Lecture 13

Given a nontrivial character $\chi : (\mathbb{Z}/f\mathbb{Z})^\times \to \mathbb{C}^\times$, we can extend χ to a function $\chi : \mathbb{Z} \to \mathbb{C}$ by defining $\chi(n) = 0$ if $(n, f) \neq 1$. (We shall apply this convention frequently below without formally restating it each time.) Let $L(\chi, s) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$. As we noted in Lecture 12, Lemma 24 implies that $L(\chi, s)$ converges and defines an analytic function in the region $\mathrm{Re}(s) > 0$. This allows us to restate an improvement of the converse to Corollary 34 which we proved in Lecture 6.

Theorem 65. *(Dirichlet's Reinterpretation of Quadratic Reciprocity) Let k be a quadratic field with discriminant d. Then there is a character $\chi : (\mathbb{Z}/d\mathbb{Z})^\times \to \{\pm 1\}$ (i.e., a character $\chi : (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^\times$ such that $\chi \neq 1$ and $\chi^2 = 1$) such that for every prime number p*

1. *p splits in k iff $\chi(p) = 1$*
2. *p is inert in k iff $\chi(p) = -1$*
3. *p is ramified in k iff $\chi(p) = 0$*

Moreover, we have a factorization $\zeta_k(s) = \zeta(s) L(\chi, s)$ in the region $Re(s) > 1$, and $\zeta_k(s)$ has a meromorphic continuation to the region $Re(s) > 0$ whose unique pole is at the point $s = 1$ and is simple.

Example. If $d = \ell \equiv 1 \pmod 4$ is a prime, then, as noted in Lecture 6, the character is exactly $\chi(n) = (\frac{n}{\ell})$, as

1. p splits in k iff $(\frac{p}{\ell}) = (\frac{\ell}{p}) = 1$
2. p is inert in k iff $(\frac{p}{\ell}) = (\frac{\ell}{p}) = -1$
3. p is ramified in k iff $p = \ell$.

Proof. The algebraic properties were proved in Lecture 6. Furthermore, we demonstrated that for any prime number p (ramified or unramified) the factor of $\zeta_k(s)$ corresponding to p is

$$\left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

so that

$$\zeta_k(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

Since

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

it suffices to show that

$$L(\chi, s) = \sum_n \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

Factor any positive integer $n = \prod_i p_i^{a_i}$ so that $\chi(n) = \prod_i \chi(p_i)^{a_i}$. Since

$$(1 - \chi(p)p^{-s})^{-1} = 1 + \chi(p)p^{-s} + \chi(p^2)p^{-2s} + \cdots = \sum_{n=0}^{\infty} \chi(p^n)p^{-ns}$$

a bit of bookkeeping implies the desired result. □

Corollary 66. *(Class Number Formula for Quadratic k) Let k be a quadratic field with discriminant d. If $d > 0$, then the regulator is given by $R = \log|\epsilon|$ for some $\epsilon > 1$ generating $A^\times/\{\pm 1\}$. If $d < 0$, then $R = 1$. Moreover,*

$$L(\chi, 1) = h\kappa = \begin{cases} \frac{2h\log|\epsilon|}{\sqrt{d}} & \text{if } d > 0 \\ \frac{2\pi h}{w\sqrt{|d|}} & \text{if } d < 0 \end{cases}$$

Proof. If $d > 0$ then $r_1 = 2$ and $r_2 = 0$, so $A^\times$ has rank 1. With the notation of Lecture 12, let $H \subset \mathbb{R}^2$ denote the hyperplane which contains the image E of $A^\times$ under the map $\lambda : A^\times \to \mathbb{R}^2$ given by $\epsilon \mapsto (\log|\epsilon|_{v_1}, \log|\epsilon|_{v_2})$. Fix an element $\epsilon \in A^\times$ which maps to a basis element of E. Without loss of generality, assume that $|\epsilon|_{v_1} > 1$, as $\epsilon \neq \pm 1$. Then $R = \log|\epsilon|_{v_1}$, by definition. If $d < 0$ then $A^\times$ has rank 0, and the definition of R shows that R is the "empty determinant" which is 1.

The final claim follows from considering the Taylor expansions of $\zeta(s)$ and $\zeta_k(s)$, and the facts that $\mathrm{Res}_{s=1}\zeta(s) = 1$ and $\mathrm{Res}_{s=1}\zeta_k(s) = h\kappa$ by Lemma 54 and Theorem 61. □

Corollary 67. $L(\chi, 1) > 0$

Proof. Immediate. □

We desire an independent evaluation of $L(\chi, 1)$ for any nontrivial character $\chi : (\mathbb{Z}/f\mathbb{Z})^\times \to \mathbb{C}^\times$. Our first tool shall be the *Gauss sum* of χ which we define to be the complex number

$$g(\chi) = \sum_{a=1}^{f} \chi(a)e^{2\pi i a/f}$$

Lemma 68. *If $(n, f) = 1$, then*

$$\chi(n)g(\overline{\chi}) = \sum_{b=1}^{f} \overline{\chi}(b)e^{2\pi i bn/f}$$

where $\overline{\chi}$ denotes the conjugate character, i.e., $\overline{\chi} = \chi^{-1}$ as the image of χ is contained in the set of roots of unity in $\mathbb{C}$.

Proof. By making the substitution $b \equiv an^{-1} \pmod f$ so that $nb \equiv a \pmod f$ we have

$$\chi(n)g(\overline{\chi}) = \sum_{a=1}^{f} \chi(n)\overline{\chi(a)}e^{2\pi ia/f} = \sum_{b=1}^{f} \overline{\chi}(b)e^{2\pi ibn/f}$$

as desired. □

By definition,

$$g(\overline{\chi})L(\chi, s) = \sum_{n=1}^{\infty} g(\overline{\chi})\chi(n)n^{-s}$$

If $(n, f) \neq 1$ then $\chi(n) = 0$ and the corresponding terms in the series vanish. If $(n, f) = 1$, then the lemma applies, and it follows that

$$g(\overline{\chi})L(\chi, s) = \sum_{\substack{b=1 \\ (f,b)=1}}^{f} \overline{\chi}(b)\Big(\sum_{n=1}^{\infty} \frac{e^{2\pi ibn/f}}{n^s}\Big) = \sum_{\substack{b=1 \\ (f,b)=1}}^{f} \overline{\chi}(b)\Big(\sum_{n=1}^{\infty} \frac{(e^{2\pi ib/f})^n}{n^s}\Big)$$

and we have absolute convergence in the region $\mathrm{Re}(s) > 1$. If $\zeta = e^{2\pi i/f}$, then the formula becomes

$$g(\overline{\chi})L(\chi, s) = \sum_{\substack{b=1 \\ (f,b)=1}}^{f} \overline{\chi}(b)\Big(\sum_{n=1}^{\infty} \frac{(\zeta^b)^n}{n^s}\Big) \tag{1}$$

We recall that the branch of $-\log(1 - x)$ defined on the region $|x| < 1$ has power series expansion $-\log(1 - x) = \sum_{n \geq 1} \frac{x^n}{n}$ and that the series actually converges to $-\log(1 - x)$ for $|x| = 1$ such that $x \neq 1$. A lemma of Abel states that we may substitute the value $s = 1$ into equation (1) which yields

$$g(\overline{\chi})L(\chi, 1) = \sum_{\substack{b=1 \\ (f,b)=1}}^{f} \overline{\chi}(b)(-\log(1 - \zeta^b)) \tag{2}$$

Lecture 14

Definition. A character $\chi : (\mathbb{Z}/f\mathbb{Z})^\times \to \mathbb{C}^\times$ is *primitive* if there is no proper divisor f' of f such that χ is induced by a character $\chi' : (\mathbb{Z}/f'\mathbb{Z})^\times \to \mathbb{C}^\times$. It follows immediately that, if f is prime, then χ is primitive iff χ is nontrivial. A character χ (not necessarily primitive) is said to be *even* if $\chi(-1) = 1$, and is said to be *odd* if $\chi(-1) = -1$.

