb{R}^k$ or more generally any box in $\mathbb{R}^k$. Let $L^1(X)$ be the space of Lebesque integrable functions on X. More precisely, $L^1(X)$ is the set of equivalence classes of integrable functions where f is equivalent to g if

$$\int |f-g| = 0.$$

This is the same as saying that $f = g$ almost everywhere i.e. except on the set of Lebesgue measure 0. We shall now prove that $L^1(X)$ is a complete normed space for the norm

$$||f|| = \int |f|.$$

This result is an easy combination of the Monotone Convergence and Lebesgue Dominated Convergence Theorems. Let $\sum_{i=1}^{\infty} f_i$ be an absolutely convergent series of functions $f_i \in L^1(X)$ i.e.

$$\sum_{i=1}^{\infty} \int |f_i| = \sum_{i=1}^{\infty} ||f_i|| < \infty$$

We need to find a function $f \in L^1(X)$ to which this series converges. Consider the sequence of non-negative functions

$$\varphi_n = \sum_{i=1}^{n} |f_i|.$$

The sequence (φ_n) is clearly monotone and

$$\int \varphi_n = \sum_{i=1}^{n} \int |f_i| = \sum_{i=1}^{n} ||f_i|| < \sum_{i=1}^{\infty} ||f_i||$$

for every n. Hence, by the Monotone Convergence Theorem, there exists an integrable function φ, such that $\lim_n \varphi_n(x) = \varphi(x)$ for almost all x. i.e. except perhaps on a measure 0 set. Thus, for almost all x, the series of real numbers

$$\sum_{i=1}^{\infty} |f_i(x)| = \varphi(x)$$

is convergent. In particular, for those x, the series $\sum_{i=1}^{\infty} f_i(x)$ is also convergent, and we define

$$f(x) := \sum_{i=1}^{\infty} f_i(x).$$

For other x (in the set of measure 0) we can set $f(x) = 0$ or $f(x) = 1$ or any other value. Different choice give different functions, but the same element in $L^1(X)$.

Exercise: Explain why f is integrable.

Solution: f is a pointwise limit of the sequence $\sum_{i=1}^{n} f_i$ of measurable functions so it is measurable. Any measurable function f, such that $|f|$ is bounded by an integrable function φ, is integrable.

It remains to prove that the series $\sum_{i=1}^{\infty} f_i$ converges to f in $L^1(X)$, that is, for the sequence of partial sums

$$g_n = \sum_{i=1}^{n} f_i$$

we want to prove that

$$\lim_n ||g_n - f|| = \lim_n \int |g_n - f| = 0.$$

The sequence of functions $|g_n - f|$ converges to 0 pointwise, so we need to justify that we can switch the order of the limit and integral. Observe that, from the triangle inequality,

$$|g_n(x) - f(x)| \leq \sum_{i>n} |f_i(x)| \leq \varphi(x).$$

Thus the sequence of functions $|g_n - f|$ is bounded (dominated) by the integrable function φ. Now by the Lebesgue Dominated Convergence Theorem, we can switch the order of limit and integral,

$$\lim_n \int |g_n - f| = \int \lim_n |g_n - f| = \int 0 = 0.$$

Thus we have proved the following:

Theorem 0.1. *$L^1(X)$ is a complete normed space.*

Complete normed spaces are also called Banach spaces.