Lecture 13

Given a nontrivial character $\chi : (\mathbb{Z}/f\mathbb{Z})^\times \to \mathbb{C}^\times$, we can extend $\chi$ to a function $\chi : \mathbb{Z} \to \mathbb{C}$ by defining $\chi(n) = 0$ if $(n, f) \neq 1$. (We shall apply this convention frequently below without formally restating it each time.) Let $L(\chi, s) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$. As we noted in Lecture 12, Lemma 24 implies that $L(\chi, s)$ converges and defines an analytic function in the region $\text{Re}(s) > 0$. This allows us to restate an improvement of the converse to Corollary 34 which we proved in Lecture 6.

**Theorem 65.** (Dirichlet’s Reinterpretation of Quadratic Reciprocity) Let $k$ be a quadratic field with discriminant $d$. Then there is a character $\chi : (\mathbb{Z}/d\mathbb{Z})^\times \to \{\pm 1\}$ (i.e., a character $\chi : (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^\times$ such that $\chi \neq 1$ and $\chi^2 = 1$) such that for every prime number $p$

1. $p$ splits in $k$ iff $\chi(p) = 1$
2. $p$ is inert in $k$ iff $\chi(p) = -1$
3. $p$ is ramified in $k$ iff $\chi(p) = 0$

Moreover, we have a factorization $\zeta_k(s) = \zeta(s)L(\chi, s)$ in the region $\text{Re}(s) > 1$, and $\zeta_k(s)$ has a meromorphic continuation to the region $\text{Re}(s) > 0$ whose unique pole is at the point $s = 1$ and is simple.

**Example.** If $d = \ell \equiv 1 \pmod{4}$ is a prime, then, as noted in Lecture 6, the character is exactly $\chi(n) = \left(\frac{n}{\ell}\right)$, as

1. $p$ splits in $k$ iff $\left(\frac{\ell}{p}\right) = \left(\frac{\ell}{p}\right) = 1$
2. $p$ is inert in $k$ iff $\left(\frac{\ell}{p}\right) = \left(\frac{\ell}{p}\right) = -1$
3. $p$ is ramified in $k$ iff $p = \ell$.

**Proof.** The algebraic properties were proved in Lecture 6. Furthermore, we demonstrated that for any prime number $p$ (ramified or unramified) the factor of $\zeta_k(s)$ corresponding to $p$ is

$$
\left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}
$$

so that

$$
\zeta_k(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}
$$

Since

$$
\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}
$$
it suffices to show that
\[
L(\chi, s) = \sum_n \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}
\]

Factor any positive integer \( n = \prod_i p_i^{a_i} \) so that \( \chi(n) = \prod_i \chi(p_i)^{a_i} \). Since
\[
(1 - \chi(p)p^{-s})^{-1} = 1 + \chi(p)p^{-s} + \chi(p^2)p^{-2s} + \cdots = \sum_{n=0}^{\infty} \chi(p^n)p^{-ns}
\]
a bit of bookkeeping implies the desired result.  

\[
\text{Corollary 66. (Class Number Formula for Quadratic k)} \text{ Let } k \text{ be a quadratic field with discriminant } d. \text{ If } d > 0, \text{ then the regulator is given by } R = \log |\epsilon| \text{ for some } \epsilon > 1 \text{ generating } A^\times/\{\pm 1\}. \text{ If } d < 0, \text{ then } R = 1. \text{ Moreover,}
\]
\[
L(\chi, 1) = h\kappa = \begin{cases} 
\frac{2h\log|\epsilon|}{\sqrt{d}} & \text{if } d > 0 \\
\frac{2\pi h}{w\sqrt{|d|}} & \text{if } d < 0
\end{cases}
\]

\[
\text{Proof. If } d > 0 \text{ then } r_1 = 2 \text{ and } r_2 = 0, \text{ so } A^\times \text{ has rank 1. With the notation of Lecture 12, let } H \subset \mathbb{R}^2 \text{ denote the hyperplane which contains the image } E \text{ of } A^\times \text{ under the map } \lambda : A^\times \to \mathbb{R}^2 \text{ given by } \epsilon \mapsto (\log |\epsilon|_{\nu_1}, \log |\epsilon|_{\nu_2}). \text{ Fix an element } \epsilon \in A^\times \text{ which maps to a basis element of } E. \text{ Without loss of generality, assume that } |\epsilon|_{\nu_1} > 1, \text{ as } \epsilon \neq \pm 1. \text{ Then } R = \log |\epsilon|_{\nu_1}, \text{ by definition. If } d < 0 \text{ then } A^\times \text{ has rank 0, and the definition of } R \text{ shows that } R \text{ is the "empty determinant" which is 1.}
\]

The final claim follows from considering the Taylor expansions of \( \zeta(s) \) and \( \zeta_k(s) \), and the facts that \( \text{Res}_{s=1} \zeta(s) = 1 \) and \( \text{Res}_{s=1} \zeta_k(s) = h\kappa \) by Lemma 54 and Theorem 61.  

\[
\text{Corollary 67. } L(\chi, 1) > 0
\]

\[
\text{Proof. Immediate.}
\]

We desire an independent evaluation of \( L(\chi, 1) \) for any nontrivial character \( \chi : (\mathbb{Z}/f\mathbb{Z})^\times \to \mathbb{C}^\times \). Our first tool shall be the Gauss sum of \( \chi \) which we define to be the complex number
\[
ge(\chi) = \sum_{a=1}^{f} \chi(a)e^{2\pi ia/f}
\]

\[
\text{Lemma 68. If } (n, f) = 1, \text{ then}
\]
\[
\chi(n)e(\chi) = \sum_{b=1}^{f} \chi(b)e^{2\pi ibn/f}
\]

where \( \chi \) denotes the conjugate character, i.e., \( \chi = \chi^{-1} \) as the image of \( \chi \) is contained in the set of roots of unity in \( \mathbb{C} \).
Proof. By making the substitution \( b \equiv an^{-1} \mod f \) so that \( nb \equiv a \mod f \) we have
\[
\chi(n)g(\chi) = \sum_{a=1}^{f} \chi(n)\overline{\chi(a)}e^{2\pi i an/f} = \sum_{b=1}^{f} \chi(b)e^{2\pi ibn/f}
\]
as desired. \( \square \)

By definition,
\[
g(\chi)L(\chi, s) = \sum_{n=1}^{\infty} g(\chi)\chi(n)n^{-s}
\]
If \((n, f) \neq 1\) then \(\chi(n) = 0\) and the corresponding terms in the series vanish. If \((n, f) = 1\), then the lemma applies, and it follows that
\[
g(\chi)L(\chi, s) = \sum_{\substack{b=1 \atop (f, b) = 1}}^{f} \chi(b) \left( \sum_{n=1}^{\infty} \frac{e^{2\pi ibn/f}}{n^s} \right) = \sum_{\substack{b=1 \atop (f, b) = 1}}^{f} \chi(b) \left( \sum_{n=1}^{\infty} \frac{(e^{2\pi ib/f})^n}{n^s} \right)
\]
and we have absolute convergence in the region \(\Re(s) > 1\). If \(\zeta = e^{2\pi i/f}\), then the formula becomes
\[
g(\chi)L(\chi, s) = \sum_{\substack{b=1 \atop (f, b) = 1}}^{f} \chi(b) \left( \sum_{n=1}^{\infty} \frac{(\zeta^b)^n}{n^s} \right) \tag{1}
\]