Lemma 69. *(Gauss' Formula) If χ is primitive, then $g(\chi)g(\overline{\chi}) = \chi(-1)f$. In particular, $g(\chi) \neq 0$ and $|g(\chi)| = \sqrt{f}$.*

We shall prove this lemma in the case where f is prime.

Proof. Assume that $f = \ell$ is prime. For any function $F : \mathbb{Z}/\ell\mathbb{Z} \to \mathbb{C}$, the Fourier transform of F is the function $\hat{F} : \mathbb{Z}/\ell\mathbb{Z} \to \mathbb{C}$ given by the formula

$$\hat{F}(b) = \sum_{a=1}^{\ell} F(a) e^{2\pi i a b/\ell}$$

In particular, a character $\chi : (\mathbb{Z}/\ell\mathbb{Z})^\times \to \mathbb{C}^\times$ can be extended to such a function, and we have

$$\hat{\chi}(b) = \sum_{a=1}^{\ell} \chi(a) e^{2\pi i a b/\ell}$$

For $b = 0$ the fact that χ is a nontrivial character implies that

$$\hat{\chi}(0) = \sum_{a=1}^{\ell-1} \chi(a) = 0$$

If $b \neq 0$, then we make the substitution $c \equiv ab \pmod{\ell}$ so that $a \equiv cb^{-1} \pmod{\ell}$ and

$$\hat{\chi}(b) = \sum_{c \neq 0} \chi(cb^{-1}) e^{2\pi i c/\ell} = \chi(b)^{-1} \sum_{c \neq 0} \chi(c) e^{2\pi i c/\ell} = \chi(b)^{-1} g(\chi) \tag{3}$$

Thus, $\hat{\chi} = g(\chi)\chi^{-1}$. If $g(\chi) = 0$, then $\hat{\chi} = 0$, but this can not hold because of the Fourier inversion formula

$$\widehat{(\hat{F})}(-a) = \ell F(a)$$

Applying formula (3) twice yields

$$\hat{\hat{\chi}} = g(\chi)\widehat{\chi^{-1}} = g(\chi) g(\chi^{-1}) \chi$$

and Fourier inversion implies that

$$g(\chi) g(\chi^{-1}) \chi(-a) = \chi(a)\ell$$

Letting $a = -1$ we get

$$\chi(-1)\ell = g(\chi) g(\chi^{-1}) \chi(1) = g(\chi) g(\overline{\chi})$$

as desired. If we conjugate the definition of $g(\chi)$ and make the substitution $a \mapsto -a$, this yields

$$\overline{g(\chi)} = \sum_{a=1}^{\ell} \overline{\chi(a)} e^{-2\pi i a/\ell} = \sum_{a=1}^{\ell} \overline{\chi(-a)} e^{2\pi i a/\ell} = \overline{\chi(-1)} \sum_{a=1}^{\ell} \overline{\chi(a)} e^{-2\pi i a/\ell} = \chi(-1) g(\overline{\chi})$$

so that

$$|g(\chi)| = [g(\chi)\overline{g(\chi)}]^{1/2} = [g(\chi)\chi(-1)g(\overline{\chi})]^{1/2} = [\chi(-1)\chi(-1)\ell]^{1/2} = \sqrt{\ell}$$

as desired. $\square$

Assume that ℓ is a prime number and that χ is nontrivial. Then

$$\chi(n)g(\overline{\chi}) = \sum_{b=1}^{\ell} \overline{\chi}(b)e^{2\pi ibn/\ell}$$

for all $n \geq 1$: if $(n, \ell) = 1$ then this is Lemma 68, otherwise each side is zero. Thus

$$\begin{aligned} g(\overline{\chi})L(\chi, s) &= \sum_{n\geq 1}\Big(\sum_{b=1}^{\ell} \overline{\chi}(b)e^{2\pi ibn/\ell}\Big)n^{-s} = \sum_{b=1}^{\ell} \overline{\chi}(b)\Big(\sum_{n\geq 1} \frac{e^{2\pi ibn/\ell}}{n^s}\Big) \\ &= \sum_{b=1}^{\ell-1} \overline{\chi}(b)\Big(\sum_{n\geq 1} \frac{(\zeta^b)^n}{n^s}\Big) \end{aligned} \tag{4}$$

where $\zeta = e^{2\pi i/\ell}$. This series converges (conditionally) for $\mathrm{Re}(s) > 0$ by Lemma 64. The power series

$$-\log(1-x) = \sum_{n\geq 1} \frac{x^n}{n}$$

gives the branch of log given by

$$\log(z) = \log|z| + i\arg(z) \qquad \text{for} \qquad -\pi/2 < \arg(z) < \pi/2$$

Thus, substituting $s = 1$ into equation (4) yields

$$g(\overline{\chi})L(\chi, 1) = \sum_{b=1}^{\ell-1} \overline{\chi}(b)\Big(\sum_{n\geq 1} \frac{(\zeta^b)^n}{n}\Big) = \sum_{b=1}^{\ell-1} \overline{\chi}(b)(-\log(1-\zeta^b))$$

To simplify the formula for $L(\chi, 1)$, we consider two cases.

If χ is even, then $\chi(-b) = \chi(b)$ for all b so that

$$g(\overline{\chi})L(\chi, 1) = -\sum_{b=1}^{\ell-1} \overline{\chi}(b)\log(1-\zeta^b) = -\sum_{b=1}^{\ell-1} \overline{\chi}(b)\log|1-\zeta^b|$$

by our choice of the branch of $\log x$, since the argument terms cancel. By Lemma 69, this implies that

$$L(\chi, 1) = -\frac{g(\chi)}{\ell} \sum_{b=1}^{\ell-1} \overline{\chi}(b)\log|1-\zeta^b| \tag{5}$$

If χ is odd, then $\chi(-b) = -\chi(b)$ for all b so that

$$g(\overline{\chi})L(\chi, 1) = -i\sum_{b=1}^{\ell-1} \overline{\chi}(b)\arg(1-\zeta^b)$$

as in the even case. Thus,

$$L(\chi, 1) = \frac{ig(\chi)}{\ell} \sum_{b=1}^{\ell-1} \overline{\chi}(b)\arg(1-\zeta^b)$$

This is not quite as nice as the final formula in the even case since $\arg(1-\zeta^b)$ is more complicated than $\log|1-\zeta^b|$. For $b=1,\ldots,\ell-1$ the fact that arg is additive (up to multiple of 2π) implies that

$$\arg(1-\zeta^b) = \arg(\zeta^{b/2}(\zeta^{-b/2}-\zeta^{b/2})) = \arg(\zeta^{b/2}) + \arg(\zeta^{-b/2}-\zeta^{b/2})$$

