ALGEBRA - LECTURE VI

1. Localization

Let R be a commutative ring with 1. Let $S \subseteq R$ be a subset of R closed under multiplication. It will be convenient to assume that 1 is in S, but not necessary. Let M be an R-module. The S-torsion of M is

$$Tor_S(M) = \{m \in M \mid a \cdot m = 0 \text{ for some } a \in S\}.$$

Note that $Tor_S(M)$ is a submodule. (This follows since S is closed under multiplication.) Note that $M/Tor_S(M)$ has no S-torsion. A typical example of S is $R \setminus 0$ if R is an integral domain. In this case we write $Tor(M)$ for $Tor_S(M)$.

Let M an R-module. In particular, any element of M can be multiplied by an element of R. We would like to define a module where we can also divide by any element in S. This is called a localisation of M by S. Roughly speaking, elements of $S^{-1}M$ are fractions m/a where m is in M and a in S. Then

$$\frac{m}{a} = \frac{n}{b}$$

if $bm = an$ (note the analogy with rational numbers). Formally, $S^{-1}M$ is constructed as follows. Define a relation $\sim$ in $M \times S$ by $(m, a) \sim (n, b)$ if

$$x(bm - an) = 0$$

for some x in S. This is an equivalnce relation. If $Tor_S(M) = 0$ then the factor x can be omitted in the definition of the relation. The addition in $S^{-1}M$ is defined by

$$[(m, a)] + [n, b] = [(bm + an, ab)]$$

which corresponds of the usual addition of fractions. The zero element is $[(0, 1)]$, the class of $(0, 1)$. Note that $S^{-1}R$ is a ring and $S^{-1}M$ is an $S^{-1}R$-module where multiplication is given by

$$[(c, b)] \cdot [(m, a)] = [(cm, ab)].$$

Example: Let R be an integral domain and let $S = R \setminus 0$. Then $Tor(R) = 0$, clealry, and $S^{-1}R$ is the field of fractions of R.

Proposition 1.1. *If $M \neq Tor_S(M)$, then $S^{-1}M \neq 0$.*

Proof. Let $m \neq 0$ be in $M \setminus Tor_S(M)$. Then $(m, 1)$ is not equivalent to $(0, 1)$, the proposition follows. □

Corollary 1.2. *If m is an element in $M \setminus Tor_S(M)$ then $1 \otimes m$ is a non-zero element in $S^{-1}R \otimes_R M$. In particular, if $M \neq Tor_S(M)$, then $S^{-1}R \otimes_R M \neq 0$.*

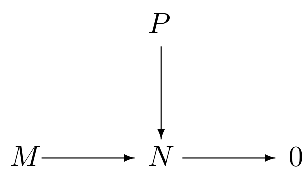
Proof. There is a natural bilinear form on $S^{-1}R \times M$ with values in $S^{-1}M$:

$$([(r, s)], m) \mapsto [(rm, s)].$$

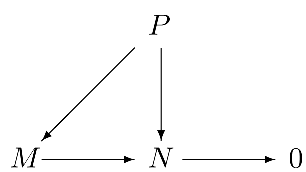
□

2. Projective Modules

An R-module P is projective if for every diagram of R-modules



there exists a homomorphism $h : P \to M$ such that the following diagram is commutative.



Example: Any free module is projective!

Note that for *every* R-module P, every exact sequence of R-modules $0 \to L \to M \to N \to 0$ gives rise to an exact sequence

$$0 \to Hom_R(P, L) \to Hom_R(P, M) \to Hom_R(P, N).$$

Proposition 2.1. *The following are equivalent:*

(i) *P is projective.*
(ii) *Every exact sequence $0 \to L \to M \to P \to 0$ splits. This means that $M \cong P \oplus L$.*
(iii) *For every exact sequence of R-modules $0 \to L \to M \to N \to 0$, the sequence*

$$0 \to Hom_R(P, L) \to Hom_R(P, M) \to Hom_R(P, N) \to 0$$

is exact. One says that the functor $Hom_R(P, \cdot)$ is exact!

Proof. The first and the last statement are trivially equivalent as exactness of the sequence is equivalent to surjectivity of the map from $Hom(M, P)$ to $Hom(N, P)$.

(i) implies (ii): Let f denote the map from M to P in the given exact sequence. Since P is projective, there exists $h : P \to M$ such that $f \circ h : P \to P$ is the identity. This shows that h is injective and $Im(h) \cap ker(f) = 0$. Also, every m in M can be written as

$$m = h(f(m)) + (m - h(f(m))) \in Im(h) + ker(f).$$

This shows that $M = Im(h) \oplus ker(f) \cong I \oplus L$.

