MATH 6370, LECTURE 1
MARCH 18

GORDAN SAVIN

Let $[F:\mathbb{Q}]=n$ be a number field, and let A be the ring of all algebraic integers in F. Let Δ be the discriminant of A. The Dedekind zeta function attached to F is

$$\zeta_F(s)=\sum_{I\subseteq A}\frac{1}{N(I)^s}$$

where the sum is taken over all non-zero ideals in A. Observe that $\zeta_{\mathbb{Q}}(s)$ is the Riemann zeta function. The series $\zeta_F(s)$ converges absolutely on the half plane $\Re(s)>1$ and has a factorization

$$\zeta_F(s)=\prod_{P\subset A}\frac{1}{1-\frac{1}{N(P)^s}}$$

where the product is over maximal (non-zero prime) ideals in A. If $S\subset\mathbb{N}$ is a set of primes, consider the partial product

$$\zeta_{F,S}(s)=\prod_{P|p\in S}\frac{1}{1-\frac{1}{N(P)^s}}.$$

In words, this is a product over P appearing in factorizations of $p\in S$. The Dedekind zeta function has a meromorphic continuation on the half-plane $\Re(s)>1-1/n$, with a simple pole at $s=1$. We shall use this fact to prove a number of results on distribution of primes. If $f(s)$ and $g(s)$ are two meromorphic functions, we shall write $f(s)\approx g(s)$ if $f(s)/g(s)$ is holomorphic and non-zero at $s=1$. In particular, since the factors $1/(1-\frac{1}{N(P)^s})$ are holomorphic and non-zero at $s=1$, for any S containing almost all primes we have

$$\zeta_F(s)\approx\zeta_{F,S}(s).$$

As a warm-up, in order to illustrate main ideas, let's prove the following case of primes in progression result.

Proposition 0.1. *There are infinitely many primes $p\equiv 1 \pmod 4$ and infinitely many primes $p\equiv 3 \pmod 4$.*

Proof. □

Let $F=\mathbb{Q}(\sqrt{-1})$. The proof is based on the fact that an odd prime p splits in this quadratic extension if $p\equiv 1 \pmod 4$ and is inert (stays prime) if $p\equiv 3 \pmod 4$. Let S be the set of primes $p\equiv 1 \pmod 4$ and let T be the set of primes $p\equiv 3 \pmod 4$. We have

$$\zeta_F(s)\approx\zeta_{F,S}(s)\zeta_{F,T}(s).$$

If $p \in S$, then $(p) = P\bar{P}$ where $P \neq \bar{P}$ are two prime ideals such that $N(P) = N(\bar{P}) = p$. Hence

$$\zeta_{F,S}(s) = \prod_{p \in S} \frac{1}{(1 - \frac{1}{p^s})^2}.$$

If $p \in T$, then (p) is prime and $N((p)) = p^2$. Hence

$$\zeta_{F,T}(s) = \prod_{p \in T} \frac{1}{(1 - \frac{1}{p^{2s}})}.$$

Notice that these are the factors of the Riemann zeta function at s (squared) and at $2s$, respectively.

Now we can prove that S and T are both infinite. If S is finite, then T contain almost all primes, hence $\zeta_{F,T}(s) \approx \zeta_{\mathbb{Q}}(2s)$, and

$$\zeta_F(s) \approx \zeta_{\mathbb{Q}}(2s).$$

The left hand side has a pole at $s = 1$, while the right hand side is holomorphic, a contradiction. On the other hand, if T is finite, then S contain almost all primes, hence $\zeta_{F,S}(s) \approx \zeta_{\mathbb{Q}}(s)^2$ and

$$\zeta_F(s) \approx \zeta_{\mathbb{Q}}(s)^2.$$

The left hand side has a pole at $s = 1$, while the right hand side has a double pole, a contradiction.

Let's do this more conceptually. We say that a set $S \subset \mathbb{N}$ of primes has polar density $\frac{m}{d}$ if $\zeta_{\mathbb{Q},S}(s)^d$ has a pole of order m at $s = 1$. For example, if S is a set of almost all primes p, then its polar density is 1. On the other hand, if S is finite, then its polar density is 0. Let F be a quadratic extension of $\mathbb{Q}$, and let S be the set of all primes p that split completely, i.e. $(p) = P_1P_2$ where P_1 and P_2 are two prime ideals with norm p. Then

$$\zeta_{F,S}(s) = \prod_{p \in S} \frac{1}{(1 - \frac{1}{p^s})^2} = \zeta_{\mathbb{Q},S}(s)^2.$$

Let T be the set of all primes p that stay inert, i.e. (p) is a prime ideal in A with norm p^2. Then

$$\zeta_{F,T}(s) = \prod_{p \in T} \frac{1}{(1 - \frac{1}{p^{2s}})} = \zeta_{\mathbb{Q},T}(2s).$$

For $s > 1$, real, we have the following obvious inequalities:

$$1 < \zeta_{\mathbb{Q},T}(2s) < \zeta_{\mathbb{Q}}(2s).$$

Thus $\zeta_{F,T}(s)$ has neither zero nor a pole at $s = 1$. Hence

$$\zeta_F(s) \approx \zeta_{F,S}(s)\zeta_{F,T}(s) \approx \zeta_{\mathbb{Q},S}(s)^2.$$

It follows that the polar density of the set of primes that split in a quadratic extension is $1/2$.

We now push this idea a step further. Assume that $[F : \mathbb{Q}] = n$ is Galois. Let p be an unramified prime, so we have a factorization $(p) = P_1 \cdots P_g$ in A where $P_1, \ldots, P_g$ are prime ideals such that $A/P_1 \cong \ldots \cong A/P_g \cong \mathbb{F}_{p^f}$, and $n = fg$. The prime p is said to split completely if $f = 1$ and $g = n$.

Exercise: Prove that the polar density of the set of primes that split completely in F is $1/n$.