The complex number $\zeta^{-b/2}-\zeta^{b/2}$ is purely imaginary, so its argument is $\pm\pi/2$. By definition, $\zeta^{b/2} = e^{2\pi ib/2\ell} = e^{\pi ib/\ell}$ and therefore has argument $\pi b/\ell$ which is in the interval $(0,\pi)$. It follows that, in order for $\arg(1-\zeta^b)$ to fall in the correct interval, $\zeta^{-b/2}-\zeta^{b/2}$ must have argument $-\pi/2$. Thus,

$$\arg(1-\zeta^b) = \pi(\tfrac{b}{\ell}-\tfrac{1}{2})$$

which implies that

$$L(\chi,1) = \frac{\pi i g(\chi)}{\ell}\sum_{b=1}^{\ell-1}\overline{\chi}(b)(\tfrac{b}{\ell}-\tfrac{1}{2}) \tag{6}$$

We note that, if $B_1(x) = x-\frac{1}{2}$ is the first Bernoulli polynomial, then

$$\sum_{b=1}^{\ell-1}\overline{\chi}(b)(\tfrac{b}{\ell}-\tfrac{1}{2}) = \sum_{b=1}^{\ell-1}\overline{\chi}(b)B_1(b/\ell)$$

which we shall denote by $B_{1,\overline{\chi}}$, and we may write

$$L(\chi,1) = \frac{\pi i g(\chi)}{\ell}B_{1,\overline{\chi}}$$

Example. Let $k=\mathbb{Q}(\sqrt{\ell})$ where ℓ is a positive prime such that $\ell\equiv 1 \pmod 4$. Then Corollary 66 implies that

$$\mathrm{Res}_{s=1}\zeta_k(s) = \frac{2h\log|\epsilon|}{\sqrt{\ell}} = L(\chi,1)$$

where $\chi(b) = (\frac{b}{\ell})$, as we noted in Lecture 6. Since χ is real-valued, we see that $\chi=\overline{\chi}$ and has order 2. By equation (5) it follows that

$$L(\chi,1) = -\frac{g(\chi)}{\ell}\log\left|\frac{\prod_{(\frac{b}{\ell})=1}(1-\zeta^b)}{\prod_{(\frac{b}{\ell})=-1}(1-\zeta^b)}\right|$$

We know that $g(\chi)^2=\ell$, so the coefficient is $\pm\frac{1}{\sqrt{\ell}}$. In fact, $g(\chi)=+\sqrt{\ell}$, but this is not easy to verify. (For a proof, see Borevich and Shafarevich, *Number Theory* Theorem 5.4.7.) If we let

$$u = \frac{\prod_{(\frac{b}{\ell})=1}(1-\zeta^b)}{\prod_{(\frac{b}{\ell})=-1}(1-\zeta^b)}$$

then this implies that

$$\frac{2h\log|\epsilon|}{\sqrt{\ell}} = -\frac{g(\chi)}{\ell}\log|u| = -\frac{1}{\sqrt{\ell}}\log|u|$$

Cancellation implies that

$$2h \log |\epsilon| = -\log |u|$$

and exponentiation yields

$$u = \pm \epsilon^{-2h}$$

Example. Let $k = \mathbb{Q}(\sqrt{-\ell})$ where ℓ is a positive prime such that $\ell \equiv 3 \pmod 4$. Then

$$\begin{aligned}\operatorname{Res}_{s=1}\zeta_k(s) = \frac{2\pi h}{w\sqrt{\ell}} = L(\chi, 1) &= \frac{\pi i g(\chi)}{\ell} \sum_{b=1}^{\ell} \chi(b)\left(\tfrac{b}{\ell} - \tfrac{1}{2}\right) \\ &= \frac{\pi i g(\chi)}{\ell} \left(\sum_{b=1}^{\ell} \chi(b)(b/\ell) - \sum_{b=1}^{\ell} \chi(b)(1/2) \right) = \frac{\pi i g(\chi)}{\ell} \sum_{b=1}^{\ell} \chi(b)(b/\ell)\end{aligned}$$

as

$$\sum_{b=1}^{\ell} \chi(b)(1/2) = \frac{1}{2} \sum_{b=1}^{\ell} \chi(b) = 0$$

Here $\chi(b) = (\frac{b}{\ell})$ as above, and $g(\chi)^2 = -\ell$ so that $g(\chi) = \pm i\sqrt{\ell}$. In fact, $g(\chi) = +i\sqrt{\ell}$, so

$$\frac{ig(\chi)}{\ell} = -\frac{1}{\sqrt{\ell}}$$

and therefore

$$\frac{2h}{w} = -\sum_{b=1}^{\ell} \chi(b)(b/\ell)$$

It should be noted that $w = 2$ unless $\ell = 3$.

Example. Let $k = \mathbb{Q}(i)$. We can recover the fact that the class number of k is $h = 1$ from the ideas above. Let $\chi : (\mathbb{Z}/4\mathbb{Z})^\times \to \{\pm 1\}$ be the character described in Theorem 65, so that $\zeta_k(s) = \zeta(s)L(\chi, s)$. By Corollary 66,

$$L(\chi, 1) = \frac{2\pi h}{w\sqrt{|d|}} = \frac{\pi h}{4}$$

By definition

$$L(\chi, 1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

which lies between 1 and 2/3. It follows that $h = 1$.

Proposition 70. *Let ℓ be an odd prime number, $\zeta = \zeta_\ell$ a primitive ℓth root of unity, and $k = \mathbb{Q}(\zeta)$. Then*

$$\zeta_k(s) = \zeta(s) \prod_{\chi \neq 1} L(\chi, s)$$

where the product is taken over all nontrivial characters $\chi : (\mathbb{Z}/\ell\mathbb{Z})^\times \to \mathbb{C}^\times$.

Proof. The Euler products of each function show that the desired equation is

$$\prod_p \prod_{P|p} (1 - \mathbb{N}(P)^{-s})^{-1} = \prod_p (1 - p^{-s}) \prod_{\chi \neq 1} \prod_{p \neq \ell} (1 - \chi(p)p^{-s})^{-1}$$

and we check equality factor-by-factor at each prime. The prime ℓ is completely ramified, so that $\ell A = P^{\ell-1}$ and $\mathbb{N}(P) = \ell$. In the infinite product on the left, this contributes a factor of $(1 - \ell^{-s})^{-1}$. In the factorization of $\zeta_k(s)$ we have a factor of $(1 - \ell^{-s})^{-1}$ contributed by ℓ, and no factor contributed to any of the $L(\chi, s)$. Thus, "over ℓ" the left and right-hand sides agree.