We recall that the branch of \(- \log(1 - x)\) defined on the region \(|x| < 1\) has power series expansion \(- \log(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n}\) and that the series actually converges to \(- \log(1 - x)\) for \(|x| = 1\) such that \(x \neq 1\). A lemma of Abel states that we may substitute the value \(s = 1\) into equation (1) which yields
\[
g(\chi)L(\chi, 1) = \sum_{\substack{b=1 \atop (f, b) = 1}}^{f} \chi(b)(- \log(1 - \zeta^b)) \tag{2}
\]

Lecture 14

Definition. A character \(\chi : (\mathbb{Z}/f\mathbb{Z})^\times \to \mathbb{C}^\times\) is primitive if there is no proper divisor \(f'\) of \(f\) such that \(\chi\) is induced by a character \(\chi' : (\mathbb{Z}/f'\mathbb{Z})^\times \to \mathbb{C}^\times\). It follows immediately that, if \(f\) is prime, then \(\chi\) is primitive iff \(\chi\) is nontrivial. A character \(\chi\) (not necessarily primitive) is said to be even if \(\chi(-1) = 1\), and is said to be odd if \(\chi(-1) = -1\).

Lemma 69. (Gauss’ Formula) If \(\chi\) is primitive, then \(g(\chi)g(\chi) = \chi(-1)f\). In particular, \(g(\chi) \neq 0\) and \(|g(\chi)| = \sqrt{f}\).

We shall prove this lemma in the case where \(f\) is prime.
Proof. Assume that $f = \ell$ is prime. For any function $F : \mathbb{Z}/\ell\mathbb{Z} \to \mathbb{C}$, the Fourier transform of $F$ is the function $\hat{F} : \mathbb{Z}/\ell\mathbb{Z} \to \mathbb{C}$ given by the formula

$$\hat{F}(b) = \sum_{a=1}^{\ell} F(a) e^{2\pi i ab/\ell}$$

In particular, a character $\chi : (\mathbb{Z}/\ell\mathbb{Z})^* \to \mathbb{C}^*$ can be extended to such a function, and we have

$$\hat{\chi}(b) = \sum_{a=1}^{\ell} \chi(a) e^{2\pi i ab/\ell}$$

For $b = 0$ the fact that $\chi$ is a nontrivial character implies that

$$\hat{\chi}(0) = \sum_{a=1}^{\ell-1} \chi(a) = 0$$

If $b \neq 0$, then we make the substitution $c \equiv ab \pmod{\ell}$ so that $a \equiv cb^{-1} \pmod{\ell}$ and

$$\hat{\chi}(b) = \sum_{c \neq 0} \chi(cb^{-1}) e^{2\pi ic/\ell} = \chi(b^{-1}) \sum_{c \neq 0} \chi(c) e^{2\pi ic/\ell} = \chi(b^{-1}) g(\chi) \tag{3}$$

Thus, $\chi = g(\chi) \chi^{-1}$. If $g(\chi) = 0$, then $\chi = 0$, but this cannot hold because of the Fourier inversion formula

$$\hat{(\hat{F})}(-a) = \ell F(a)$$

Applying formula (3) twice yields

$$\hat{\chi} = g(\chi) \hat{\chi}^{-1} = g(\chi) g(\chi^{-1}) \chi$$

and Fourier inversion implies that

$$g(\chi) g(\chi^{-1}) \chi(-a) = \chi(a) \ell$$

Letting $a = -1$ we get

$$\chi(-1) \ell = g(\chi) g(\chi^{-1}) \chi(1) = g(\chi) g(\chi)$$

as desired. If we conjugate the definition of $g(\chi)$ and make the substitution $a \to -a$, this yields

$$\overline{g(\chi)} = \sum_{a=1}^{\ell} \overline{\chi(a)} e^{-2\pi i a/\ell} = \sum_{a=1}^{\ell} \overline{\chi(-a)} e^{2\pi i a/\ell} = \overline{\chi(-1)} \sum_{a=1}^{\ell} \overline{\chi(a)} e^{-2\pi i a/\ell} = \overline{\chi(-1)} g(\overline{\chi})$$

so that

$$|g(\chi)| = |g(\chi) g(\chi)|^{1/2} = |g(\chi) \chi(-1) g(\chi)|^{1/2} = |\chi(-1) \chi(-1)|^{1/2} = \sqrt{\ell}$$

as desired. □
Assume that $\ell$ is a prime number and that $\chi$ is nontrivial. Then

$$\chi(n)g(\chi) = \sum_{b=1}^{\ell} \chi(b) e^{2\pi ibn/\ell}$$

for all $n \geq 1$: if $(n, \ell) = 1$ then this is Lemma 68, otherwise each side is zero. Thus

$$g(\chi)L(\chi, s) = \sum_{n \geq 1} \left( \sum_{b=1}^{\ell} \chi(b) e^{2\pi ibn/\ell} \right) n^{-s} = \sum_{b=1}^{\ell} \chi(b) \left( \sum_{n \geq 1} \frac{e^{2\pi ibn/\ell}}{n^s} \right)$$

$$= \sum_{b=1}^{\ell-1} \chi(b) \left( \sum_{n \geq 1} \frac{(\zeta^b)^n}{n^s} \right)$$

(4)

where $\zeta = e^{2\pi i/\ell}$. This series converges (conditionally) for $\Re(s) > 0$ by Lemma 64. The power series

$$-\log(1-x) = \sum_{n \geq 1} \frac{x^n}{n}$$

gives the branch of $\log$ given by

$$\log(z) = \log|z| + i\arg(z) \quad \text{for} \quad -\pi/2 < \arg(z) < \pi/2$$

Thus, substituting $s = 1$ into equation (4) yields

$$g(\chi)L(\chi, 1) = \sum_{b=1}^{\ell-1} \chi(b) \left( \sum_{n \geq 1} \frac{(\zeta^b)^n}{n} \right) = \sum_{b=1}^{\ell-1} \chi(b)(-\log(1 - \zeta^b))$$

To simplify the formula for $L(\chi, 1)$, we consider two cases.

If $\chi$ is even, then $\chi(-b) = \chi(b)$ for all $b$ so that

$$g(\chi)L(\chi, 1) = -\sum_{b=1}^{\ell-1} \chi(b) \log(1 - \zeta^b) = -\sum_{b=1}^{\ell-1} \chi(b) \log|1 - \zeta^b|$$

by our choice of the branch of $\log x$, since the argument terms cancel. By Lemma 69, this implies that

$$L(\chi, 1) = -\frac{g(\chi)}{\ell} \sum_{b=1}^{\ell-1} \chi(b) \log|1 - \zeta^b|$$

(5)

If $\chi$ is odd, then $\chi(-b) = -\chi(b)$ for all $b$ so that

$$g(\chi)L(\chi, 1) = -i \sum_{b=1}^{\ell-1} \chi(b) \arg(1 - \zeta^b)$$

as in the even case. Thus,

$$L(\chi, 1) = \frac{ig(\chi)}{\ell} \sum_{b=1}^{\ell-1} \chi(b) \arg(1 - \zeta^b)$$
This is not quite as nice as the final formula in the even case since \( \arg(1 - \zeta^b) \) is more complicated than \( \log |1 - \zeta^b| \). For \( b = 1, \ldots, \ell - 1 \) the fact that \( \arg \) is additive (up to multiple of \( 2\pi \)) implies that

\[
\arg(1 - \zeta^b) = \arg(\zeta^{b/2}(\zeta^{-b/2} - \zeta^{b/2})) = \arg(\zeta^{b/2}) + \arg(\zeta^{-b/2} - \zeta^{b/2})
\]

