Some Variational Problems in the Space of Functions of Bounded Variation

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Outline

1. Introduction to functions of bounded variation.

2. Sets of finite perimeter - Minimization problem in BV space.

3. Relaxation - Variational inequality in BV space.

Functions of bounded variations (BV functions)

Definition. ([Ziemer]) Ω: open set in $\mathbf{R}^N$. $u \in L^1(\Omega)$.

$u \in BV(\Omega) \iff$ Distributional derivatives of u are measures with finite total variation on Ω: $\exists$ Radon (signed) measures $\mu_1, \dots, \mu_N$ with $\|\mu_i\|(\Omega) < \infty$:

$$\int_\Omega u \partial_i \phi dx = -\int_\Omega \phi d\mu_i, \ \forall \phi \in C_c^1(\Omega),$$

$(i = 1, \dots, N)$.

Put $\partial_i u = \mu_i$ and

$$\nabla u = (\partial_1 u, \dots, \partial_N u) = (\mu_1, \dots, \mu_N).$$

Norm of ∇u:

$$\int_\Omega |\nabla u| = \sup\{\int_\Omega u \mathrm{div} \phi dx : \phi \in C_c^1(\Omega, \mathbf{R}^N),$$

$$|\phi(x)| \leq 1, \forall x \in \Omega\}.$$

Remark. For $u \in C^1(\overline{\Omega})$, $\phi \in C_c^1(\Omega)$,

$$\int_\Omega u \mathrm{div} \phi dx = -\int_\Omega \nabla u \cdot \phi dx.$$

$\Rightarrow$

$$\int_\Omega |\nabla u| dx = \text{sup} \left\{ \int_\Omega u \text{div} \phi : \phi \in C_c^1(\Omega), |\phi| \leq 1 \right\}.$$

(**Proof.** $(\geq)$ $|\phi| \leq 1 \Rightarrow \left| \int_\Omega u \text{div} \phi dx \right|$

$$\leq \int_\Omega |\nabla u||\phi| dx \leq \int_\Omega |\nabla u| dx$$

$(\leq)$ Choose $\{\phi_n\} \subset C_c^1(\Omega, \mathbf{R}^N)$: $\phi_n \to \nabla u / |\nabla u|$ in $[L^2(\Omega)]^N$ and $|\phi_n| \leq 1$.

$$\int_\Omega \nabla u \cdot \phi_n dx \to \int_\Omega \nabla u \cdot \frac{\nabla u}{|\nabla u|} dx = \int_\Omega |\nabla u| dx.$$

- If $f = \tilde{f}$ a.e. on Ω, then $\int_\Omega |\nabla (f - \tilde{f})| = 0$

$\Rightarrow$ Can identify $f, \tilde{f}$ in $BV(\Omega)$.

Properties of BV functions

(Property 1) $u \in BV(\Omega) \iff$

$$\sup\left\{\int_\Omega u\,\mathrm{div}\phi dx : \phi \in C_c^1(\Omega, \mathbf{R}^N), |\phi(x)| \le 1, \forall x \in \Omega\right\} < \infty.$$

Proof.

$(\Rightarrow)$ $u \in BV(\Omega)$, $\partial_i u = \mu_i$. For $\phi \in C_c^1(\Omega)$, $\sup_{x\in\Omega} |\phi(x)| \le 1$,

$$\begin{aligned}
\int_\Omega u\,\mathrm{div}\phi dx &= \sum \int_\Omega u \partial_i \phi_i dx \\
&= -\sum \int_\Omega \phi_i d\mu_i \\
&= -\sum \int_\Omega \phi_i \sigma_i d\|\mu_i\| \\
&\quad (\|\mu_i\| = \text{total variation of } \mu_i, \\
&\quad \sigma_i = \text{density function}) \\
&\le \sum \int_\Omega |\phi_i||\sigma_i| d\|\mu_i\| \\
&\le \sum \|\mu_i\|(\Omega)(< \infty).
\end{aligned}$$

($\Rightarrow$) Assume

$$\sup\left\{\int_\Omega u \text{div}\phi dx : \phi \in C_c^1(\Omega, \mathbf{R}^N), \sup_{x\in\Omega} |\phi(x)| \leq 1\right\} < \infty. \tag{1}$$

Theorem. (Structure theorem for BV functions) Assume $u \in L^1(\Omega)$ satisfies (1). Then $\exists$ Radon (outer) measure μ on Ω, $\sigma : \Omega \to \mathbf{R}^N$ μ-measurable:

(i) $|\sigma(x)| = 1$ μ-a.e on Ω,

(ii) $\int_\Omega u \text{div}\phi dx = -\int_\Omega \phi \cdot \sigma d\mu$, $\forall \phi \in C_c^1(\Omega, \mathbf{R}^N)$,

(iii) $\mu(\Omega) < \infty$.

Proof based on Riesz Representation Theorem (vector version, Theorem 1, Sec. 1.8, [Evans], with trivial modifications)

Theorem. $\Omega \subset \mathbf{R}^N$ open.

$L : C_c(\Omega, \mathbf{R}^N) \to \mathbf{R}$ linear functional: $\forall K \subset \Omega$ compact,

$$\sup\{L(f) : f \in C_c(\Omega, \mathbf{R}^N), \sup_{x\in\Omega} |f(x)| \leq 1,$$

$$\text{supp}(f) \subset K\} < \infty. \tag{2}$$

$\Rightarrow \exists$ Radon (outer) measure μ on Ω, $\sigma : \Omega \to \mathbf{R}^N$ μ-measurable:

(i) $|\sigma(x)| = 1$ μ-a.e on Ω,

(ii) $L(f) = \int_\Omega f \cdot \sigma d\mu, \ \forall f \in C_c(\Omega, \mathbf{R}^N)$.