Let $p \neq \ell$, so that p is unramified. Then $pA = P_1 \cdots P_g$, and $\mathbb{N}(P_i) = p^f$ where f is the order of p in $(\mathbb{Z}/\ell\mathbb{Z})^\times$. In particular, $g = \frac{\ell-1}{f}$. In $\zeta_k(s)$, p contributes factors

$$\prod_{i=1}^{g} (1 - p^{-fs})^{-1} = ((1 - p^{-fs})^{(\ell-1)/f})^{-1}$$

On the right-hand side of the desired equation, p contributes

$$(1 - p^{-s})^{-1} \prod_{\chi \neq 1} (1 - \chi(p)p^{-s})^{-1}$$

and we want to know that these expressions are equal. Let $X = p^{-s}$. Then the desired equality is

$$(1 - X^f)^{(\ell-1)/f} = \prod_{\text{all } \chi} (1 - \chi(p)X)$$

which we shall show is an identity on polynomials. We note that the zeroes of each side are fth roots of unity, so this is reasonable. The subgroup $\langle p \rangle \subseteq (\mathbb{Z}/\ell\mathbb{Z})^\times$ has order f. Let G denote the quotient $(\mathbb{Z}/\ell\mathbb{Z})^\times / \langle p \rangle$, which has order $g = \frac{\ell-1}{f}$. The short exact sequence

$$1 \to \langle p \rangle \to (\mathbb{Z}/\ell\mathbb{Z})^\times \to G \to 1$$

implies that the right-hand side of the desired equality is

$$\prod_{\substack{\text{all chars.} \\ \chi \text{ of } (\mathbb{Z}/\ell\mathbb{Z})^\times}} (1 - \chi(p)X) = \prod_{\substack{\text{all chars.} \\ \psi \text{ of } \langle p \rangle}} (1 - \psi(p)X)^{(\ell-1)/f}$$

and the factorization of cyclotomic polynomials implies that this is exactly the left-hand side of the desired equality. □

Next, we have some consequences of the proposition.

Corollary 71. *For nontrivial characters* $\chi : (\mathbb{Z}/\ell\mathbb{Z})^\times \to \mathbb{C}^\times$, $L(\chi, 1) \neq 0$.

Proof. Immediate from the proposition, as $L(\chi, s)$ is regular at $s = 0$, and the fact that $\zeta_k(s)$ and $\zeta(s)$ both have simple pole at $s = 1$. □

Corollary 72. *(Dirichlet) If* $\ell \nmid a$*, then there are infinitely many prime numbers* p *such that* $p \equiv a \pmod{\ell}$.

This is the statement that there are infinitely many primes in an arithmetic progression modulo ℓ. Before we prove the corollary, we need a pair of lemmas.

Lemma 73. *For a nontrivial character χ, the function*

$$f_\chi(s) = \sum_{p \neq \ell} \frac{\chi(p)}{p^s}$$

is bounded as $s \to 1$.

Proof. For $\mathrm{Re}(s) > 1$, the power series expansion of $\log(1-x)$ implies that

$$\log(L(\chi, s)) = \log\Big(\prod_{p \neq \ell}\Big(1 - \frac{\chi(p)}{p^s}\Big)^{-1}\Big) = \sum_{p \neq \ell}\sum_{n \geq 1} \frac{\chi(p)^n}{np^{ns}}$$

The rest of the proof is similar to that of Corollary 56. □

Definition. Let G be a finite abelian group and fix $g \in G$. Define the characteristic function of g, $\mathrm{char}(g) : G \to \mathbb{C}$, as $g \mapsto 1$ and $h \mapsto 0$ for $h \neq g$.

Lemma 74. *The characteristic function of g can be described by the formula*

$$char(g) = \frac{1}{\#G} \sum_{all\ \chi} \overline{\chi}(g)\chi$$

Proof of Lemma 74. For $h \in G$

$$\mathrm{char}(g)(h) = \frac{1}{\#G} \sum_{\text{all } \chi} \overline{\chi}(g)\chi(h) = \frac{1}{\#G} \sum_{\text{all } \chi} \chi(h/g)$$

If $h = g$, then this is

$$\mathrm{char}(g)(g) = \frac{1}{\#G} \sum_{\text{all } \chi} \chi(1) = 1$$

since the number of characters of G is exactly $\#G$. If $h \neq g$, then $h/g \neq 1$ and there exists a fixed character χ' of G such that $\chi'(h/g) \neq 1$ (c.f. Borevich and Shafarevich *Number Theory*, p. 417), and of course $\chi'(h/g) \neq 0$. As χ runs through all characters of G, so does $\chi'\chi$, so that

$$\sum_{\text{all } \chi} \chi(h/g) = \sum_{\text{all } \chi} \chi'(h/g)\chi(h/g) = \chi'(h/g) \sum_{\text{all } \chi} \chi(h/g)$$

which implies that the sum is zero. □

Proof of Corollary 72. In the notation of the lemmas, let $G = (\mathbb{Z}/\ell\mathbb{Z})^\times$. Then

$$\frac{1}{\ell - 1} \sum_{\text{all } \chi} \overline{\chi}(a) f_\chi(s) = \frac{1}{\#G} \sum_{\text{all } \chi} \overline{\chi}(a) \sum_{p \neq \ell} \frac{\chi(p)}{p^s} = \sum_{p \neq \ell} \frac{1}{p^s} \frac{1}{\#G} \sum_{\text{all } \chi} \overline{\chi}(a)\chi(p)$$
$$= \sum_{p \neq \ell} \frac{1}{p^s} \mathrm{char}(a)(p) = \sum_{p \equiv a (\bmod\ \ell)} \frac{1}{p^s}$$

For each $\chi \neq 1$, $f_\chi(s)$ is bounded as $s \to 1$, and $f_1(s) \sim \log(\frac{1}{s-1})$. Thus, up to a bounded function

$$\sum_{p \equiv a (\mathrm{mod}\ \ell)} \frac{1}{p^s} \sim \frac{1}{\ell - 1} \log\Big(\frac{1}{s-1}\Big)$$

as $s \to 1$, which implies that $\sum_{p \equiv a} \frac{1}{p}$ is infinite. In particular, the number of terms in the sum must be infinite, as desired. □

Notice that the proof of the corollary gives us a stronger result. Up to a bounded function

$$\sum_{p \equiv a (\mathrm{mod}\ \ell)} \frac{1}{p^s} \sim \frac{1}{\ell - 1} \log\Big(\frac{1}{s-1}\Big)$$

as $s \to 1$. This says that the density of such primes is $1/(\ell - 1)$.

Corollary 75. *(Kummer's Class Number Formula) For the cyclotomic field* $k = \mathbb{Q}(\zeta_\ell)$

$$\Big(\prod_{\chi\ odd} \frac{\pi i g(\chi)}{\ell} B_{1,\overline{\chi}}\Big) \cdot \Big(\prod_{\substack{\chi\ even \\ \chi \neq 1}} \frac{-g(\chi)}{\ell} \sum_{b=1}^{\ell} \overline{\chi}(b) \log|1 - \zeta^b|\Big) = \frac{(2\pi)^{(\ell-1)/2}}{\ell^{(\ell-2)/2}} \cdot \frac{hR}{2\ell}$$

Proof. The right-hand side of the desired equality is

$$\mathrm{Res}_{s=1} \zeta_k(s) = \mathrm{Res}_{s=1}\Big(\zeta(s) \prod_{\chi \neq 1} L(\chi, s)\Big) = \mathrm{Res}_{s=1}\Big(\prod_{\text{all } \chi} L(\chi, s)\Big)$$

which is the left-hand side by Theorem 61. □

Lecture 15

We define the gamma function

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}$$

for $\mathrm{Re}(s) > 0$. By integrating by parts, we see that $\Gamma(s)$ satisfies the functional equation $\Gamma(s+1) = s\Gamma(s)$. Thus, for $\mathrm{Re}(s) > -1$, we may define

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

to obtain a meromorphic continuation to the region $\mathrm{Re}(s) > -1$, with simple pole at $s = 0$. We may continue this process inductively to obtain a meromorphic continuation to all of $\mathbb{C}$ with only simple poles located at $s = 0, -1, -2, \ldots$. Note that $\Gamma(1) = 1$, so that $\Gamma(m) = (m-1)!$ for $m = 1, 2, 3, \ldots$.