The complex number \( \zeta^{-b/2} - \zeta^{b/2} \) is purely imaginary, so its argument is \( \pm \pi/2 \). By definition, \( \zeta^{b/2} = e^{2\pi i b/2\ell} = e^{\pi ib/\ell} \) and therefore has argument \( \pi b/\ell \) which is in the interval \((0, \pi)\). It follows that, in order for \( \arg(1 - \zeta^b) \) to fall in the correct interval, \( \zeta^{-b/2} - \zeta^{b/2} \) must have argument \(-\pi/2\). Thus,

\[
\arg(1 - \zeta^b) = \pi \left( \frac{b}{\ell} - \frac{1}{2} \right)
\]

which implies that

\[
L(\chi, 1) = \frac{\pi i g(\chi)}{\ell} \sum_{b=1}^{\ell-1} \chi(b) \left( \frac{b}{\ell} - \frac{1}{2} \right)
\]  

(6)

We note that, if \( B_1(x) = x - \frac{1}{2} \) is the first Bernoulli polynomial, then

\[
\sum_{b=1}^{\ell-1} \chi(b) \left( \frac{b}{\ell} - \frac{1}{2} \right) = \sum_{b=1}^{\ell-1} \chi(b) B_1(b/\ell)
\]

which we shall denote by \( B_1, \chi \), and we may write

\[
L(\chi, 1) = \frac{\pi i g(\chi)}{\ell} B_1, \chi
\]

**Example.** Let \( k = \mathbb{Q}(\sqrt{\ell}) \) where \( \ell \) is a positive prime such that \( \ell \equiv 1 \pmod{4} \). Then Corollary 66 implies that

\[
\Res_{s=1} \zeta_k(s) = \frac{2h \log |\ell|}{\sqrt{\ell}} = L(\chi, 1)
\]

where \( \chi(b) = \left( \frac{b}{\ell} \right) \), as we noted in Lecture 6. Since \( \chi \) is real-valued, we see that \( \chi = \overline{\chi} \) and has order 2. By equation (5) it follows that

\[
L(\chi, 1) = \frac{g(\chi)}{\ell} \log \left| \prod_{\ell \mid \ell} \frac{1 - \zeta^b}{1 - \zeta^{b/\ell}} \right|
\]

We know that \( g(\chi)^2 = \ell \), so the coefficient is \( \pm \frac{1}{\sqrt{\ell}} \). In fact, \( g(\chi) = +\sqrt{\ell} \), but this is not easy to verify. (For a proof, see Borevich and Shafarevich, *Number Theory* Theorem 5.4.7.) If we let

\[
u = \frac{\prod_{\ell \mid \ell} (1 - \zeta^b)}{\prod_{\ell \mid \ell} (1 - \zeta^{b/\ell})}
\]

then this implies that

\[
\frac{2h \log |\ell|}{\sqrt{\ell}} = \frac{g(\chi)}{\ell} \log |\nu| = \frac{1}{\sqrt{\ell}} \log |\nu|
\]
Cancellation implies that

\[ 2h \log |\varepsilon| = -\log |u| \]

and exponentiation yields

\[ u = \pm e^{-2h} \]

**Example.** Let \( k = \mathbb{Q}(\sqrt{-\ell}) \) where \( \ell \) is a positive prime such that \( \ell \equiv 3 \pmod{4} \). Then

\[
\text{Res}_{s=1} \mathcal{Q}_k(s) = \frac{2\pi h}{w \sqrt{\ell}} = L(\chi, 1) = \frac{\pi ig(\chi)}{\ell} \sum_{b=1}^{\ell} \chi(b) \left( \frac{b}{\ell} - \frac{1}{2} \right)
\]

\[
= \frac{\pi ig(\chi)}{\ell} \left( \sum_{b=1}^{\ell} \chi(b)(b/\ell) - \sum_{b=1}^{\ell} \chi(b)(1/2) \right) = \frac{\pi ig(\chi)}{\ell} \sum_{b=1}^{\ell} \chi(b)(b/\ell)
\]

as

\[
\sum_{b=1}^{\ell} \chi(b)(1/2) = \frac{1}{2} \sum_{b=1}^{\ell} \chi(b) = 0
\]

Here \( \chi(b) = \left( \frac{b}{\ell} \right) \) as above, and \( g(\chi)^2 = -\ell \) so that \( g(\chi) = \pm i \sqrt{\ell} \). In fact, \( g(\chi) = +i \sqrt{\ell} \), so

\[
\frac{ig(\chi)}{\ell} = -\frac{1}{\sqrt{\ell}}
\]

and therefore

\[
2h \frac{w}{\ell} = \sum_{b=1}^{\ell} \chi(b)(b/\ell)
\]

It should be noted that \( w = 2 \) unless \( \ell = 3 \).

**Example.** Let \( k = \mathbb{Q}(i) \). We can recover the fact that the class number of \( k \) is \( h = 1 \) from the ideas above. Let \( \chi : (\mathbb{Z}/4\mathbb{Z})^\times \to \{\pm 1\} \) be the character described in Theorem 65, so that \( \mathcal{Q}_k(s) = \zeta(s)L(\chi, s) \). By Corollary 66,

\[
L(\chi, 1) = \frac{2\pi h}{w \sqrt{|\Delta|}} = \frac{\pi h}{4}
\]

By definition

\[
L(\chi, 1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots
\]

which lies between 1 and 2/3. It follows that \( h = 1 \).

**Proposition 70.** Let \( \ell \) be an odd prime number, \( \zeta = \zeta_\ell \) a primitive \( \ell \)th root of unity, and \( k = \mathbb{Q}(\zeta) \). Then

\[
\zeta_h(s) = \zeta(s) \prod_{\chi \neq 1} L(\chi, s)
\]

where the product is taken over all nontrivial characters \( \chi : (\mathbb{Z}/\ell\mathbb{Z})^\times \to \mathbb{C}^\times \).
Proof. The Euler products of each function show that the desired equation is
\[
\prod_p \prod_{p|P} (1 - N(P)^{-s})^{-1} = \prod_p (1 - p^{-s}) \prod_{\chi \neq 1, p \neq \ell} (1 - \chi(p)p^{-s})^{-1}
\]
and we check equality factor-by-factor at each prime. The prime \( \ell \) is completely ramified, so that \( \ell A = P^{f-1} \) and \( N(P) = \ell \). In the infinite product on the left, this contributes a factor of \((1 - \ell^{-s})^{-1}\). In the factorization of \( \zeta_k(s) \) we have a factor of \((1 - \ell^{-s})^{-1} \) contributed by \( \ell \), and no factor to contribute to any of the \( L(\chi, s) \). Thus, “over \( \ell \)” the left and right-hand sides agree.