- μ (variation measure of L) is defined by

$$\mu(V) = \sup\{L(f) : f \in C_c(\Omega, \mathbf{R}^N),$$

$$\sup_{x\in\Omega} |f(x)| \leq 1, \text{supp}(f) \subset V\},$$

for $V \subset \Omega$, open, and

$$\mu(A) = \inf\{\mu(V) : A \subset V \subset \Omega, V \text{ open}\},$$

for $A \subset \Omega$.

In particular,

$$\mu(\Omega) = \sup\{L(f) : f \in C_c(\Omega, \mathbf{R}^N), \sup_{x\in\Omega} |f(x)| \leq 1\}.$$

Proof. (Structure theorem) Define

$L : C_c^1(\Omega, \mathbf{R}^N) \to \mathbf{R}$ by

$$L(f) = -\int_\Omega u \text{div} f dx, \ \forall f \in C_c^1(\Omega, \mathbf{R}^N).$$

$M := \sup\{L(f) : f \in C_c^1(\Omega, \mathbf{R}^N), \|f\|_\infty \le 1\} < \infty.$ $\Rightarrow$

$$L(f) \le M\|f\|_\infty, \forall f \in C_c^1(\Omega, \mathbf{R}^N) \qquad (3)$$

$(\|f\|_\infty = \sup\{|f(x)| : x \in \Omega\})$.

$\Rightarrow$ Can extend L to $\bar{L} : C_c(\Omega, \mathbf{R}^N) \to \mathbf{R}$,

$$\bar{L}(f) = \lim L(f_n),$$

for $f \in C_c(\Omega, \mathbf{R}^N), \{f_n\} \subset C_c^1(\Omega, \mathbf{R}^N), \ f_n \to f$ uniformly on Ω.

$\bar{L}$ also satisfies (3) and for $K \subset \Omega$ compact,

$\sup\{\bar{L}(f) : f \in C_c(\Omega, \mathbf{R}^N), \|f\|_\infty \le 1, \text{supp} f \subset K\} < \infty.$

- Riesz Representation theorem $\Rightarrow \exists\ \mu$ and σ:

$$L(\phi) = \bar{L}(\phi) = \int_\Omega \phi \cdot \sigma d\mu = -\int_\Omega u \text{div} \phi dx,$$

$$\forall \phi \in C_c^1(\Omega, \mathbf{R}^N).$$

$$\mu(\Omega) = \sup\{\bar{L}(\phi) : \phi \in C_c(\Omega, \mathbf{R}^N), \|\phi\|_\infty \leq 1\} \leq M$$

Proof. (Property 1)

Put $d\mu_i = \sigma_i d\mu$: μ_i is signed measure (with bounded variation)

$$\|\mu_i\|(\Omega) \leq \mu(\Omega) < \infty.$$

- $\phi \in C_c^1(\Omega, \mathbf{R})$, put $f = (0, \ldots, \phi, \ldots, 0) \in C_c^1(\Omega, \mathbf{R}^N)$.

$$\mathrm{div} f = \partial_i \phi,$$

$$\int_\Omega u \partial_i \phi dx = \int_\Omega u \mathrm{div} f dx = -\int_\Omega f \cdot \sigma d\mu$$

$$= -\int_\Omega \phi \sigma_i d\mu = -\int_\Omega \phi d\mu_i.$$

$\Rightarrow \partial_i u = \mu_i$. Put $\|\nabla u\| = \mu$.

$\to d(\nabla u) = \sigma d\mu = \sigma d\|\nabla u\|$.

$\to \int_\Omega u \mathrm{div} \phi dx = -\int_\Omega \phi \cdot d(\nabla u)$.

(Property 2) $BV(\Omega)$ is Banach space with norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + \int_\Omega |\nabla u|.$$

(Property 3) (Lower semicontinuity) $\Omega \subset \mathbf{R}^N$ open, $\{f_n\} \subset BV(\Omega)$, $f \in L^1(\Omega)$,

$$f_n \to f \text{ in } L^1_{loc}(\Omega).$$

Then, $\int_\Omega |\nabla f| \le \liminf \int_\Omega |\nabla f_n|$.

Proof. Let $\phi \in C_c^1(\Omega, \mathbf{R}^N)$, $\|\phi\|_\infty \le 1$.

$$\int_\Omega f \text{div}\phi dx = \lim_{n\to\infty} \int_\Omega f_n \text{div}\phi dx.$$

Since

$$\int_\Omega f_n \text{div}\phi dx \le \int_\Omega |\nabla f_n|, \ \forall n,$$

$$\Rightarrow \int_\Omega f \text{div}\phi dx = \lim_{n\to\infty} \int_\Omega f_n \text{div}\phi dx \le \liminf \int_\Omega |\nabla f_n|, \forall \phi.$$

$$\Rightarrow \int_\Omega |\nabla f| = \sup\{\int_\Omega f \text{div}\phi : \phi \in C_c^1(\Omega, \mathbf{R}^N), \|\phi\|_\infty \le 1\}$$

$$\le \liminf \int_\Omega |\nabla f_n|.$$

(Property 4) (Approximation by smooth functions) $f \in BV(\Omega)$. Then, $\exists \{f_n\} \subset BV(\Omega) \cap C^\infty(\Omega)$:

(i) $f_n \to f$ in $L^1(\Omega)$,

(ii) $\int_\Omega |\nabla f_n| \to \int_\Omega |\nabla f|$

(Not $\int_\Omega |\nabla (f_n - f)| \to 0$ in general. Approximation in BV-norm is not expected since the closure of C^∞ in this norm is $W^{1,1}$ space).