Theorem 76. *(Hecke's Thesis) Let k be a number field.*

1. *$\zeta_k(s)$ has a meromorphic continuation all of $\mathbb{C}$, with a unique pole at $s = 1$ which is simple.*

2. *If we define*

$$\xi_k(s) = \zeta_k(s)(\pi^{-s/2}\Gamma(s/2))^{r_1}((2\pi)^{-s}\Gamma(s))^{r_2}$$

   then $\xi_k(s)$ has a meromorphic continuation to all of $\mathbb{C}$, with unique poles at $s = 0, 1$ which are simple. Furthermore, $\xi_k(s)$ satisfies the functional equation

$$\xi_k(s) = |d|^{\frac{1}{2}-s}\xi_k(1-s)$$

   where d is the discriminant of k.

For a complete proof, see Weil *Basic Number Theory*, Chapter VII. We shall continue with these ideas below. First, we give some motivation for the Gamma factors in this product by considering curves over finite fields.

Let $\mathbb{F}_q$ denote the finite field with $q = p^m$ elements, and let X be a complete, nonsingular curve over $\mathbb{F}_q$. Let $k = \mathbb{F}_q(X)$ be the field of rational functions on X, and let $A \subset k$ be the integral domain of functions regular outside a single place ∞. We define the zeta function of A as

$$\zeta_A(s) = \sum_{I \subseteq A} \frac{1}{\mathbb{N}(I)^s} = \prod_{\substack{P \subset A \\ P \neq 0}} \left(1 - \frac{1}{\mathbb{N}(P)^s}\right)^{-1}$$

for $\mathrm{Re}(s) > 1$, where the product is taken over all nonzero prime ideals of A.

For example, if $X = \mathbb{P}^1 = \mathbb{P}^1_{\mathbb{F}_q}$, then $k = \mathbb{F}_q(X) = \mathbb{F}_q(t) \supset \mathbb{F}_q[t] = A$. The nonzero prime ideals P of A are in bijection with the collection of irreducible, monic polynomials $f(t)$ over $\mathbb{F}_q$. Under this correspondence, if $f(t)$ has degree d, then $\mathbb{N}(P) = q^d$. Since $\mathbb{F}_q$ sits inside A/P for every such prime, we see that every P has characteristic p. (This is true in general, of course.) Furthermore, $A^\times = \mathbb{F}_q^\times$.

If q is odd and X is an elliptic curve, say given by the equation $y^2 = x^3 + Ax + B$, then

$$A = \mathbb{F}_q[x, y]/(y^2 - x^3 - Ax - B)$$

and $k = \mathbb{F}_q(X)$ is the quotient field of A.

Let X, k and A be as above. Since $\mathbb{F}_q \hookrightarrow A \subseteq K$, A has characteristic p. For every prime ideal P, $\mathbb{N}(P) = q^d$ for some positive integer d, which we call the *degree* of P and denote by $d = \deg(P)$. We define the expression $Z_A(T)$ by

$$Z_A(T) = \prod_{P \neq 0} (1 - T^{\deg(P)})^{-1}$$

and note that, by definition, $\zeta_A(s) = Z_A(q^{-s})$. As in the number field situation, we can use P to define a valuation on k by

$$|\alpha|_P = \mathbb{N}(P)^{-\mathrm{ord}_P(\alpha)}$$

For a valuation v on k, let A_v denote the ring of elements α of k such that $v(\alpha) \leq 1$. Then A_v is a local ring with maximal ideal equal to the set of elements α of k such that $v(\alpha) < 1$. In addition, A_v is a discrete valuation ring so that the maximal ideal is principal, generated by a uniformizing parameter which we denote by π_v. The degree of v is defined to be the dimension of the quotient $A_v/\pi_v A_v$ over $\mathbb{F}_q$, and this is exactly the degree of the prime ideal P which induced v. We define the complete zeta function of X

$$Z_X(T) = \prod_v (1 - T^{\deg(v)})^{-1} = Z_A(T)(1 - T^{\deg(\infty)})^{-1}$$

by using *all* the valuations of k, including the valuation at ∞, which does not correspond to a prime ideal of A. Using the power series $-\log(1-x) = \sum_{n\geq 1} \frac{x^n}{n}$, which holds for $|x| < 1$, we see that in $\mathbb{Q}[\![T]\!]$

$$\log(Z_X(T)) = \sum_v \sum_{n\geq 1} \frac{T^{n\deg(v)}}{n} = \sum_v \sum_{n\geq 1} \frac{\deg(v) T^{n\deg(v)}}{n\deg(v)} = \sum_{n\geq 1} \frac{T^n}{n} A_n$$

where

$$A_n = \sum_{\deg(v)|n} \deg(v) \in \mathbb{Z}$$

A prime ideal P is determined by an orbit of the action of $G = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on sets of points of $\overline{X} = X(\overline{\mathbb{F}}_q)$ (the points of X defined over $\overline{\mathbb{F}}_q$) and the degree of P is the number of elements in the orbit. Fix a point x of X. G acts on $\overline{X}$ and therefore determines an orbit of x. The orbit is finite and appears as a set of points in $X(\mathbb{F}_{q^m})$ for some m. The sum $d = \sum_{\sigma\in G} x^\sigma$ is a prime divisor on X and therefore determines a prime ideal of A. It follows that $A_n = \#X(\mathbb{F}_{q^n})$ and so

$$\log(Z_X(T)) = \sum_{n\geq 1} \frac{T^n}{n}(\#X(\mathbb{F}_{q^n}))$$

Exponentiation yields the equation

$$Z_X(T) = \exp\Big(\sum_{n\geq 1} \frac{T^n}{n}(\#X(\mathbb{F}_{q^n}))\Big)$$

For example, if $X = \mathbb{P}^1$, then $\#X(\mathbb{F}_{q^n}) = q^n + 1$ and

$$\begin{aligned}\log(Z_X(T)) &= \sum_{n\geq 1} \frac{(q^n+1)T^n}{n} = \sum \frac{T^n}{n} + \sum \frac{(qT)^n}{n} \\ &= \log(1-T)^{-1} + \log(1-qT)^{-1} = \log\Big(\frac{1}{(1-T)(1-qT)}\Big)\end{aligned}$$

so that

$$Z_{\mathbb{P}^1}(T) = \frac{1}{(1-T)(1-qT)}$$

and the substitution $T = q^{-s}$ implies that

$$\zeta_X(s) = \frac{1}{(1-q^{-s})(1-q^{1-s})}$$

Notice that this is a rational function in q^{-s}, which is meromorphic with poles only where $q^{-s} = 1$ and $q^{1-s} = 1$. This is similar to Theorem 76.