Let \( p \neq \ell \), so that \( p \) is unramified. Then \( pA = P_1 \cdots P_g \), and \( N(P_i) = p^f \) where \( f \) is the order of \( p \) in \((\mathbb{Z}/\ell\mathbb{Z})^\times\). In particular, \( g = \frac{\ell - 1}{f} \). In \( \zeta_k(s) \), \( p \) contributes factors
\[
\prod_{i=1}^g (1 - p^{-f s})^{-1} = ((1 - p^{-f s})(\ell^{-1})/f)^{-1}
\]
On the right-hand side of the desired equation, \( p \) contributes
\[
(1 - p^{-s})^{-1} \prod_{\chi \neq 1} (1 - \chi(p)p^{-s})^{-1}
\]
and we want to know that these expressions are equal. Let \( X = p^{-s} \). Then the desired equality is
\[
(1 - X^f)^{(\ell-1)/f} = \prod_{\text{all } \chi} (1 - \chi(p)X)
\]
which we shall show is an identity on polynomials. We note that the zeroes of each side are \( f \)th roots of unity, so this is reasonable. The subgroup \( \langle p \rangle \subseteq (\mathbb{Z}/\ell\mathbb{Z})^\times \) has order \( f \). Let \( G \) denote the quotient \((\mathbb{Z}/\ell\mathbb{Z})^\times/\langle p \rangle\), which has order \( g = \frac{\ell - 1}{f} \). The short exact sequence
\[
1 \rightarrow \langle p \rangle \rightarrow (\mathbb{Z}/\ell\mathbb{Z})^\times \rightarrow G \rightarrow 1
\]
implies that the right-hand side of the desired equality is
\[
\prod_{\text{all chars.}} (1 - \chi(p)X) = \prod_{\text{all chars.}} (1 - \psi(p)X)^{(\ell-1)/f}
\]
and the factorization of cyclotomic polynomials implies that this is exactly the left-hand side of the desired equality. \( \square \)

Next, we have some consequences of the proposition.

**Corollary 71.** For nontrivial characters \( \chi : (\mathbb{Z}/\ell\mathbb{Z})^\times \rightarrow \mathbb{C}^\times \), \( L(\chi, 1) \neq 0 \).

*Proof.* Immediate from the proposition, as \( L(\chi, s) \) is regular at \( s = 0 \), and the fact that \( \zeta_k(s) \) and \( \zeta(s) \) both have simple pole at \( s = 1 \). \( \square \)

**Corollary 72.** (Dirichlet) If \( \ell \nmid a \), then there are infinitely many prime numbers \( p \) such that \( p \equiv a \pmod{\ell} \).
This is the statement that there are infinitely many primes in an arithmetic progression modulo \( \ell \). Before we prove the corollary, we need a pair of lemmas.

**Lemma 73.** For a nontrivial character \( \chi \), the function

\[
f_\chi(s) = \sum_{p \neq \ell} \frac{\chi(p)}{p^s}
\]

is bounded as \( s \to 1 \).

**Proof.** For \( \text{Re}(s) > 1 \), the power series expansion of \( \log(1 - x) \) implies that

\[
\log(L(\chi, s)) = \log \left( \prod_{p \neq \ell} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}\right) = \sum_{p \neq \ell} \sum_{n \geq 1} \frac{\chi(p)^n}{np^{ns}}
\]

The rest of the proof is similar to that of Corollary 56. \(\square\)

**Definition.** Let \( G \) be a finite abelian group and fix \( g \in G \). Define the characteristic function of \( g \), \( \text{char}(g) : G \to \mathbb{C} \), as \( g \mapsto 1 \) and \( h \mapsto 0 \) for \( h \neq g \).

**Lemma 74.** The characteristic function of \( g \) can be described by the formula

\[
\text{char}(g) = \frac{1}{\#G} \sum_{\chi} \chi(g)\chi
\]

**Proof of Lemma 74.** For \( h \in G \)

\[
\text{char}(g)(h) = \frac{1}{\#G} \sum_{\chi} \chi(g)\chi(h) = \frac{1}{\#G} \sum_{\chi} \chi(h/g)
\]

If \( h = g \), then this is

\[
\text{char}(g)(g) = \frac{1}{\#G} \sum_{\chi} \chi(1) = 1
\]

since the number of characters of \( G \) is exactly \( \#G \). If \( h \neq g \), then \( h/g \neq 1 \) and there exists a fixed character \( \chi' \) of \( G \) such that \( \chi'(h/g) \neq 1 \) (c.f. Borevich and Shafarevich *Number Theory*, p. 417), and of course \( \chi'(h/g) \neq 0 \). As \( \chi \) runs through all characters of \( G \), so does \( \chi' \chi \), so that

\[
\sum_{\chi} \chi(h/g) = \sum_{\chi} \chi'(h/g)\chi(h/g) = \chi'(h/g) \sum_{\chi} \chi(h/g)
\]

which implies that the sum is zero. \(\square\)

**Proof of Corollary 72.** In the notation of the lemmas, let \( G = (\mathbb{Z}/\ell\mathbb{Z})^\times \). Then

\[
\frac{1}{\ell - 1} \sum_{a} \overline{\chi(a)} f_\chi(s) = \frac{1}{\#G} \sum_{\chi} \sum_{p \neq \ell} \frac{\chi(p)}{p^s} = \sum_{p \neq \ell} \frac{1}{\#G} \sum_{\chi} \overline{\chi(a)}\chi(p)
\]

\[=
\sum_{p \neq \ell} \frac{1}{p^s} \text{char}(a)(p) = \sum_{p \equiv a \pmod{\ell}} \frac{1}{p^s}
\]
For each $\chi \neq 1$, $f_\chi(s)$ is bounded as $s \to 1$, and $f_1(s) \sim \log(\frac{1}{s-1})$. Thus, up to a bounded function

$$\sum_{p \equiv a \pmod{\ell}} \frac{1}{p^s} \sim \frac{1}{\ell - 1} \log \left( \frac{1}{s - 1} \right)$$

as $s \to 1$, which implies that $\sum_{p \equiv a} \frac{1}{p}$ is infinite. In particular, the number of terms in the sum must be infinite, as desired. \qed

Notice that the proof of the corollary gives us a stronger result. Up to a bounded function

$$\sum_{p \equiv a \pmod{\ell}} \frac{1}{p^s} \sim \frac{1}{\ell - 1} \log \left( \frac{1}{s - 1} \right)$$
as $s \to 1$. This says that the density of such primes is $1/(\ell - 1)$.

**Corollary 75. (Kummer’s Class Number Formula) For the cyclotomic field $k = \mathbb{Q}(\zeta_\ell)$**

$$\left( \prod_{\chi \, \text{odd}} \frac{\pi i g(\chi)}{\ell} B_{1,\chi} \right) \cdot \left( \prod_{\chi \, \text{even}} \frac{-g(\chi)}{\ell} \sum_{b=1}^{\ell} \chi(b) \log |1 - \zeta_\ell^b| \right) = \frac{2 \pi (\ell - 1)}{\ell (\ell - 2)/2} \cdot \frac{hR}{2\ell}$$

**Proof.** The right-hand side of the desired equality is

$$\text{Res}_{s=1} \zeta_k(s) = \text{Res}_{s=1} \left( \zeta(s) \prod_{\chi \neq 1} L(\chi, s) \right) = \text{Res}_{s=1} \left( \prod_{\chi \neq 1} L(\chi, s) \right)$$

which is the left-hand side by Theorem 61. \qed

**Lecture 15**

We define the gamma function

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

for $\text{Re}(s) > 0$. By integrating by parts, we see that $\Gamma(s)$ satisfies the functional equation $\Gamma(s + 1) = s \Gamma(s)$. Thus, for $\text{Re}(s) > -1$, we may define

$$\Gamma(s) = \frac{\Gamma(s + 1)}{s}$$
to obtain a meromorphic continuation to the region $\text{Re}(s) > -1$, with simple pole at $s = 0$. We may continue this process inductively to obtain a meromorphic continuation to all of $\mathbb{C}$ with only simple poles located at $s = 0, -1, -2, \ldots$. Note that $\Gamma(1) = 1$, so that $\Gamma(m) = (m - 1)!$ for $m = 1, 2, 3, \ldots$.