– Theorem 1.17, [Giusti] (or Theorem 2, 5.2.2, [Evans]). Result by Anzelotti and Giaquinta (Funzioni BV e tracce, Rend. Sem. Mat. Padova, 60 (1978), 1-21).

(Property 5) (Weak compactness (Theorem 1.19, [Giusti])) $\Omega \subset \mathbf{R}^N$ open, bounded with Lipschitz boundary $\partial\Omega$.

$\{f_n\}$ is bounded sequence in $BV(\Omega)$,

$$\left(\text{sup}\left\{\int_\Omega |f_n|dx + \int_\Omega |\nabla f_n| : n \in \mathbf{N}\right\} < \infty\right).$$

Then, $\exists\{f_{n_k}\} \subset \{f_n\}$ and $f \in BV(\Omega)$:

$$f_{n_k} \to f \text{ in } L^1(\Omega).$$

Proof. From approximation property, $\forall n \in N$, $\exists g_n \in BV(\Omega) \cap C^\infty(\Omega)$:

$$\int_\Omega |f_n - g_n|dx < \frac{1}{n},$$

and

$$\int_\Omega |\nabla g_n|dx < \int_\Omega |\nabla f_n| + 1.$$

$\Rightarrow \sup_n \int_\Omega |\nabla g_n|dx < \infty.$

$\Rightarrow$ $\{g_n\}$ is bounded sequence in $W^{1,1}(\Omega)$.

Rellich–Kondrachov Theorem $\Rightarrow \{g_n : n \in \mathbf{N}\}$ is relatively compact in $L^1(\Omega)$

$\Rightarrow\ \exists f \in L^1(\Omega),\ \{g_{n_k}\} \subset \{g_n\} : g_{n_k} \to f$ in $L^1(\Omega)$

$\Rightarrow\ f_{n_k} \to f$ in $L^1(\Omega)$.

(Lower semicontinuity property) $\Rightarrow f \in BV(\Omega)$.

Hausdorff Measure

(Measure for lower dimensional sets: surfaces, lines, etc. in $\mathbf{R}^N$)

Definition. $0 \leq s < \infty$, $0 < \delta \leq \infty$. $A \subset \mathbf{R}^N$. Define

$$\mathcal{H}^s_\delta(A) = \inf\{ \sum_{j=1}^{\infty} \alpha(s) \left(\frac{\text{diam} C_j}{2} \right)^s : A \subset \bigcup_{j=1}^{\infty} C_j,$$

$$\text{diam} C_j \leq \delta \},$$

$\alpha(s) = \dfrac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}+1\right)}$ (= (Lebesgue) volume of the unit ball in $\mathbf{R}^s$, when s is positive integer).

$\alpha(s)$ is included to make $\mathcal{H}^s$ agree with $\mathcal{L}^s$ in $\mathbf{R}^N (s = N)$ and on smooth surfaces.

- $\mathcal{H}^s_\delta$ is decreasing in δ.

Definition. $s \geq 0$, $A \subset \mathbf{R}^N$,

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

Properties of $\mathcal{H}^s$.

(1) $\mathcal{H}^s$ is a Borel regular measure.

(Borel sets are measurable and $\forall A \subset \mathbf{R}^N$, $\exists$ Borel set $B \supset A$: $\mathcal{H}^s(A) = \mathcal{H}^s(B)$.)

$\mathcal{H}^s$ is not a Radon measure if $0 \leq s < N$ ($\mathbf{R}^N$ is not σ-finite with respect to $\mathcal{H}^s$).

(2) In $\mathbf{R}^N$, $\mathcal{H}^N = \mathcal{L}^N$ ($\mathcal{H}^N(A) = \mathcal{L}^N(A)$, $\forall A \subset \mathbf{R}^N$, $\mathcal{L}^N =$ (outer) Lebesgue measure on $\mathbf{R}^N$).

(3) • $\mathcal{H}^s(A) = 0$, $\forall A \subset \mathbf{R}^N$ if $s > N$.

• Let $0 \leq s < t < \infty$. For $A \subset \mathbf{R}^N$,

$$\begin{cases} \mathcal{H}^s(A) < \infty \Rightarrow \mathcal{H}^t(A) = 0 \\ \mathcal{H}^t(A) > 0 \Rightarrow \mathcal{H}^s(A) = \infty. \end{cases}$$

• $\exists d \geq 0$:

$$\begin{cases} \mathcal{H}^s(A) = 0 & \text{if } \ s > d \\ \mathcal{H}^s(A) = \infty & \text{if } \ s < d, \end{cases}$$

($d =$ Hausdorff dimension of A).

(4) S is C^1 (($N-1$)-dimensional) surface in $\mathbf{R}^N$. $\mathcal{H}^{N-1}$ agrees with the (($N-1$)-dimensional) surface Lebesgue measure on S.

Trace of BV functions

• Ω open, bounded in $\mathbf{R}^N$ with Lipschitz Boundary $\partial\Omega$.

• Outer unit normal vector ν exists $d\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$.