In general, $Z_X(T) = \frac{P(T)}{(1-T)(1-qT)}$, where $P(T)$ is a polynomial in T with integer coefficients having degree $2g$ where g is the genus of X. More specifically,

$$P(T) = 1 + \cdots + q^g T^{2g} = \prod_{i=1}^{2g}(1-\alpha_i T)$$

for some algebraic integers α_i, and we may write

$$\begin{aligned}\exp\Big(\sum_{n\geq 1}\frac{T^n}{n}(\#X(\mathbb{F}_{q^n}))\Big) = Z_X(T) &= \exp\Big(\sum_{n\geq 1}\frac{\#X(\mathbb{F}_{q^n})T^n}{n}\Big)\\ &= \exp\Big(\#X(\mathbb{F}_q)T + \#X(\mathbb{F}_{q^2})\frac{T^2}{2} + \cdots\Big)\end{aligned}$$

If X is an elliptic curve, then $P(T) = 1 - aT + qT^2$ since $g_X = 1$. The only unknown coefficient is a. In fact, $a = 1 + q - \#X(\mathbb{F}_q)$, as can easily be verified by checking the coefficient of T in $Z_X(T)$.

Now, we return to the case where k is a number field. The definition of $\xi_k(s)$ may be restated, as follows. Every valuation v on k corresponds to a factor $\zeta_v(s)$ in the Euler product of $\xi_k(s)$. For the finite valuations (those corresponding to nonzero prime ideals of A) $\zeta_v(s) = (1-\mathbb{N}(P)^{-s})^{-1}$, which is exactly the factor of $\zeta_k(s)$ corresponding to P. For a valuation arising from a real place, $\zeta_v(s) = \pi^{-s/2}\Gamma(s/2)$, which is meromorphic in $\mathbb{C}$ with (simple) poles at $s = 0, -2, -4, \ldots$. For a valuation arising from a complex place, $\zeta_v(s) = (2\pi)^{-s}\Gamma(s)$, which is meromorphic in $\mathbb{C}$ with (simple) poles at $s = 0, -1, -2, \ldots$.

We shall prove part of Theorem 76 for the case $k = \mathbb{Q}$. In this case, $d = 1$, $r_1 = 1$ and $r_2 = 0$. As with the zeta function of $\mathbb{Q}$, we denote $\xi_{\mathbb{Q}}(s)$ as $\xi(s)$.

Theorem 77. *The function*

$$\xi(s) = \zeta(s)\pi^{-s/2}\Gamma(s/2)$$

has a meromorphic continuation to all of $\mathbb{C}$ *with simple poles at* $s = 0, 1$. *Furthermore,* $\xi(s)$ *satisfies the functional equation* $\xi(1-s) = \xi(s)$.

Proof. For a fixed positive integer, we make the substitution $t = n^2\pi x$ into the definition of

$\Gamma(s/2)$. Here, $\frac{dt}{t} = \frac{dx}{x}$ so that

$$\begin{aligned}
\Gamma(s/2) &= \int_0^\infty t^{s/2} e^{-t} \frac{dt}{t} \\
&= \int_0^\infty (n^2 \pi x)^{s/2} e^{-n^2 \pi x} \frac{dx}{x} \\
&= \int_0^\infty n^s \pi^{s/2} x^{s/2} e^{-n^2 \pi x} \frac{dx}{x} \\
n^{-s} \pi^{-s/2} \Gamma(s/2) &= \int_0^\infty x^{s/2} e^{-n^2 \pi x} \frac{dx}{x}
\end{aligned}$$

and summing over all $n \geq 1$ yields

$$\begin{aligned}
\xi(s) &= \zeta(s) \pi^{-s/2} \Gamma(s/2) \\
&= \pi^{-s/2} \Gamma(s/2) \sum_n \frac{1}{n^s} \\
&= \sum_{n \geq 1} \pi^{-s/2} \Gamma(s/2) n^{-s} \\
&= \sum_{n \geq 1} \int_0^\infty x^{s/2} e^{-n^2 \pi x} \frac{dx}{x}
\end{aligned}$$

which converges absolutely for $\mathrm{Re}(s) > 1$, so we may interchange the sum and integral

$$\xi(s) = \int_0^\infty x^{s/2} \Big(\sum_{n \geq 1} e^{-n^2 \pi x} \Big) \frac{dx}{x}$$

Let $\omega(x) = \sum_{n \geq 1} e^{-n^2 \pi x}$.

Here, we introduce Jacobi's theta function. For $x > 0$, let

$$\theta(x) = \sum_{n \in \mathbb{Z}} e^{-n^2 \pi x} = 1 + 2\omega(x)$$

If we write $f(t) = e^{-t^2 \pi x}$ and let $\hat{f}(t)$ denote the Fourier transform of f, then by the Poisson summation formula

$$\theta(x) = \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

Furthermore, the identity

$$\hat{f}(t) = \frac{e^{-t^2 \pi / x}}{x^{1/2}}$$

implies that

$$\theta(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) = \frac{\theta(x^{-1})}{x^{1/2}} \tag{7}$$

which is Jacobi's famous theta identity. Thus,

$$\begin{aligned}\xi(s) &= \int_0^\infty x^{s/2}\omega(x)\frac{dx}{x} \\ &= \int_0^1 x^{s/2}\omega(x)\frac{dx}{x} + \int_1^\infty x^{s/2}\omega(x)\frac{dx}{x}\end{aligned}$$

we substitute $x \mapsto x^{-1}$ into the first integral

$$= \int_1^\infty x^{-s/2}\omega(x^{-1})\frac{dx}{x} + \int_1^\infty x^{s/2}\omega(x)\frac{dx}{x}$$

The fact that $\omega(x) = \frac{1}{2}(\theta(x) - 1)$ implies that

$$\omega(x^{-1}) = \frac{1}{2}(\theta(x^{-1}) - 1) = \frac{1}{2}(x^{1/2}\theta(x) - 1) = x^{1/2}\omega(x) - \frac{1}{2} + \frac{x^{1/2}}{2}$$

so that

$$\begin{aligned}\xi(s) &= \int_1^\infty x^{1/2}\omega(x)x^{-s/2}\frac{dx}{x} + \int_1^\infty x^{s/2}\omega(x)\frac{dx}{x} \\ &\quad - \frac{1}{2}\int_1^\infty x^{s/2}\frac{dx}{x} + \frac{1}{2}\int_1^\infty x^{1/2-s/2}\frac{dx}{x} \\ &= \int_1^\infty (x^{(1-s)/2} + x^{s/2})\omega(x)\frac{dx}{x} - \frac{1}{s} + \frac{1}{s-1} \\ &= \frac{1}{s(s-1)} + \int_1^\infty (x^{(1-s)/2} + x^{s/2})\omega(x)\frac{dx}{x}\end{aligned}$$

To see that $\xi(s)$ has the proper meromorphic continuation, it suffices to show that the final integral has an *analytic* continuation to all of $\mathbb{C}$. The essential fact is that, as $x \to \infty$, $\omega(x)$ decays exponentially so the integral converges. To see that $\xi(s)$ satisfies the desired functional equation, we simply observe that if we replace s by $1 - s$ in the final expression for $\xi(s)$, it remains the same. □

Lecture 16

Next, we will discuss the analytic continuation of $L(\chi, s)$ and a functional equation satisfied by $L(\chi, s)$. Let $\tau = x + yi$ be a complex number with $y > 0$, and let $q = e^{\pi i \tau}$. We define a function $\Theta(\tau)$ similar to Jacobi's theta function, as

$$\Theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} = \sum_{n\in\mathbb{Z}} q^{n^2}$$

It readily follows from the definitions that $\Theta(iy) = \theta(y)$.