**Theorem 76. (Hecke’s Thesis) Let $k$ be a number field.**
1. \( \zeta_k(s) \) has a meromorphic continuation all of \( \mathbb{C} \), with a unique pole at \( s = 1 \) which is simple.

2. If we define
\[
\xi_k(s) = \zeta_k(s)(\pi^{-s/2}\Gamma(s/2))^{\nu_1}((2\pi)^{-s}\Gamma(s))^{\nu_2}
\]
then \( \xi_k(s) \) has a meromorphic continuation to all of \( \mathbb{C} \), with unique poles at \( s = 0, 1 \) which are simple. Furthermore, \( \xi_k(s) \) satisfies the functional equation
\[
\xi_k(s) = |d|^{1-2s}\xi_k(1-s)
\]
where \( d \) is the discriminant of \( k \).

For a complete proof, see Weil *Basic Number Theory*, Chapter VII. We shall continue with these ideas below. First, we give some motivation for the Gamma factors in this product by considering curves over finite fields.

Let \( F_q \) denote the finite field with \( q = p^m \) elements, and let \( X \) be a complete, nonsingular curve over \( F_q \). Let \( k = F_q(X) \) be the field of rational functions on \( X \), and let \( A \subset k \) be the integral domain of functions regular outside a single place \( \infty \). We define the zeta function of \( A \) as
\[
\zeta_A(s) = \sum_{I \subseteq A} \frac{1}{\mathcal{N}(I)^s} = \prod_{P \in \mathcal{A}, P \neq 0} \left( 1 - \frac{1}{\mathcal{N}(P)^s} \right)^{-1}
\]
for \( \text{Re}(s) > 1 \), where the product is taken over all nonzero prime ideals of \( A \).

For example, if \( X = \mathbb{P}^1 = \mathbb{P}^1_{F_q} \), then \( k = F_q(X) = F_q(t) \supset F_q[t] = A \). The nonzero prime ideals \( P \) of \( A \) are in bijection with the collection of irreducible, monic polynomials \( f(t) \) over \( F_q \). Under this correspondence, if \( f(t) \) has degree \( d \), then \( \mathcal{N}(P) = q^d \). Since \( F_q \) sits inside \( A/P \) for every such prime, we see that every \( P \) has characteristic \( p \). (This is true in general, of course.) Furthermore, \( A^\times = F_q^\times \).

If \( q \) is odd and \( X \) is an elliptic curve, say given by the equation \( y^2 = x^3 + Ax + B \), then
\[
A = F_q[x,y]/(y^2 - x^3 - Ax - B)
\]
and \( k = F_q(X) \) is the quotient field of \( A \).

Let \( X, k \) and \( A \) be as above. Since \( F_q \hookrightarrow A \subseteq K \), \( A \) has characteristic \( p \). For every prime ideal \( P \), \( \mathcal{N}(P) = q^d \) for some positive integer \( d \), which we call the degree of \( P \) and denote by \( d = \text{deg}(P) \). We define the expression \( Z_A(T) \) by
\[
Z_A(T) = \prod_{P \neq 0} (1 - T^\text{deg}(P))^{-1}
\]
and note that, by definition, \( \zeta_A(s) = Z_A(q^{-s}) \). As in the number field situation, we can use \( P \) to define a valuation on \( k \) by
\[
|\alpha|_P = \mathcal{N}(P)^{-\text{ord}_P(\alpha)}
\]
For a valuation \( v \) on \( k \), let \( A_v \) denote the ring of elements \( \alpha \) of \( k \) such that \( v(\alpha) \leq 1 \). Then \( A_v \) is a local ring with maximal ideal equal to the set of elements \( \alpha \) of \( k \) such that \( v(\alpha) < 1 \). In addition, \( A_v \) is a discrete valuation ring so that the maximal ideal is principal, generated by a uniformizing parameter which we denote by \( \pi_v \). The degree of \( v \) is defined to be the dimension of the quotient \( A_v / \pi_v A_v \) over \( \mathbb{F}_q \), and this is exactly the degree of the prime ideal \( P \) which induced \( v \). We define the complete zeta function of \( X \)

\[
Z_X(T) = \prod_v (1 - T^{\deg(v)})^{-1} = Z_A(T)(1 - T^{\deg(\infty)})^{-1}
\]

by using all the valuations of \( k \), including the valuation at \( \infty \), which does not correspond to a prime ideal of \( A \). Using the power series \( -\log(1 - x) = \sum_{n \geq 1} \frac{x^n}{n} \), which holds for \( |x| < 1 \), we see that in \( \mathbb{Q}[T] \)

\[
\log(Z_X(T)) = \sum_v \sum_{n \geq 1} \frac{T^{n \deg(v)}}{n} = \sum_v \sum_{n \geq 1} \frac{\deg(v) T^{n \deg(v)}}{n \deg(v)} = \sum_{n \geq 1} \frac{T^n}{n} A_n
\]

where

\[
A_n = \sum_{\deg(v) \mid n} \deg(v) \in \mathbb{Z}
\]

A prime ideal \( P \) is determined by an orbit of the action of \( G = \text{Gal}(\mathbb{F}_q / \mathbb{F}_q) \) on sets of points of \( \mathbb{X} = X(\mathbb{F}_q) \) (the points of \( X \) defined over \( \mathbb{F}_q \)) and the degree of \( P \) is the number of elements in the orbit. Fix a point \( x \) of \( X \). \( G \) acts on \( \mathbb{X} \) and therefore determines an orbit of \( x \). The orbit is finite and appears as a set of points in \( X(\mathbb{F}_{q^m}) \) for some \( m \). The sum \( d = \sum_{\sigma \in G} x^\sigma \) is a prime divisor on \( X \) and therefore determines a prime ideal of \( A \). It follows that \( A_n = \#X(\mathbb{F}_{q^n}) \) and so

\[
\log(Z_X(T)) = \sum_{n \geq 1} \frac{T^n}{n} (\#X(\mathbb{F}_{q^n}))
\]

Exponentiation yields the equation

\[
Z_X(T) = \exp \left( \sum_{n \geq 1} \frac{T^n}{n} (\#X(\mathbb{F}_{q^n})) \right)
\]

For example, if \( X = \mathbb{P}^1 \), then \( \#X(\mathbb{F}_{q^n}) = q^n + 1 \) and

\[
\log(Z_X(T)) = \sum_{n \geq 1} \frac{(q^n + 1)T^n}{n} = \sum \frac{T^n}{n} + \sum \frac{(qT)^n}{n} = \log(1 - T)^{-1} + \log(1 - qT)^{-1} = \log \left( \frac{1}{(1 - T)(1 - qT)} \right)
\]

so that

\[
Z_{\mathbb{P}^1}(T) = \frac{1}{(1 - T)(1 - qT)}
\]
and the substitution $T = q^{-s}$ implies that

$$\zeta_X(s) = \frac{1}{(1-q^{-s})(1-q^{1-s})}$$

Notice that this is a rational function in $q^{-s}$, which is meromorphic with poles only where $q^{-s} = 1$ and $q^{1-s} = 1$. This is similar to Theorem 76.

