Variational Inequalities of Elliptic and Parabolic Type

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May 20, 2002

Abstract

These lectures constitute a short introduction to the subject of variational inequalities of elliptic and parabolic types. After discussing several illustrative examples in detail, we present basic existence results for elliptic variational inequalities which will subsequently also be used to study problems of parabolic type.

Contents

1 Introduction

The study of evolution problems where the state of the system is subject to some set of constraints has a long history and its beginnings are nearly simultaneous to the early studies of variational inequalities.

Since such problems are, by their very nature, nonlinear problems, methods complementing the semigroup theoretic approach ([4], [16], [32], [52]), used for the study of evolution equations had to be devised.

These methods are mainly based on existence results for static variational inequalities and go back to theories presented in [6], [7], [45], and have been discussed in detail in various other places, e.g., [33], [59], [62].

The lectures are based on a recent survey [57] and some earlier lecture notes [58]. We begin in Section 2 by presenting some illustrative examples of elliptic and parabolic variational inequalities and establish some notation to be used throughout this paper. We then present a brief survey and some examples of results about static elliptic variational inequalities which will subsequently be used to derive existence results for parabolic variational inequalities. We then discuss a method for establishing the existence of solutions to parabolic inequalities. This method is based on the forward Euler method for the numerical solution of initial value problems and is often referred to as *the method of lines* or *Rothe's* method.

Although the study of variational inequalities dates back to the origins of the calculus of variations, their systematic development began in the sixties with the work of Fichera ([25]) and Stampacchia ([60], [61]), which was motivated by problems in mechanics (obstacle problems in elasticity - the Signorini problem) and potential theory (the study of the capacity of sets). After the fundamental work of Lions and Stampacchia ([48]), the study of variational inequalities intensified and became an important subject in nonlinear analysis. The rapid growth of the theory was made possible by the work of Brézis ([5], [6]), Browder ([9], [10]), Kinderlehrer ([35]), Duvaut and Lions ([21]), $\cdots$. It brought about important contributions to nonlinear analysis, calculus of variations, optimization theory, optimal control, and to many branches of mechanics, mathematical physics, and engineering.

For additional material on elliptic and parabolic variational inequalities, see [4], [26], [33], [44], [45], [46], [51], and [59].

2 Examples

This section presents several examples that motivate the study of parabolic variational inequalities and indicate their range of applicability. In Section 2.2, we introduce the subject with a linear diffusion equation whose nonlinear boundary conditions represent a semipermeable boundary. We then examine two problems for the p–Laplacian, a parabolic obstacle problem (Section 2.3) and a nonlinear evolution equation (Section 2.4). These three examples guide the way to the general formulation of parabolic variational inequalities discussed in Section 3.

2.1 The bending of an elastic beam encountering an obstacle

An elementary example of a variational inequality is the following simple deformation problem of a beam constrained by an obstacle. If we consider a homogeneous elastic beam occupying an interval $[a, b]$ with clamped ends and subject to a force f, and lying above an obstacle ψ, where ψ is a measurable function, the displacement of the beam is then constrained, and the set of admissible displacements is described by the convex set

$$K = \{v : v \geq \psi, \text{ a.e. on } [a, b]\}.$$

Using the principle of energy minimization, the deflection u of the beam must satisfy the following minimization problem:

$$u \in K : E(u) \leq E(v),\ \forall v \in K, \tag{2.1}$$

where

$$E(v) = \frac{1}{2}\int_a^b (v'')^2 - \int_a^b fv$$

denotes the potential energy. Using the fact that K is a convex set, we must have that

$$(1-t)u + tv \in K,\ \forall v \in K,\ \forall t \in [0,1],$$

and, hence, the function

$$i(t) = E((1-t)u + tv),$$

must have a minimum at $t = 0$, and $i'(0) \geq 0$, i.e.

$$i'(0) = \int_a^b u''(v-u)'' - \int_a^b f(v-u) \geq 0,\ \forall v \in K, \tag{2.2}$$

which can be viewed as the Euler-Lagrange inequality corresponding to (2.1). On the other hand, if u_1 and u_2 both satisfy (2.2), then, after an elementary calculation, we find that

$$-\int_a^b (u_1'' - u_2'')^2 \geq 0$$

or

$$u_1'' = u_2'', \text{ on } [a,b],$$

and, because u_1 and u_2 both satisfy the same boundary conditions, we conclude that

$$u_1 = u_2, \text{ on } [a,b].$$

Thus, the minimization problem (2.1) is equivalent to the variational inequality (2.2).