Example. Let $k = \mathbb{Q}(i)$. Then $A = \mathbb{Z}[i]$ and

$$\zeta_k(s) = \sum_{\substack{\mathfrak{a}\subseteq A \\ \mathfrak{a}=(a+bi)}} \frac{1}{\mathbb{N}(\mathfrak{a})^s} = \frac{1}{4}\sum_{(a,b)\neq(0,0)} \frac{1}{(a^2+b^2)^s}$$

as $\mathbb{N}(\mathfrak{a}) = a^2 + b^2$. If we define $f(\tau)$ as

$$f(\tau) = \sum_{(a,b)} q^{a^2+b^2} = (\Theta(\tau))^2$$

then f satisfies an equation similar to the equation satisfied by θ: $f(iy) = \frac{1}{y} f(iy^{-1})$.

Assume that $\chi : (\mathbb{Z}/f\mathbb{Z})^\times \to \mathbb{C}^\times$ is a primitive character. In order to uncover the functional equations for $L(\chi, s)$ we introduce functions $\xi(\chi, s)$ as we did for $\zeta_k(s)$. There are two cases.

If χ is even, then we define

$$\xi(\chi, s) = \pi^{-s/2}\Gamma(s/2)L(\chi, s)f^{s/2}$$

In this case, let $w_\chi = g(\chi)/\sqrt{f}$, where $g(\chi)$ is the Gauss sum from Lecture 13. Then $|w_\chi| = 1$ and $\xi(\chi, s)$ has a holomorphic continuation to all of $\mathbb{C}$ which satisfies the functional equation

$$\xi(\chi, s) = w_\chi \xi(\overline{\chi}, 1 - s)$$

If χ is odd, then we define

$$\xi(\chi, s) = \pi^{-(s+1)/2}\Gamma((s + 1)/2)L(\chi, s)f^{s/2}$$

Let $w_\chi = g(\chi)/i\sqrt{f}$. Again $|w_\chi| = 1$ and $\xi(\chi, s)$ satisfies the functional equation

$$\xi(\chi, s) = w_\chi \xi(\overline{\chi}, 1 - s)$$

If $\chi^2 = 1$, then we say that χ is a *quadratic* character. In this case $\overline{\chi} = \chi$ and it can be shown that

$$g(\chi) = \begin{cases} \sqrt{f} & \text{if } \chi \text{ is even} \\ i\sqrt{f} & \text{if } \chi \text{ is odd} \end{cases}$$

(c.f., Borevich and Shafarevich *Number Theory*, Theorem 5.4.7) so that $w_\chi = 1$. Thus, the simpler functional equation

$$\xi(\chi, s) = \xi(\chi, 1 - s)$$

is satisfied. We shall only prove that χ satisfies the desired functional equation when χ is an even character. First, however, we note that there are ways of slightly changing the functions $\xi_k(s)$ to get the same meromorphic continuation properties and simpler functional equations.

For example, assume that k is a quadratic field, with character χ as is Theorem 65. It can be shown that k is real if and only if χ is even (c.f., Borevich and Shafarevich *Number Theory*, Chapter 5,SS4). Furthermore, it can be shown that every quadratic character $(\mathbb{Z}/f\mathbb{Z})^\times \to \{\pm 1\}$ corresponds to a quadratic field in this manner. If k is a real field, then we define

$$\xi_k^*(s) = (\pi^{-s/2}\Gamma(s/2))^2 f^{s/2}\zeta_k(s)$$

which is very similar to the definition of $\xi_k(s)$. It is straightforward to check that $\xi_k(s)$ satisfies the functional equation

$$\xi_k(s) = |d|^{\frac{1}{2}-s}\xi_k(1-s)$$

if and only if $\xi_k^*(s)$ satisfies the functional equation

$$\xi_k^*(s) = \xi_k^*(1-s)$$

which is simpler. To see that this second equation is satisfied, notice that the definitions of $\xi_k^*(s)$, $\xi(s)$ and $\xi(\chi, s)$ imply that

$$\xi_k^*(s) = \xi(s)\xi(\chi, s)$$

In the quadratic case, $\xi(\chi, s) = \xi(\chi, 1-s)$ and by Theorem 77, $\xi(1-s) = \xi(s)$. Thus,

$$\xi_k^*(s) = \xi(s)\xi(\chi, s) = \xi(1-s)\xi(\chi, 1-s) = \xi_k^*(1-s)$$

as desired. A similar change works for the case when k is imaginary.

Before we prove the real case, we need some notation and a lemma. Let g be an element of the real Schwarz space, so that g is C^∞, and g and all its derivatives decrease rapidly at ∞. The Fourier transform of g, denoted $\hat{g}$, is given by

$$\hat{g}(y) = \int_{\mathbb{R}} g(x)e^{-2\pi ixy}dx = \int_{\mathbb{R}} g(x)\overline{\langle x, y\rangle}, dx$$

where $\langle x, y\rangle = e^{2\pi ixy}$. (The assumption that g is in the real Schwarz space implies that we may take iterated Fourier transforms of g.) Define $\Theta_g(t; x, y)$ as

$$\Theta_g(t; x, y) = \sum_{n\in\mathbb{Z}} g((x+n)t)\langle y, n\rangle$$

It is immediate from this definition that $\Theta_g(t; 0, 0) = \sum_n g(nt)$.

Lemma 78. *(Generalized Jacobi Identity)*

$$\Theta_g(t; x, y) = \frac{\overline{\langle x, y\rangle}}{t}\Theta_{\hat{g}}(t^{-1}; -y, x)$$

Proof. To simplify notation, let $G(w) = g((x+w)t)\langle y, w\rangle$. Then by the Poisson summation formula

$$\Theta_g = \sum_{n\in\mathbb{Z}} G(n) = \sum_{n\in\mathbb{Z}} \hat{G}(n)$$

so we calculate $\hat{G}$. By definition

$$\begin{aligned}\hat{G}(u) &= \int_{\mathbb{R}} g((x+w)t)\langle y, w\rangle\overline{\langle u, w\rangle}\, dw \\ &= \int_{\mathbb{R}} g((x+w)t)\overline{\langle u-y, w\rangle}\, dw\end{aligned}$$

and we make the substitution $z = (x+w)t$ so that $w = \frac{z}{t} - x$, $dw = \frac{dz}{t}$ and

$$\begin{aligned}\hat{G}(u) &= \int_{\mathbb{R}} g(z)\overline{\left\langle u-y, \frac{z}{t} - x\right\rangle}\,\frac{dz}{t} \\ &= \frac{1}{t}\overline{\langle x,y\rangle}\langle u,x\rangle \int_{\mathbb{R}} g(z)\overline{\left\langle z, \frac{u-v}{t} - x\right\rangle}\,dz \\ &= \frac{\overline{\langle x,y\rangle}\langle u,x\rangle}{t}\hat{g}\left(\frac{u-y}{t}\right)\end{aligned}$$