In general, $Z_X(T) = \frac{P(T)}{(1-T)(1-qT)}$, where $P(T)$ is a polynomial in $T$ with integer coefficients having degree $2g$ where $g$ is the genus of $X$. More specifically,

$$P(T) = 1 + \cdots + q^g T^{2g} = \prod_{i=1}^{2g} (1 - \alpha_i T)$$

for some algebraic integers $\alpha_i$, and we may write

$$\exp\left( \sum_{n \geq 1} \frac{T^n}{n} (\# X(F_{q^n})) \right) = Z_X(T) = \exp\left( \sum_{n \geq 1} \frac{\# X(F_{q^n}) T^n}{n} \right)$$

$$= \exp\left( \# X(F_q)T + \# X(F_{q^2}) \frac{T^2}{2} + \cdots \right)$$

If $X$ is an elliptic curve, then $P(T) = 1 - aT + qT^2$ since $g_X = 1$. The only unknown coefficient is $a$. In fact, $a = 1 + q - \# X(F_q)$, as can easily be verified by checking the coefficient of $T$ in $Z_X(T)$.

Now, we return to the case where $k$ is a number field. The definition of $\xi_k(s)$ may be restated, as follows. Every valuation $v$ on $k$ corresponds to a factor $\zeta_v(s)$ in the Euler product of $\zeta_k(s)$. For the finite valuations (those corresponding to nonzero prime ideals of $A$) $\zeta_v(s) = (1 - N(P)^{-s})^{-1}$, which is exactly the factor of $\zeta_k(s)$ corresponding to $P$. For a valuation arising from a real place, $\zeta_v(s) = \pi^{-s/2} \Gamma(s/2)$, which is meromorphic in $\mathbb{C}$ with (simple) poles at $s = 0, -2, -4, \ldots$. For a valuation arising from a complex place, $\zeta_v(s) = (2\pi)^{-s} \Gamma(s)$, which is meromorphic in $\mathbb{C}$ with (simple) poles at $s = 0, -1, -2, \ldots$.

We shall prove part of Theorem 76 for the case $k = \mathbb{Q}$. In this case, $d = 1$, $r_1 = 1$ and $r_2 = 0$. As with the zeta function of $\mathbb{Q}$, we denote $\xi_k(s)$ as $\xi(s)$.

**Theorem 77.** The function

$$\xi(s) = \zeta(s) \pi^{-s/2} \Gamma(s/2)$$

has a meromorphic continuation to all of $\mathbb{C}$ with simple poles at $s = 0, 1$. Furthermore, $\xi(s)$ satisfies the functional equation $\xi(1-s) = \xi(s)$.

**Proof.** For a fixed positive integer, we make the substitution $t = n^2 \pi x$ into the definition of
\[ \Gamma(s/2). \text{ Here, } \frac{dt}{t} = \frac{dx}{x} \text{ so that} \]

\[
\begin{align*}
\Gamma(s/2) &= \int_0^\infty t^{s/2} e^{-t} \frac{dt}{t} \\
&= \int_0^\infty (n^2 \pi x)^{s/2} e^{-n^2 \pi x} \frac{dx}{x} \\
&= \int_0^\infty n^{-s} s^{s/2} x^{s/2} e^{-n^2 \pi x} \frac{dx}{x}
\end{align*}
\]

\[ n^{-s} n^{-s/2} \Gamma(s/2) = \int_0^\infty x^{s/2} e^{-n^2 \pi x} \frac{dx}{x} \]

and summing over all \( n \geq 1 \) yields

\[
\xi(s) = \zeta(s) n^{-s/2} \Gamma(s/2) \\
= \pi^{-s/2} \Gamma(s/2) \sum_n \frac{1}{n^s} \\
= \sum_{n\geq1} \pi^{-s/2} \Gamma(s/2) n^{-s} \\
= \sum_{n\geq1} \int_0^\infty x^{s/2} e^{-n^2 \pi x} \frac{dx}{x}
\]

which converges absolutely for \( \text{Re}(s) > 1 \), so we may interchange the sum and integral

\[
\xi(s) = \int_0^\infty x^{s/2} \left( \sum_{n\geq1} e^{-n^2 \pi x} \right) \frac{dx}{x}
\]

Let \( \omega(x) = \sum_{n\geq1} e^{-n^2 \pi x} \).

Here, we introduce Jacobi’s theta function. For \( x > 0 \), let

\[
\theta(x) = \sum_{n\in\mathbb{Z}} e^{-n^2 \pi x} = 1 + 2\omega(x)
\]

If we write \( f(t) = e^{-t^2 \pi x} \) and let \( \hat{f}(t) \) denote the Fourier transform of \( f \), then by the Poisson summation formula

\[
\theta(x) = \sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)
\]

Furthermore, the identity

\[
\hat{f}(t) = \frac{e^{-t^2 \pi/x}}{x^{1/2}}
\]

implies that

\[
\theta(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) = \frac{\theta(x^{-1})}{x^{1/2}} \tag{7}
\]

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which is Jacobi’s famous theta identity. Thus,
\[
\xi(s) = \int_0^\infty x^{s/2} \omega(x) \frac{dx}{x} = \int_0^1 x^{s/2} \omega(x) \frac{dx}{x} + \int_1^\infty x^{s/2} \omega(x) \frac{dx}{x}
\]
we substitute \( x \mapsto x^{-1} \) into the first integral
\[
\int_1^\infty x^{-s/2} \omega(x^{-1}) \frac{dx}{x} + \int_1^\infty x^{s/2} \omega(x) \frac{dx}{x}
\]
The fact that \( \omega(x) = \frac{1}{2}(\theta(x) - 1) \) implies that
\[
\omega(x^{-1}) = \frac{1}{2}(\theta(x^{-1}) - 1) = \frac{1}{2}(x^{1/2} \theta(x) - 1) = x^{1/2} \omega(x) - \frac{1}{2} + \frac{x^{1/2}}{2}
\]
so that
\[
\xi(s) = \int_1^\infty x^{1/2} \omega(x) x^{-s/2} \frac{dx}{x} + \int_1^\infty x^{s/2} \omega(x) \frac{dx}{x}
\]
\[
- \frac{1}{2} \int_1^\infty x^{s/2} \frac{dx}{x} + \frac{1}{2} \int_1^\infty x^{1/2-s/2} \frac{dx}{x}
\]
\[
= \int_1^\infty \left( x^{(1-s)/2} + x^{s/2} \right) \omega(x) \frac{dx}{x} - \frac{1}{s} + \frac{1}{s-1}
\]
\[
= \frac{1}{s(s-1)} + \int_1^\infty \left( x^{(1-s)/2} + x^{s/2} \right) \omega(x) \frac{dx}{x}
\]
To see that \( \xi(s) \) has the proper meromorphic continuation, it suffices to show that the final integral has an analytic continuation to all of \( \mathbb{C} \). The essential fact is that, as \( x \to \infty \), \( \omega(x) \) decays exponentially so the integral converges. To see that \( \xi(s) \) satisfies the desired functional equation, we simply observe that if we replace \( s \) by \( 1 - s \) in the final expression for \( \xi(s) \), it remains the same.

\[\square\]

**Lecture 16**

Next, we will discuss the analytic continuation of \( L(\chi, s) \) and a functional equation satisfied by \( L(\chi, s) \). Let \( \tau = x + yi \) be a complex number with \( y > 0 \), and let \( q = e^{2\pi i \tau} \). We define a function \( \Theta(\tau) \) similar to Jacobi’s theta function, as
\[
\Theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau} = \sum_{n \in \mathbb{Z}} q^{n^2}
\]
It readily follows from the definitions that \( \Theta(iy) = \theta(y) \).