Theorem. (Theorem 1, Sect. 5.3, [Evans])

(a) $\exists$ linear bounded mapping

$$T : BV(\Omega) \to L^1(\partial\Omega, \mathcal{H}^{N-1})$$

such that

$$\int_\Omega f \text{div}\phi dx = -\int_\Omega \phi \cdot d(\nabla f) + \int_{\partial\Omega} (\phi\cdot\nu)[T(f)] d\mathcal{H}^{N-1},$$

$\forall f \in BV(\Omega), \phi \in C^1(\mathbf{R}^N, \mathbf{R}^N)$.

(b) Function $T(f) \in L^1(\partial\Omega, \mathcal{H}^{N-1})$ is uniquely defined up to set of measure 0 in $d\mathcal{H}^{N-1}$ on $\partial\Omega$.

(c) For $f \in W^{1,1}(\Omega)$, $T(f)$ = usual Sobolev trace.

Definition. $T(f) = f|_{\partial\Omega}$: trace of f on $\partial\Omega$.

Extension theorem for BV functions

Theorem. $\Omega \subset \mathbf{R}^N$ open, bounded with $\partial\Omega$ Lipschitz.

$f_1 \in BV(\Omega)$, $f_2 \in BV(\mathbf{R}^N \setminus \overline{\Omega})$. Define

$$f(x) = \begin{cases} f_1(x), & x \in \Omega \\ f_2(x), & x \in \mathbf{R}^N \setminus \overline{\Omega}. \end{cases}$$

$\Rightarrow f \in BV(\mathbf{R}^N)$ and

$$\int_{\mathbf{R}^N} |\nabla f| = \int_\Omega |\nabla f_1| + \int_{\mathbf{R}^N \setminus \overline{\Omega}} |\nabla f_2|$$

$$+ \int_{\partial\Omega} |f_1|_{\partial\Omega} - f_2|_{\partial\Omega}| d\mathcal{H}^{N-1}.$$

Corollary. $\Omega, \mathcal{B} \subset \mathbf{R}^N$ open, bounded with Lipschitz boundary, and $\overline{\Omega} \subset \mathcal{B}$.

$f_1 \in BV(\Omega)$. Put

$$f(x) = \begin{cases} f_1(x), & x \in \Omega \\ 0, & x \in \mathcal{B} \setminus \Omega. \end{cases}$$

$\Rightarrow f \in BV(\mathcal{B})$ and

$$\int_{\mathcal{B}} |\nabla f| = \int_\Omega |\nabla f_1| + \int_{\partial\Omega} |f_1|_{\partial\Omega}| d\mathcal{H}^{N-1}.$$

Sets of finite perimeter

Motivation

From the Extension theorem with (Ω open, bounded, Lipschitz boundary) $f_1 = 1$ on Ω, $f_2 = 0$ on $\mathbf{R}^N \setminus \Omega$. Extension $f = \chi_\Omega$ and

$$\int_{\mathbf{R}^N} |\nabla \chi_\Omega| = \int_{\partial\Omega} d\mathcal{H}^{N-1} = \mathcal{H}^{N-1}(\partial\Omega).$$

$\Rightarrow \displaystyle\int_{\mathbf{R}^N} |\nabla \chi_\Omega| =$ perimeter of Ω.

Definition. $E \subset \mathbf{R}^N$ measurable, $\Omega \subset \mathbf{R}^N$ open.

E is set of finite perimeter in Ω if $\chi_E \in BV(\Omega)$.

Put $P(E, \Omega) = \displaystyle\int_\Omega |\nabla \chi_E|$.

Minimal partitioning surfaces

Problem. Ω open, bounded. Divide Ω in 2 regions with equal volume such that the perimeter of the interface is smallest.

Theorem. (Theorem 1.4, [Struwe])

$\Omega \subset \mathbf{R}^N$ open, bounded, with Lipschitz boundary. Then, $\exists$ measurable subset $G \subset \Omega$ such that

$$m(G) = m(\Omega \setminus G)(= \frac{m(\Omega)}{2}),$$

$(m = \mathcal{L}^N)$ such that

$$P(G,\Omega)\left(= \int_\Omega |\nabla \chi_G|\right) \le P(F,\Omega)\left(= \int_\Omega |\nabla \chi_F|\right),$$

for all $F \subset \Omega$ measurable such that

$$m(F) = m(\Omega \setminus F)(= \frac{m(\Omega)}{2}).$$

Proof. Put

$$M = \{\chi_F : F \subset \Omega, \text{ measurable and } m(F) = m(\Omega \backslash F)\}$$

$M \neq \emptyset$ (Intermediate value theorem). Put

$$E : M \to [0,\infty],\ E(u) = \int_\Omega |\nabla u|,$$

$(E(\chi_F) = P(F,\Omega))$. $E \geq 0$. $E \not\equiv \infty$.

Choose sequence $\{u_k := \chi_{F_k}\} \subset M$:

$E(u_k) \left(= \int_\Omega |\nabla u_k|\right) \to \inf\{E(u) : u \in M\}(< \infty)$.

$$\int_\Omega |u_k| dx = m(F_k) = \frac{m(\Omega)}{2}(< \infty),\ \forall k.$$

$\Rightarrow$ $\{u_k\}$ is bounded in $BV(\Omega)$.

$\Rightarrow$ (Passing to subsequences) $\exists f \in L^1(\Omega)$:

$$\begin{cases} u_k(= \chi_{F_k}) \to f \ \text{ in } \ L^1(\Omega) \\ u_k(= \chi_{F_k}) \to f \ \text{ a.e. on } \ \Omega. \end{cases}$$

$\Rightarrow$ $f = \chi_G$, G measurable.

$$m(G) = \int_\Omega \chi_G = \lim \int_\Omega \chi_{F_k} = \frac{m(\Omega)}{2}.$$

$\Rightarrow$ $f = \chi_G \in M$.