In a similar vein, minimization problems on a more abstract level lead to variational inequalities. For example, if F is a real convex functional of class C^1, defined on a Banach space V, and K is a closed convex subset of V, then, the minimization problem

$$u \in K : F(u) \leq F(v), \ \forall v \in K, \tag{2.3}$$

is equivalent to the variational inequality

$$u \in K : \langle F'(u), v-u \rangle \geq 0, \ \forall v \in K, \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between V and its dual space V^*; the argument for the equivalence may be found in ([21]) and is very similar to the one given above.

Detailed presentations and surveys of the theory of variational inequalities and their applications may be found in [35] and [45] (general theory and applications), [3] and [26] (applications to free boundary problems), [21] and [56] (applications of variational inequalities in physics and mechanics), [44] (bifurcation theory).

Some other recent work which is concerned with variational inequalities connected with quasilinear operators may be found in [42], [40], [41], [43], [28], [27], [15].

2.2 Diffusion with a semipermeable membrane

The next example is a model problem describing diffusion in a domain with a semi–permeable boundary ([45], [51]). Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with smooth boundary Γ, let the final time $T < \infty$ be given, and consider the problem of finding $u = u(x,t)$ such that

$$\frac{\partial u}{\partial t} - \Delta u = f \quad \text{for} \quad (x,t) \in \Omega \times (0,T), \tag{2.5}$$

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \overline{\Omega}, \tag{2.6}$$

$$u \geq 0, \quad \frac{\partial u}{\partial \nu} \geq 0, \quad \text{and} \quad u \frac{\partial u}{\partial \nu} = 0 \quad \text{for} \quad (x,t) \in \Gamma \times (0,T), \quad (2.7)$$

where Δ is the Laplacian with respect to x. With $V = H^1(\Omega)$, we look for $u \in \mathcal{V} = L^2(0,T;V)$, the Banach space of functions $v : [0,T] \to V$ with norm

$$\|v\|_{\mathcal{V}} = \left(\int_0^T \|v(s)\|_V^2 \, dt \right)^{1/2}. \quad (2.8)$$

Furthermore, we require that $f(t) \in V^*$ for a.e. $t \in (0,T)$ and that the initial datum $u_0 \in H = L^2(\Omega)$.

The nonlinear boundary conditions (2.7) lead to this problem's formulation as a variational inequality. In fact, if u solves (2.5)–(2.7) and t is an arbitrarily chosen point in $(0,T)$, then $u(t)$ clearly belongs to the closed convex set $K \subset V$ defined by

$$K = \{ v \in V \mid v(x) \geq 0 \quad \text{for} \quad x \in \Gamma \}. \quad (2.9)$$

For any $v \in \mathcal{V}$ with $v(t) \in K$, multiplying both sides of (2.5) by $v(t) - u(t)$ and integrating over Ω produces the identity

$$\int_\Omega \big(u'(t) - f(t)\big) \left(v(t) - u(t)\right) \, dx = \int_\Omega \Delta u(t) \left(v(t) - u(t)\right) \, dx, \quad (2.10)$$

where we now write u' for the derivative $\partial u / \partial t$, since we view u as a function of time with values in V.

Using the divergence theorem and the boundary conditions (2.7), we have

$$\int_\Omega \Delta u(t) \left(v(t) - u(t)\right) + \nabla u(t) \cdot \nabla \left(v(t) - u(t)\right) \, dx = \int_\Gamma \left(v(t) - u(t)\right) \frac{\partial u(t)}{\partial \nu} \, dx \; \geq \; 0, \quad (2.11)$$

from which we see that

$$\int_\Omega \Delta u(t) \left(v(t) - u(t)\right) \, dx \geq - \int_\Omega \nabla u(t) \cdot \nabla \left(v(t) - u(t)\right) \, dx. \quad (2.12)$$

Combining (2.12) with (2.10), we see that u belongs to $\mathcal{K}$ and satisfies the *parabolic variational inequality*

$$\int_\Omega u'(t) \left(v(t) - u(t)\right) + \nabla u(t) \cdot \nabla \left(v(t) - u(t)\right) \, dx \geq \int_\Omega f(t) \left(v(t) - u(t)\right) \, dx, \; \forall \, v \in \mathcal{K}, \; a.e. \; t \in (0,T), \quad (2.13)$$

where $\mathcal{K}$ denotes the collection of functions $v \in \mathcal{V}$ such that $v(t) \in K$ for a.e. $t \in (0,T)$.