(ii) implies (i): We need the following lemma:

Lemma 2.2. *Let P be an R-module. Then there exists a projective module Q such that $Q \to P \to 1$.*

Proof. Let S be a set of generators of P. (You could take S to be the whole P, for example.) Let F_S be the free module generated by elements e_s for s in S. Then F_S is projective and $f : F_S \to P$ given by

$$f(\sum r_s e_s) = \sum r_s s$$

surjective. □

Let L be the kernel of the projection $Q \to P \to 0$. Then (ii) imples that $Q \cong P \oplus L$. Note that a diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

can be extended to a diagram

$$\begin{array}{ccccc} & & P \oplus L & & \\ & & \downarrow & & \\ M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

so that L maps trivially to N. Since $P \oplus L \cong Q$ is projective, there exists a map $h' : P \oplus L \to M$ making the diagram commutative. Now put h to be the restriction of h' to P. This shows that P is projective.

□

In the course of the proof we have also proved:

Corollary 2.3. *Every projective module is a direct summand of a free module.*

Corollary 2.4. *The following are easy consequeces:*

(i) *A finitely generated $\mathbb{Z}$-module is projective if and only if it is free.*
(ii) *$\mathbb{Q}$ is not projective as a $\mathbb{Z}$-module.*

Proof. Clearly, a direct summand of a free $\mathbb{Z}$-module has no torsion. Therefore a finitely generated projective $\mathbb{Z}$-module cannot have any torsion. Thus it must be free. This shows the first statement. The second is as easy, since $\mathbb{Q}$ cannot be a summand of a free $\mathbb{Z}$-module. (A free $\mathbb{Z}$-module is not divisible.) □

Corollary 2.5. *Let P and Q be two projective R-modules. Then $P \otimes_R Q$ is also a projective R-module.*

Proof. We need to show that the functor $Hom_R(P \otimes_R Q, \cdot)$ is exact. Recall from the definition of the tensor product that, for any R-module K, we have

$$Hom_R(P \otimes Q, K) \cong Bil_R(P \times Q, K)$$

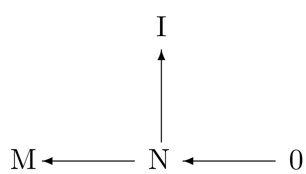
where $Bil_R(P \times Q, K)$ is the set of R-bilinear maps from $P \times Q$ to K. Since, trivially,

$$Bil_R(P \times Q, K) \cong Hom_R(P, Hom_R(Q, K))$$

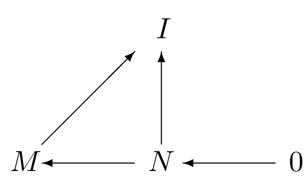
it follows that the functor $Hom_R(P \otimes_R Q, \cdot)$ is a composition of functors $Hom_R(Q, \cdot)$ and $Hom_R(P, \cdot)$. Since P and Q are projective, these two functors are exact. So is the functor $Hom_R(P \otimes_R Q, \cdot)$. The corollary is proved. □

3. Injective modules

The theory of injective R-modules is obtained by reversing arrows and replacing the letter P by I. An R-module I is injective if for every diagram of R-modules



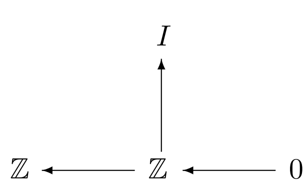
there exists a homomorphism $h : M \to I$ such that the following diagram is commutative.



In order to give some examples of injective modules, let us assume that $R = \mathbb{Z}$. In this case we have the following

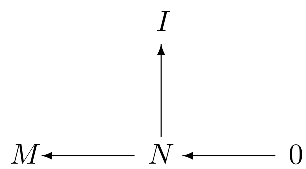
Proposition 3.1. *A $\mathbb{Z}$-module I is injective if an only if it is divisible.*

Proof. Let I be an injective module. We want to show that I is divisible, that is, for every x in I and n in $\mathbb{Z}$ there is y in I such that $x = ny$. Consider the diagram



where the vertical map is given by $1 \mapsto x$ and horizontal by $1 \mapsto n$. Then there exists $h : \mathbb{Z} \to I$ making the diagram commutative. Now $y = h(1)$ satisifies $ny = x$.