(Lower Semicontinuity property) $\Rightarrow$

$$P(G,\Omega) = \int_\Omega |\nabla \chi_G| \le \liminf \int_\Omega |\nabla \chi_{F_k}|$$

$$= \inf\{P(F,\Omega) : \chi_F \in M\}.$$

$\Rightarrow$ $P(G,\Omega) = \min\{P(F,\Omega) : \chi_F \in M\}.$

Abstract existence result - Direct method

Theorem. (Theorem 1.1, [Struwe])

M: Hausdorff topological space, $E : M \to \mathbf{R} \cup \{\infty\}$:

$$K_\alpha = \{u \in M : E(u) \le \alpha\}$$

is compact (or sequentially compact), $\forall \alpha \in \mathbf{R}$.

$\Rightarrow$ E is bounded below and attains $\inf\{E(u) : u \in M\}$.

(Minimization problem: $w \in M$, $E(w) = \min_{u \in M} E(u)$ has a solution.)

Proof. (for sequentially compact case) Let $\{u_k\} \subset M$:

$$E(u_k) \to \alpha_0 := \inf\{E(u) : u \in M\} \ge -\infty.$$

For $\alpha > \alpha_0$, $u_k \in K_\alpha$ for almost all $k \in \mathbf{N}$.

$\Rightarrow$ (Passing to a subsequence) $u_k \to u \in M$.

$u \in K_\alpha$ $(E(u) \le \alpha)$, $\forall \alpha > \alpha_0$.

$\Rightarrow E(u) \leq \alpha_0 \Rightarrow \alpha_0 > -\infty$ and $E(u) = \alpha_0$.

Remark. In Minimal partitioning surface problem, K_α is sequentially compact with respect to the $L^1(\Omega)$ topology.

$(\{u_k\} \subset K_\alpha \Rightarrow \int_\Omega |\nabla u_k| = E(u_k) \leq \alpha, \ \forall k$

$\Rightarrow \{u_k\}$ is bounded in $BV(\Omega)$

$\Rightarrow \exists \{u_{k_j}\} \subset \{u_k\}$: $u_{k_j} \to u$ in $L^1(\Omega)$.

$u \in K_\alpha$ by lower semicontinuity property.)

Direct method and Relaxation

Abstract existence result

Find conditions such that the sub-level sets K_α are compact (closed and relatively compact).

(X, τ) is a Hausdorff topological space.

$F : X \to \mathbf{R} \cup \{\infty\}$.

Definition.

(i) F is τ-lower semicontinuous (τ-l.s.c.) if $\{x \in X : F(x) \le \alpha\} (= K_\alpha) = F^{-1}((-\infty, \alpha])$ is closed in (X, τ).

(ii) F is τ-sequentially lower semicontinuous (τ-seq.l.s.c.) if for $\{x_k\} \subset X$, $x \in X$, $x_k \xrightarrow{\tau} x \Rightarrow F(x) \le \liminf F(x_k)$.

(iii) F is τ-coercive (τ-sequentially coercive) if for each $\alpha \in \mathbf{R}$, $\exists$ C_α compact (sequentially compact) in X such that

$$\{x \in X : F(x) \le \alpha\} (= K_\alpha) \subset C_\alpha.$$

Theorem. If F is τ-l.s.c (τ-seq.l.s.c) and τ-coercive (τ-seq. coercive) then the minimization problem

$$F(u) = \min_{v \in X} F(v)$$

has a solution.

Proof. For each $\alpha \in \mathbf{R}$, $\{x \in X : F(x) \leq \alpha\}$ is compact (closed and relatively compact).

Proposition.

$X = V^*$: Dual of Banach space V (in particular, X is reflexive), τ : weak* topology in X.

$F : X \to \mathbf{R} \cup \{\infty\}$ is τ-coercive

$\Leftrightarrow \lim_{\|x\|\to\infty} F(x) = \infty$.

Proof. ($\Leftarrow$) Assume $\lim_{\|x\|\to\infty} F(x) = \infty$.

$\Rightarrow \forall \alpha \in \mathbf{R}$, $\exists m_\alpha > 0$: $\|x\| > m_\alpha \Rightarrow F(x) > \alpha$

$\Rightarrow \{x : F(x) \leq \alpha\} \subset \{x : \|x\| \leq m_\alpha\}$- weak* compact.

($\Rightarrow$) Assume $\exists \{x_n\} \subset X$: $\|x_n\| \to \infty$ and $F(x_n) \leq \alpha$, $\forall n$.

$\Rightarrow \{x : F(x) \leq \alpha\}$ is not bounded $\Rightarrow$ not subset of weak* compact set.

Relaxation

• In Existence theorem, if F is not τ-coercive
$\rightarrow$ used regularization and recession arguments
($\rightarrow$ compact-coercive, semi-coercive, P-coercive, property (P), etc.)

• If F is not τ-l.s.c. $\Rightarrow$ there are minimizing sequences without leading to minimizers

$\rightarrow$ Replace F by τ-l.s.c. functional which still preserves minimum and other properties of F
$\rightarrow$ Relaxed functional

Definition. (X, τ): Hausdorff topological space.
$F : X \rightarrow \mathbf{R} \cup \{\infty\}$.

τ-relaxed functional (τ-relaxation) of F,

$$\tilde{F}(x) = (\Gamma(\tau^-)F(x)) = \sup\{G(x) : G \text{ is } \tau\text{-l.s.c. and } G \leq F \text{ on } X\}.$$

Properties of relaxed functional $\tilde{F}$

(i) $\tilde{F}$ is τ-l.s.c.