Substituting $u = n$ and summing over all n yields

$$\begin{aligned}\Theta_g(t;x,y) &= \sum_{n\in\mathbb{Z}} \hat{G}(n) = \sum_n \frac{\overline{\langle x,y\rangle}\langle n,x\rangle}{t}\hat{g}\left(\frac{n-y}{t}\right) \\ &= \frac{\overline{\langle x,y\rangle}}{t}\sum_n \langle n,x\rangle\hat{g}\left(\frac{n-y}{t}\right) = \frac{\overline{\langle x,y\rangle}}{t}\Theta_{\hat{g}}(t^{-1};-y,x)\end{aligned}$$

as desired. □

Corollary 79. *For real numbers y and t*

$$\sum_{n\in\mathbb{Z}} e^{(-\pi n^2 t + 2\pi i n y)} = \frac{1}{\sqrt{t}}\sum_{n\in\mathbb{Z}} e^{-\pi(ny)^2/t}$$

Proof. Let $g(\alpha) = e^{-\pi\alpha^2}$. Then it is a standard fact that $\hat{g} = g$, and Lemma 78 implies that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} e^{(-\pi n^2 t + 2\pi i n y)} &= \Theta_g(\sqrt{t};0,y) \\ &= \frac{\overline{\langle 0,y\rangle}}{\sqrt{t}}\Theta_{\hat{g}}(t^{-1/2};-y,0) \\ &= \frac{\overline{\langle 0,y\rangle}}{\sqrt{t}}\Theta_g(t^{-1/2};-y,0) \\ &= \frac{1}{\sqrt{t}}\sum_{n\in\mathbb{Z}} e^{-\pi(ny)^2/t}\end{aligned}$$

as desired. □

Theorem 80. *Assume that $\chi : (\mathbb{Z}/f\mathbb{Z})^\times \to \{\pm 1\}$ is a primitive, even character. Let*

$$\xi(\chi,s) = \pi^{-s/2}\Gamma(s/2)L(\chi,s)f^{s/2}$$

and let $w_\chi = g(\chi)/\sqrt{f}$, where $g(\chi)$ is the Gauss sum. Then $\xi(\chi,s)$ has a holomorphic continuation to all of $\mathbb{C}$, and satisfies the functional equation

$$\xi(\chi,s) = w_\chi \xi(\overline{\chi}, 1-s)$$

Proof. Start with the identity

$$f^{s/2}\pi^{-s/2}\Gamma(s/2)n^{-s} = \int_0^\infty e^{-\pi n^2 x/f} x^{s/2}\frac{dx}{x}$$

which is proved as in the proof of Theorem 77. Let

$$\theta_\chi(x) = \sum_{n\in\mathbb{Z}} \chi(n)e^{-\pi n^2 x/f}$$

so that, when we sum over all $n \geq 1$ we have

$$\begin{aligned}\xi(\chi, s) &= \pi^{-s/2}\Gamma(s/2)L(\chi, s)f^{s/2}\\ &= \sum_{n\geq 1} \chi(n)n^{-s}\pi^{-s/2}\Gamma(s/2)f^{s/2}\\ &= \int_0^\infty x^{s/2}\Big(\sum_{n\geq 1}\chi(n)e^{-\pi n^2 x/f}\Big)\frac{dx}{x}\\ &= \int_0^\infty x^{s/2}\Big(\frac{1}{2}\theta_\chi(x)\Big)\frac{dx}{x}\end{aligned}$$

as $\chi(-n) = \chi(n)$ and $\chi(0) = 0$

$$= \frac{1}{2}\Big(\int_0^1 x^{s/2}\theta_\chi(x)\frac{dx}{x} + \int_1^\infty x^{s/2}\theta_\chi(x)\frac{dx}{x}\Big) \tag{8}$$

The first integral converges for $\mathrm{Re}(s) > 0$, while the second converges for all s. To obtain the desired properties, we need to find an equation relating $\theta_\chi(x)$ to $\theta_\chi(x^{-1})$.

By Lemma 68,

$$\chi(n)g(\overline{\chi}) = \sum_{b=1}^{f} \overline{\chi}(b)e^{2\pi i bn/f}$$

so that

$$\begin{aligned}g(\overline{\chi})\theta_\chi(x) &= \sum_{n\in\mathbb{Z}}\Big(\sum_{b=1}^{f}\overline{\chi}(b)e^{2\pi i bn/f}\Big)e^{-\pi n^2 x/f}\\ &= \sum_{b=1}^{f}\overline{\chi}(b)\Big(\sum_{n\in\mathbb{Z}} e^{(-\pi n^2 x/f)+(2\pi i bn/f)}\Big)\end{aligned}$$

and the substitutions $t = x/f$ and $y = b/f$ yield

$$\begin{aligned}&= \sum_{b=1}^{f}\overline{\chi}(b)\Big(\frac{1}{\sqrt{x/f}}\sum_{n\in\mathbb{Z}} e^{-\pi(n+b/f)^2 f^2/(xf)}\Big)\\ &= \sqrt{\frac{f}{x}}\sum_{b=1}^{f}\overline{\chi}(b)\sum_{n\in\mathbb{Z}} e^{-\pi(nf+b)^2/(xf)}\end{aligned}$$

and the replacement $m = nf + b$ gives

$$= \sqrt{\frac{f}{x}} \sum_{n \in \mathbb{Z}} \overline{\chi}(m) e^{-\pi m^2/(fx)}$$

$$= \sqrt{\frac{f}{x}} \theta_{\overline{\chi}}(x^{-1})$$

If we substitute this final expression into (8), we find

$$\xi(\chi, s) = \frac{1}{2}\Big(\int_1^\infty x^{(1-s)/2} \frac{\sqrt{f}}{g(\overline{\chi})} \theta_{\overline{\chi}}(x) \frac{dx}{x} + \int_1^\infty x^{s/2} \theta_\chi(x) \frac{dx}{x}\Big)$$

which gives our analytic continuation. Furthermore, the substitution $s \mapsto 1 - s$ gives the desired functional equation. □

We have shown that $\zeta_k(s)$ has order of vanishing -1 at $s = 1$ and residue

$$\mathrm{Res}_{s=1} \zeta_k(s) ds = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}} \frac{hR}{w}$$

We shall use Hecke's functional equation for $\xi_k(s)$ to calculate the order of vanishing and residue at $s = 0$. By Theorem 76, we know that

$$\zeta_k(s)(\pi^{-s/2}\Gamma(s/2))^{r_1}((2\pi)^{-s}\Gamma(s))^{r_2} = \xi_k(s) = |d|^{\frac{1}{2}-s}\xi_k(1-s)$$

and that $\xi_k(s)$ has a simple pole (i.e., vanishing of order -1) at $s = 0$. Also, $\Gamma(s)$ has a simple pole at $s = 0$. Thus,

$$-1 = \mathrm{ord}_{s=0}\xi_k(s) = \mathrm{ord}_{s=0}\zeta_k(s) + r_1 \mathrm{ord}_{s=0}\Gamma(s) + r_2 \mathrm{ord}_{s=0}\Gamma(s) = \mathrm{ord}_{s=0}\zeta_k(s) - r_1 - r_2$$

so that

$$\mathrm{ord}_{s=0}\zeta_k(s) = r_1 + r_2 - 1 = \mathrm{rank}(A^\times)$$

and

$$\zeta_k(s) = cs^{r_1+r_2-1} + O(s^{r_1+r_2})$$

A further computation shows that $c = -\frac{hR}{w}$.

3.2.99–3.11.99