In the other direction, assume that I is divisible, and consider a diagram



For simplicity, assume that M is generated by N and another element m. If $M \cong N \oplus \mathbb{Z}m$, then m can be mapped to any element in I. Otherwise, there exists a smallest positive integer n such that nm is in N. Let x be the image of nm in I. Let y be in I such that $x = ny$. Now map m to y. In general, use Zorn's lemma. □

The previous characterization of injective $\mathbb{Z}$-modules shows that $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules.

Proposition 3.2. *Any $\mathbb{Z}$-module I is contained in an injective $\mathbb{Z}$-module.*

Proof. Let S be a set of generators of I and F_S a free $\mathbb{Z}$-module with a basis e_s for all s in S, and $f : F_S \to I$ as in the proof of Lemma 2.2. In particular, $I \cong F_S/K$ where K is the kernel of f. Let F'_S be the free $\mathbb{Q}$-module with the same basis e_s for all s in S. Then $K \subseteq F_S \subseteq F'_S$ and, therefore,

$$I \cong F_S/K \subseteq F'_S/K.$$

Note that F_S is divisible $\mathbb{Z}$-module. Since a quotient of a divisible module is again divisible module, it follows that F'_S/K is divisible and, therefore, injective. Proposition is proved. □

Note that for *every* R-module I, every exact sequence of R-modules $0 \to N \to M \to L \to 0$ gives rise to an exact sequence

$$0 \to Hom_R(L, I) \to Hom_R(M, I) \to Hom_R(N, I).$$

Proposition 3.3. *The following are equivalent:*

(i) *I is injective.*
(ii) *Every exact sequence $0 \to I \to M \to L \to 0$ splits. This means that $M \cong I \oplus L$.*
(iii) *For every exact sequence of R-modules $0 \to N \to M \to L \to 0$, the sequence*

$$0 \to Hom_R(L, I) \to Hom_R(M, I) \to Hom_R(N, I) \to 0$$

is exact. One says that the functor $Hom_R(\cdot, I)$ is exact!

Proof. The first and the last statement are trivially equivalent as exactness of the sequence is equivalent to surjectivity of the map from $Hom_R(M, I)$ to $Hom_R(N, I)$.

(i) implies (ii): Analogous to the case of projective modules.

(ii) implies (i): We need the following lemma:

Lemma 3.4. *Let I be an R-module. Then there exists an injective module Q such that $0 \to I \to Q$.*

Proof. Conisider I as a $\mathbb{Z}$-module. By Proposition 3.2, there exists an injective $\mathbb{Z}$-module $Q_{\mathbb{Z}}$ containing I. Note that

$$I \cong Hom_R(R, I) \subseteq Hom_{\mathbb{Z}}(R, Q_{\mathbb{Z}}).$$

Now put $Q = Hom_{\mathbb{Z}}(R, Q_{\mathbb{Z}})$. This is an R module, via the action $(rT)(r') = T(rr')$ for every T in Q. I claim that Q is an injective R module. To this end, note that, for every R-module K

$$Hom_R(K, Hom_{\mathbb{Z}}(R, Q_{\mathbb{Z}})) = Hom_{\mathbb{Z}}(K, Q_{\mathbb{Z}}).$$

This shows that the functor $Hom_R(\cdot, Q)$ is exact since $Hom_{\mathbb{Z}}(\cdot, Q_{\mathbb{Z}})$ is so. The lemma is proved. □

The rest of the proof is analogous to the case of projective modules. □

Remark: The lemma is the only difference between what we have done for projective modules and what we are doing for injective modules, except for reversing all arrows and replacing P by I. While it is very easy to see that any R-module is a quotient of a projective (even free) module, showing that any R-module is contained in an injective module is much harder. This is, I guess, an inherent feature of the category of R-modules.

Exercises

1) Prove that P_1 and P_2 are projective if and only if $P_1 \oplus P_2$ is.

2) If $0 \to K \to P \to M \to 0$ and $0 \to K' \to P' \to M \to 0$ are exact sequences of R-modules and P and P' are projective, show that $K \oplus P'$ is isomorphic to $K' \oplus P$.

3) Show that $\mathbb{Z}[\sqrt{2}]$ is not a projective $\mathbb{Z}[2\sqrt{2}]$-module. Hint: consider $\mathbb{Z}[\sqrt{2}] \otimes_{\mathbb{Z}[2\sqrt{2}]} \mathbb{Z}[\sqrt{2}]$.

4) Let R be a commutative ring with 1. Prove that the following two are equivalent

  (i) Every R-module is injective.
  (ii) Every R-module is projective.

This proof consists of two if and only if statements and the conclusion. I will NOT read anything longer.