(ii) $\tilde{F}(x) = \liminf_{y \to x} F(y)$, $\forall x \in X$.

(iii) $\inf_{y \in X} F(y) \leq \tilde{F}(x) \leq F(x)$, $\forall x \in X$,

$$\Rightarrow \quad \inf_{y \in X} F(y) = \inf_{y \in X} \tilde{F}(y).$$

(iv) F is τ-coercive $\Rightarrow$ $\tilde{F}$ is τ-coercive

(v) If $\{x_n\}$ is a minimizing sequence of F ($F(x_n) \to \inf_{y \in X} F(y)$) and $x_n \to x$,

$\Rightarrow$ x is a minimum point of $\tilde{F}$.

Proof.

(i) Supremum of family of l.s.c functionals is l.s.c.

(ii) Functional $H(x) = \liminf_{y \to x} F(y)$ is τ-l.s.c and $H \leq F$ in X.

$\Rightarrow H \le \tilde{F}$.

If $G \le F$ and G is τ-l.s.c. $\Rightarrow$

$$G(x) \le \liminf_{y \to x} G(y) \le \liminf_{y \to x} F(y) = H(x)$$

$\Rightarrow \tilde{F}(x) = \sup\{G(x) : G\} \le H(x)$.

(iii) Put $G(x) = \inf_{y \in X} F(y)$. $G =$ const. $\Rightarrow \tau$-l.s.c.

$\Rightarrow G(x) \le \tilde{F}(x)$.

(iv) For $\alpha \in \mathbf{R}$, $\exists C_\alpha$ compact: $\{x : F(x) \le \alpha\} \subset C_\alpha$.

Prove: $\{x : \tilde{F}(x) \le \alpha\} \subset C_{\alpha+1}$.

Assume $\tilde{F}(x) \le \alpha$, let $U =$ neighborhood of x.

$\tilde{F}(x) = \liminf_{y \to x} F(y) < \alpha + 1$

$\Rightarrow \exists y \in U : F(y) < \alpha + 1$

$\Rightarrow U \cap C_{\alpha+1} \neq \emptyset$ (for any neighborhood U of x)

$\Rightarrow x \in \overline{C_{\alpha+1}} = C_{\alpha+1}$.

(v)

$$\begin{aligned}
\tilde{F}(x) &\leq \liminf_{y \to x} \tilde{F}(y) \leq \liminf_{n \to \infty} \tilde{F}(x_n) \\
&\leq \liminf_{n \to \infty} F(x_n) \\
&= \inf_{y \in X} F(y) = \inf_{y \in X} \tilde{F}(y).
\end{aligned}$$

Examples of relaxation

Example 1. $X = L^1(\Omega)$, $\tau = L^1$-topology. $F : X \to [0, \infty]$,

$$F(u) = \begin{cases} \int_\Omega \sqrt{1 + |\nabla u|^2} dx, & u \in C^1(\Omega) \\ \infty, & \text{otherwise.} \end{cases}$$

• Closure of $C^1(\Omega)$ with respect to BV-norm $= W^{1,1}(\Omega)$.

• $C^1(\Omega)$ is not dense in $BV(\Omega)$ with respect to BV-norm.

• $C^1(\Omega)$ is dense in $BV(\Omega)$ with respect to the metric

$$d(u, v) = \|u - v\|_{L^1} + \left| \int_\Omega |\nabla u| - \int_\Omega |\nabla v| \right|.$$

• Relaxation of F (cf. [Buttazzo]):

$$\tilde{F}(u)$$
$$= \begin{cases} \int_\Omega \sqrt{1+|(\nabla u)_a|^2}dx + \int_\Omega |(\nabla u)_s|, & u \in BV(\Omega) \\ \infty, & \text{otherwise.} \end{cases}$$
$$= \begin{cases} \int_\Omega \sqrt{1+|\nabla u|^2}dx, & u \in BV(\Omega) \\ \infty, & \text{otherwise.} \end{cases}$$

Example 2. X, τ as above. $F : X \to [0, \infty]$,

$$F(u) = \begin{cases} \int_\Omega \sqrt{1 + |\nabla u|^2} dx, & u \in C_0^1(\Omega) \\ \infty, & \text{otherwise.} \end{cases}$$

- Relaxation of F:

$$\tilde{F}(u) = \begin{cases} \int_\Omega \sqrt{1 + |(\nabla u)_a|^2} dx + \int_\Omega |(\nabla u)_s| \\ + \int_{\partial\Omega} |u|_{\partial\Omega}| d\mathcal{H}^{N-1}, \ u \in BV(\Omega) \\ \infty, \text{ otherwise.} \end{cases}$$

$$= \begin{cases} \int_\Omega \sqrt{1 + |\nabla u|^2} dx + \int_{\partial\Omega} |u|_{\partial\Omega}| d\mathcal{H}^{N-1}, u \in BV(\Omega) \\ \infty, \text{ otherwise.} \end{cases}$$

(cf. [Buttazzo], [Giusti])

- In Examples 1, 2, can replace $C^1(\Omega)$ by $W^{1,1}(\Omega)$ and $C_0^1(\Omega)$ by $W_0^{1,1}(\Omega)$.

- F is not L^1-l.s.c.

A variational inequality in BV space

Study the existence of nonzero solutions of the BVP

$$\begin{cases} -\text{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda f(x,u) \quad \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4}$$

Variational inequality set up

(4) has weak formulation

$$\int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1+|\nabla u|^2}} dx = \lambda \int_\Omega f(x,u) v dx, \tag{5}$$

for all v with $v = 0$ on $\partial\Omega$.