**Example.** Let \( k = \mathbb{Q}(i) \). Then \( A = \mathbb{Z}[i] \) and
\[
\mathcal{Q}_k(s) = \sum_{a \in A \setminus \{0\}} \frac{1}{N(a)^s} = \frac{1}{4} \sum_{(a,b) \neq (0,0)} \frac{1}{(a^2 + b^2)^s}
\]
as \( \mathbb{N}(a) = a^2 + b^2 \). If we define \( f(\tau) \) as

\[
f(\tau) = \sum_{(a,b)} q^{a^2 + b^2} = (\Theta(\tau))^2
\]

then \( f \) satisfies an equation similar to the equation satisfied by \( \theta \): \( f(iy) = \frac{1}{y} f(iy^{-1}) \).

Assume that \( \chi : (\mathbb{Z}/f\mathbb{Z})^\times \to \mathbb{C}^\times \) is a primitive character. In order to uncover the functional equations for \( L(\chi, s) \) we introduce functions \( \xi(\chi, s) \) as we did for \( \zeta_k(s) \). There are two cases.

If \( \chi \) is even, then we define

\[
\xi(\chi, s) = \pi^{-s/2} \Gamma(s/2) L(\chi, s) f^{s/2}
\]

In this case, let \( w_\chi = g(\chi)/\sqrt{T} \), where \( g(\chi) \) is the Gauss sum from Lecture 13. Then \( |w_\chi| = 1 \) and \( \xi(\chi, s) \) has a holomorphic continuation to all of \( \mathbb{C} \) which satisfies the functional equation

\[
\xi(\chi, s) = w_\chi \xi(\overline{\chi}, 1 - s)
\]

If \( \chi \) is odd, then we define

\[
\xi(\chi, s) = \pi^{-(s+1)/2} \Gamma((s+1)/2) L(\chi, s) f^{s/2}
\]

Let \( w_\chi = g(\chi)/i\sqrt{T} \). Again \( |w_\chi| = 1 \) and \( \xi(\chi, s) \) satisfies the functional equation

\[
\xi(\chi, s) = w_\chi \xi(\overline{\chi}, 1 - s)
\]

If \( \chi^2 = 1 \), then we say that \( \chi \) is a \textit{quadratic} character. In this case \( \overline{\chi} = \chi \) and it can be shown that

\[
g(\chi) = \begin{cases} 
\sqrt{T} & \text{if } \chi \text{ is even} \\
i\sqrt{T} & \text{if } \chi \text{ is odd}
\end{cases}
\]

(c.f., Borevich and Shafarevich \textit{Number Theory}, Theorem 5.4.7) so that \( w_\chi = 1 \). Thus, the simpler functional equation

\[
\xi(\chi, s) = \xi(\chi, 1 - s)
\]

is satisfied. We shall only prove that \( \chi \) satisfies the desired functional equation when \( \chi \) is an even character. First, however, we note that there are ways of slightly changing the functions \( \xi_k(s) \) to get the same meromorphic continuation properties and simpler functional equations.

For example, assume that \( k \) is a quadratic field, with character \( \chi \) as is Theorem 65. It can be shown that \( k \) is real if and only if \( \chi \) is even (c.f., Borevich and Shafarevich \textit{Number Theory}, Chapter 5, 5.84). Furthermore, it can be shown that every quadratic character \( (\mathbb{Z}/f\mathbb{Z})^\times \to \{\pm 1\} \) corresponds to a quadratic field in this manner. If \( k \) is a real field, then we define

\[
\xi_k^2(s) = (\pi^{-s/2} \Gamma(s/2))^2 f^{s/2} \zeta_k(s)
\]
which is very similar to the definition of \( \xi_k(s) \). It is straightforward to check that \( \xi_k(s) \) satisfies the functional equation

\[
\xi_k(s) = |d|^{-s} \xi_k(1 - s)
\]

if and only if \( \xi^*_k(s) \) satisfies the functional equation

\[
\xi^*_k(s) = \xi^*_k(1 - s)
\]

which is simpler. To see that this second equation is satisfied, notice that the definitions of \( \xi^*_k(s), \xi(s) \) and \( \xi(\chi, s) \) imply that

\[
\xi^*_k(s) = \xi(s) \xi(\chi, s)
\]

In the quadratic case, \( \xi(\chi, s) = \xi(\chi, 1 - s) \) and by Theorem 77, \( \xi(1 - s) = \xi(s) \). Thus,

\[
\xi^*_k(s) = \xi(s) \xi(\chi, s) = \xi(1 - s) \xi(\chi, 1 - s) = \xi^*_k(1 - s)
\]

as desired. A similar change works for the case when \( k \) is imaginary.

Before we prove the real case, we need some notation and a lemma. Let \( g \) be an element of the real Schwarz space, so that \( g \) is \( C^\infty \), and \( g \) and all its derivatives decrease rapidly at \( \infty \). The Fourier transform of \( g \), denoted \( \hat{g} \), is given by

\[
\hat{g}(y) = \int_{\mathbb{R}} g(x) e^{-2\pi i xy} dx = \int_{\mathbb{R}} g(x) \langle x, y \rangle, dx
\]

where \( \langle x, y \rangle = e^{2\pi i xy} \). (The assumption that \( g \) is in the real Schwarz space implies that we may take iterated Fourier transforms of \( g \).) Define \( \Theta_g(t; x, y) \) as

\[
\Theta_g(t; x, y) = \sum_{n \in \mathbb{Z}} g((x + n)t) \langle y, n \rangle
\]

It is immediate from this definition that \( \Theta_g(t; 0, 0) = \sum_n g(nt) \).

**Lemma 78.** *(Generalized Jacobi Identity)*

\[
\Theta_g(t; x, y) = \frac{\langle x, y \rangle}{t} \Theta_g(t^{-1}; -y, x)
\]

**Proof.** To simplify notation, let \( \hat{G}(w) = g((x + w)t) \langle y, w \rangle \). Then by the Poisson summation formula

\[
\Theta_g = \sum_{n \in \mathbb{Z}} \hat{G}(n) = \sum_{n \in \mathbb{Z}} \hat{G}(n)
\]

so we calculate \( \hat{G} \). By definition

\[
\hat{G}(u) = \int_{\mathbb{R}} g((x + w)t) \langle y, w \rangle \langle u, w \rangle dw = \int_{\mathbb{R}} g((x + w)t) \langle u - y, w \rangle dw
\]

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and we make the substitution $z = (x + w)t$ so that $w = \frac{z}{t} - x$, $dw = \frac{dz}{t}$ and

\[
\hat{G}(u) = \int_{\mathbb{R}} g(z) \left( u - y, \frac{z}{t} - x \right) \frac{dz}{t}
\]

\[
= \frac{1}{t} \overline{(x, y)} \overline{u} = \int_{\mathbb{R}} g(z) \left( z, \frac{u - y}{t} - x \right) dz
\]

\[
= \frac{\overline{(x, y)}(u, x)}{t} \hat{g} \left( \frac{u - y}{t} \right)
\]