(4)–(5) have variational structure:

$$J_0(u) = \int_\Omega \sqrt{1+|\nabla u|^2} dx$$

is potential functional of principal operator:

$$\langle J_0'(u), v\rangle = \int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1+|\nabla u|^2}} dx.$$

Put

$$F(x,u) = \int_0^u f(x,s) ds$$

$$\mathcal{F}(u) = \int_\Omega F(x,u)dx.$$

$$\Rightarrow \langle \mathcal{F}'(u), v\rangle = \int_\Omega f(x,u)v dx.$$

- (5) $\Leftrightarrow$

$$(J_0 - \lambda\mathcal{F})'(u) = 0 \tag{6}$$

Solutions of (5) are critical points of $J_0 - \lambda\mathcal{F}$.

- J_0 is convex $\rightarrow$ Replace J_0' by subdifferential ∂J_0.

$(f \in \partial J_0(u) \Leftrightarrow J_0(v) - J_0(u) \geq \langle f, v-u\rangle, \forall v \in X)$

(6) $\Leftrightarrow \lambda\mathcal{F}'(u) \in \partial J_0(u) \Leftrightarrow$ Variational inequality

$$J_0(v) - J_0(u) \geq \lambda \int_\Omega f(x,u)(v-u)dx,\ \forall v \in X. \tag{7}$$

Case $f(x,u) = f(x)$: Linear lower order term

(7) $\Leftrightarrow J_0(v) - J_0(u) \geq \lambda \int_\Omega f(v-u)dx$

$\Leftrightarrow J_0(u) - \lambda \int_\Omega fu dx = \min_{v\in X}[J_0(v) - \lambda \int_\Omega fv dx]$

For minimization problem to have solutions: J_0 coercive and l.s.c on X

Growth of J_0: $J_0(u) \approx \|u\|$ (linear growth).

$\rightarrow$ Choice for X: $X = W_0^{1,1}(\Omega)$

$\Rightarrow$ $J_0(u) \rightarrow \infty$ as $\|u\|_{W_0^{1,1}} \rightarrow \infty$

J_0 is not L^1-l.s.c.

$\rightarrow$ Replace J_0 by relaxed functional $\tilde{J}_0$ and $W_0^{1,1}(\Omega)$ by $BV(\Omega)$.

$$\tilde{J}_0(u) = \int_\Omega \sqrt{1+|\nabla u|^2} + \int_{\partial\Omega} |u|_{\partial\Omega}| d\mathcal{H}^{N-1}, \; u \in BV(\Omega).$$

$$\int_\Omega \sqrt{1+|\nabla u|^2} = \int_\Omega \sqrt{1+|(\nabla u)_a|^2} dx + \int_\Omega |(\nabla u)_s|.$$

Theorem.

$$\int_\Omega \sqrt{1+|\nabla u|^2} = \int_\Omega \sqrt{1+|(\nabla u)_a|^2} dx + \int_\Omega |(\nabla u)_s|$$
$$= \sup\{\int_\Omega (g_{N+1} + u\text{div}(g_1, \cdots, g_N)) dx :$$
$$g \in C_c^1(\Omega, \mathbf{R}^{N+1}), \|g\|_\infty \leq 1\}.$$

(7) $\to$

$$\begin{cases} \tilde{J}_0(v) - \tilde{J}_0(u) \geq \lambda \int_\Omega f(x,u)(v-u)dx, \ \forall v \in BV(\Omega) \\ u \in BV(\Omega). \end{cases} \tag{8}$$

- $\mathcal{B} = B_R(0)$ in $\mathbf{R}^N$, R large: $\overline{\Omega} \subset \mathcal{B}$.

For $u \in BV(\Omega)$, extend

$$\tilde{u}(x) = \begin{cases} u(x), & x \in \Omega \\ 0, & x \in \mathcal{B} \setminus \Omega. \end{cases}$$

$\Rightarrow$ $\tilde{u} \in BV(\mathcal{B})$ and

$$\int_\mathcal{B} \sqrt{1+|\nabla \tilde{u}|^2} = \int_\Omega \sqrt{1+|\nabla u|^2} + \int_{\partial\Omega} |u|_{\partial\Omega}| d\mathcal{H}^{N-1}$$
$$= \tilde{J}_0(u).$$

Put $J(u) = \int_\mathcal{B} \sqrt{1+|\nabla u|^2}$, $u \in BV(\mathcal{B})$:

$J(\tilde{u}) = \tilde{J}_0(u)$.

Extend $f(x,u)$ from $x \in \Omega$ to $x \in \mathcal{B}$ by $f(x,u) = 0$ for $x \in \mathcal{B} \setminus \Omega$.

Put $X = \{u \in BV(\mathcal{B}) : u = 0$ a.e. in $\mathcal{B} \setminus \Omega\}$.

• X is closed subspace of $BV(\Omega)$ (also with respect to L^1-topology).

(8) $\rightarrow$

$$\begin{cases} J(v) - J(u) \geq \lambda \int_{\mathcal{B}} f(x,u)(v-u)dx, \ \forall v \in X \\ u \in X. \end{cases} \tag{9}$$

Conclusion.

• Equation (4) leads to variational inequality (9).

• If $f(x,0) = 0$, 0 is a trivial solution of (4) and (9). Existence of nontrivial solution for some values of λ?