Substituting $u = n$ and summing over all $n$ yields

\[
\Theta_{\beta}(t; x, y) = \sum_{n \in \mathbb{Z}} \hat{G}(n) = \sum_{n} \frac{\overline{(x, y)}(n, x)}{t} \hat{g} \left( \frac{n - y}{t} \right)
\]

\[
= \frac{\overline{(x, y)}}{t} \sum_{n} \langle n, x \rangle \hat{g} \left( \frac{n - y}{t} \right) = \frac{\overline{(x, y)}}{t} \Theta_{\beta}(t^{-1}; -y, x)
\]

as desired. \qed

**Corollary 79.** For real numbers $y$ and $t$

\[
\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t + 2\pi i ny} = \frac{1}{\sqrt{t}} \sum_{n \in \mathbb{Z}} e^{-\pi (ny)^2 / t}
\]

**Proof.** Let $g(\alpha) = e^{-\pi \alpha^2}$. Then it is a standard fact that $\hat{g} = g$, and Lemma 78 implies that

\[
\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t + 2\pi i ny} = \Theta_{\beta}(\sqrt{t}; 0, y)
\]

\[
= \frac{(0, y)}{\sqrt{t}} \Theta_{\beta}(t^{-1/2}; -y, 0)
\]

\[
= \frac{(0, y)}{\sqrt{t}} \Theta_{\beta}(t^{-1/2}; -y, 0)
\]

\[
= \frac{1}{\sqrt{t}} \sum_{n \in \mathbb{Z}} e^{-\pi (ny)^2 / t}
\]

as desired. \qed

**Theorem 80.** Assume that $\chi : (\mathbb{Z}/f\mathbb{Z})^\times \to \{\pm 1\}$ is a primitive, even character. Let

\[
\xi(\chi, s) = \pi^{-s/2} \Gamma(s/2) L(\chi, s) f^{s/2}
\]

and let $w_{\chi} = g(\chi)/\sqrt{f}$, where $g(\chi)$ is the Gauss sum. Then $\xi(\chi, s)$ has a holomorphic continuation to all of $\mathbb{C}$, and satisfies the functional equation

\[
\xi(\chi, s) = w_{\chi} \xi(\overline{\chi}, 1 - s)
\]

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Proof. Start with the identity
\[
f^{s/2} \pi^{-s/2} \Gamma(s/2) n^{-s} = \int_0^\infty e^{-\pi n^2 x / f} x^{s/2} \frac{dx}{x}
\]
which is proved as in the proof of Theorem 77. Let
\[
\theta_\chi(x) = \sum_{n \in \mathbb{Z}} \chi(n) e^{-\pi n^2 x / f}
\]
so that, when we sum over all \( n \geq 1 \) we have
\[
\xi(\chi, s) = \pi^{-s/2} \Gamma(s/2) L(\chi, s) f^{s/2}
\]
\[
= \sum_{n \geq 1} \chi(n) n^{-s} \pi^{-s/2} \Gamma(s/2) f^{s/2}
\]
\[
= \int_0^\infty x^{s/2} \left( \sum_{n \geq 1} \chi(n) e^{-\pi n^2 x / f} \right) \frac{dx}{x}
\]
\[
= \int_0^\infty x^{s/2} \left( \frac{1}{2} \theta_\chi(x) \right) \frac{dx}{x}
\]
as \( \chi(-n) = \chi(n) \) and \( \chi(0) = 0 \)
\[
= \frac{1}{2} \left( \int_0^1 x^{s/2} \theta_\chi(x) \frac{dx}{x} + \int_1^\infty x^{s/2} \theta_\chi(x) \frac{dx}{x} \right) \tag{8}
\]
The first integral converges for Re(\( s \)) > 0, while the second converges for all \( s \). To obtain the desired properties, we need to find an equation relating \( \theta_\chi(x) \) to \( \theta_\chi(x^{-1}) \).

By Lemma 68,
\[
\chi(n) g(\chi) = \sum_{b=1}^f \chi(b) e^{2\pi ib n / f}
\]
so that
\[
g(\chi) \theta_\chi(x) = \sum_{n \in \mathbb{Z}} \left( \sum_{b=1}^f \chi(b) e^{2\pi ib n / f} \right) e^{-\pi n^2 x / f}
\]
\[
= \sum_{b=1}^f \chi(b) \left( \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x / f + (2\pi ib n / f)} \right)
\]
and the substitutions \( t = x / f \) and \( y = b / f \) yield
\[
= \sum_{b=1}^f \chi(b) \left( \frac{1}{\sqrt{x / f}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 (t / f)^2 / (xf)} \right)
\]
\[
= \sqrt{\frac{x}{f}} \sum_{b=1}^f \chi(b) \sum_{n \in \mathbb{Z}} e^{-\pi (n+b/f)^2 / (xf)}
\]
and the replacement $m = nf + b$ gives

$$\sum_n \chi(m) e^{-\pi m^2/(fx)} = \sqrt{\frac{f}{x}} \sum_{n \in \mathbb{Z}} \chi(m) e^{-\pi m^2/(fx)} = \sqrt{\frac{f}{x}} \theta_\chi(x^{-1})$$

If we substitute this final expression into (8), we find

$$\xi(\chi, s) = \frac{1}{2} \left( \int_1^\infty x^{(1-s)/2} \sqrt{\frac{f}{g(x)}} \theta_\chi(x) \frac{dx}{x} + \int_1^\infty x^{s/2} \theta_\chi(x) \frac{dx}{x} \right)$$

which gives our analytic continuation. Furthermore, the substitution $s \mapsto 1 - s$ gives the desired functional equation.

We have shown that $\zeta_k(s)$ has order of vanishing $-1$ at $s = 1$ and residue

$$\text{Res}_{s=1} \zeta_k(s) ds = \frac{2^r (2\pi)^{r_2} hR}{\sqrt{|d|}} \frac{w}{w}$$

We shall use Hecke’s functional equation for $\xi_k(s)$ to calculate the order of vanishing and residue at $s = 0$. By Theorem 76, we know that

$$\zeta_k(s)(\pi^{-s/2} \Gamma(s/2))^{r_1} (2\pi)^{-s} \Gamma(s) r_2 = \xi_k(s) = |d|^{1-s} \xi_k(1-s)$$

and that $\xi_k(s)$ has a simple pole (i.e., vanishing of order $-1$) at $s = 0$. Also, $\Gamma(s)$ has a simple pole at $s = 0$. Thus,

$$-1 = \text{ord}_{s=0} \xi_k(s) = \text{ord}_{s=0} \zeta_k(s) + r_1 \text{ord}_{s=0} \Gamma(s) + r_2 \text{ord}_{s=0} \Gamma(s) = \text{ord}_{s=0} \zeta_k(s) - r_1 - r_2$$

so that

$$\text{ord}_{s=0} \zeta_k(s) = r_1 + r_2 - 1 = \text{rank}(A^\times)$$

and

$$\zeta_k(s) = cs^{r_1+r_2-1} + O(s^{r_1+r_2})$$

A further computation shows that $c = -\frac{hR}{w}$.

3.99–3.11.99