Function space X

X is Banach space with norm

$$\|u\| = \|u\|_{BV(\mathcal{B})} = \|u\|_{L^1(\mathcal{B})} + \int_{\mathcal{B}} |\nabla u|$$

Poincaré's inequality on X

For each $\beta \in \left[1, \frac{N}{N-1}\right]$, $\exists\ C_\beta > 0$:

$$(\|u\|_{L^\beta(\mathcal{B})} =) \left(\int_{\mathcal{B}} |u|^\beta dx\right)^{\frac{1}{\beta}} \leq C_\beta \int_{\mathcal{B}} |\nabla u|,\ \forall u \in X.$$

$\Rightarrow$ Norm $\|u\|_0 = \int_{\mathcal{B}} |\nabla u|$ is equivalent to $\|u\|_{BV(\mathcal{B})}$ on X.

Nontrivial solutions as local minimizers

Lemma. X: Banach space (norm vector space).

$$J : X \to \mathbf{R} \cup \{\infty\} \ \text{ convex,}$$

$\mathcal{F} : X \to \mathbf{R}$ Gâteaux differentiable.

$u \in X$ is a local minimum point of $I = J - \mathcal{F}$, then u is a solution of the variational inequality

$$J(v) - J(u) - \langle \mathcal{F}'(u), v - u\rangle \geq 0, \ \forall v \in X.$$

Assumptions on f

(A) $f : \mathcal{B} \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function ($f(x,\xi) = 0$, $\xi \in \mathcal{B} \setminus \Omega$),

$\exists q \in \left(1, \min\left\{2, \dfrac{N}{N-1}\right\}\right)$, $d_1, d_2 > 0$:

$|f(x,\xi)| \leq d_1 \xi^{q-1} + d_2$, for a.e. $x \in \mathcal{B}$, all $\xi \in \mathbf{R}$,

(B) $\exists r \in (1,2)$, $d_3, \xi_0 > 0$:

$$f(x,\xi) \geq d_3 \xi^{r-1}, \ \text{ a.e. } x \in \mathcal{B}, \text{ all } \xi \in [0, \xi_0)$$

Existence result Assume (A), (B).

- $\exists \lambda_* > 0$: $\forall \lambda \in (0, \lambda_*)$, $I_\lambda = J - \lambda \mathcal{F}$ has nonzero local minimum point u_λ in the open ball $\{u \in X : \|u\|_0 < \lambda^\alpha\}$ with some $0 < \alpha < 1$.

- u_λ is a solution of (9).

- u_λ is stable: $\int_{\mathcal{B}} |\nabla u_\lambda| \to 0$ as $\lambda \to 0$.

Existence of saddle points

Assumption

(C) $\exists \gamma > 1$ and $\xi_1 > 0$:

$$f(x,\xi)\xi \geq \gamma \int_0^\xi f(x,\eta)d\eta, \text{ a.e. } x \in \mathcal{B}, \text{ all } \xi \in [\xi_1, \infty).$$

For $\xi > 0$, put

$$g_\xi(t) = \frac{t}{[(1+t^2)^{1/2}+1](t^{q-1}+\xi)} \; (t > 0).$$

$$\Rightarrow M(\xi) = \sup_{0<t<\infty} g_\xi(t) \in (0,\infty), \; \forall \xi \in (0,\infty).$$

Existence result

Assume (A), (C), and

$$M\left(\frac{C_1 d_2 q}{d_1 C_q^q |\mathcal{B}|^{q-1}}\right) > \frac{\lambda}{q} d_1 C_q^q |\mathcal{B}|^{q-1},$$

(d_1, d_2, q in (A), C_1, C_q in Poincaré's inequality).

$\Rightarrow \exists$ nontrivial solution u^λ of inequality (9):

$$u^\lambda = (L^1) - \lim_{n\to\infty} u_n,$$

$$\lim_{n\to\infty} I_\lambda(u_n) = \inf_{f\in\Gamma} \sup_{t\in[0,1]} I_\lambda[f(t))],$$

with

$$\Gamma = \{f \in C([0,1], X) : f(0) = 0, f(1) = e\},$$

($e \in BV(\mathcal{B})$, $\|e\|_0$ large).

Corollary.

• Under (A), (C), $\exists \lambda_* > 0$: $\forall \lambda \in (0, \lambda_*)$, (9) has solution $u^\lambda \neq 0$.

• Under (A), (B), (C), and (sign condition)

$$\text{(D)}\ f(x,u) \begin{cases} \geq 0, & u \geq 0 \\ = 0, & u < 0, \end{cases} \quad \text{a.e. } x \in \mathcal{B},$$

$$\Rightarrow \lim_{\lambda \to 0} \|u^\lambda\|_0 = \infty \text{ (unstable solutions).}$$

• Under (A), (B), (C), (D), (9) has at least 2 nontrivial solutions for $\lambda > 0$ sufficiently small.

References

[Buttazzo] G. Buttazzo, Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations, Pitman Research Notes in Mathematics, Vol. 207, Longman Scientific & Technical, Harlow, 1989.

[Evans] L. C. Evans and R. F. Gariepy, Measure Theory and Fine Properties of Functions, Studies in Advanced Mathematics, CRC Press, 1992.

[Giusti] E. Giusti, Minimal Surfaces and Functions of Bounded Variations, Monographs in Mathematics, Birkhäuser, 1984.

[Struwe] M. Struwe, Variational Methods, 2nd. Ed. Springer, 1991.

[Ziemer] W. P. Ziemer, Weakly Differentiable Functions, Graduate Texts in Mathematics, Springer, 1989.