Although this cone $\mathcal{K}$ might appear to omit some of the boundary conditions posed in (2.7), we will see that these two problems are indeed equivalent. To this end, suppose that $u \in \mathcal{K}$ solves (2.13), and let

$$v(t) = u(t) + \varepsilon\,\zeta$$

for $t \in (0,T)$, $\varepsilon \neq 0$, and an arbitrary test function $\zeta \in C_0^\infty(\Omega)$. As this function v belongs to $\mathcal{K}$, we may substitute it into (2.13) to obtain the inequality

$$\varepsilon \int_\Omega \big(u'(t)\zeta + \nabla u(t)\cdot\nabla\zeta - f(t)\zeta\big)\; dx \geq 0,$$

which is actually the *equation*

$$\int_\Omega \big(u'(t)\zeta + \nabla u(t)\cdot\nabla\zeta - f(t)\zeta\big)\; dx = 0,$$

since ε may be positive or negative. In the sense of distributions, u therefore satisfies the heat equation (2.5) in $\Omega \times (0,T)$.

It remains to verify the boundary conditions that are not included in the definition of $\mathcal{K}$. Observe that the relations

$$\int_\Omega \big(u'(t)w(t) + \nabla u(t)\cdot\nabla w(t)\big)\; dx \geq \int_\Omega f(t)w(t)\,dx,\ \ \forall w \in \mathcal{K},\ \ a.e.\ t \in (0,T) \tag{2.14}$$

and

$$\int_\Omega \big(u'(t)\,(u(t)\zeta) + \nabla u(t)\cdot\nabla\,(u(t)\zeta)\big)\; dx = \int_\Omega f(t)\,(u(t)\zeta)\; dx,\ \ \forall\zeta \in C^\infty(\overline{\Omega}),\ \ a.e.\ t \in (0,T), \tag{2.15}$$

follow from (2.13) by first choosing $v = w + u$, for $w \in \mathcal{K}$, in (2.13) and then choosing $w(t) = u(t)(1 \pm \zeta)$, $\zeta \in C^\infty(\overline{\Omega}), |\zeta(x)| \leq 1$ in (2.13).

Using equation (2.5), we rewrite (2.14) as

$$\int_\Omega \big(u'(t)w(t) + \nabla u(t)\cdot\nabla w(t)\big)\; dx \geq \int_\Omega \big(u'(t) - \Delta u(t)\big)\, w(t)\,dx,\ \ \forall w \in \mathcal{K},\ \ a.e.\ t \in (0,T),$$

which is simply

$$\int_\Omega \left(\nabla w(t)\cdot\nabla u(t) + w(t)\Delta u(t)\right)\, dx \geq 0,\ \forall w \in \mathcal{K},\ a.e.\ t \in (0,T). \tag{2.16}$$

We now apply the divergence theorem to (2.16) to find that

$$\int_\Gamma w(t)\frac{\partial u(t)}{\partial \nu}\, d\sigma \geq 0,\ \forall w \in \mathcal{K},\ a.e.\ t \in [0,T],$$

i.e.,

$$\frac{\partial u}{\partial \nu} \geq 0 \quad \text{on} \quad \Gamma \times (0,T).$$

A similar argument verifies the remaining condition; we replace $f(t)$ in (2.15) with $u'(t) - \Delta u(t)$ to obtain for $a.e.\ t \in (0,T)$

$$\int_\Omega \left(\nabla u(t)\cdot\nabla\left(u(t)\zeta\right) + u(t)\zeta\Delta u(t)\right)\, dx \;=\; 0,\ \forall\zeta \;\in\; C^\infty(\overline{\Omega}), \tag{2.17}$$

to which the divergence theorem applies to deduce that

$$\int_\Gamma u(t)\zeta\frac{\partial u(t)}{\partial \nu}\, d\sigma = 0,\ \forall\zeta \in C^\infty(\overline{\Omega}),\ a.e.\ t \in (0,T).$$

This means precisely that

$$u\,\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times (0,T).$$

The boundary conditions (2.7) thus hold, so the original diffusion problem may either be formulated as the boundary value problem (2.5), (2.6), (2.7) or as the parabolic variational inequality (2.13).

2.3 A nonlinear obstacle problem

In many problems the diffusion coefficient will not be constant but rather will depend upon the dependent variable in some manner. A class of problems that has received much attention in recent years is obtained by replacing the Laplacian term in the integral (2.13) with a term corresponding to the p–Laplacian. To do so, we let $V = W_0^{1,p}(\Omega)$ for $p > 1$, and we use $W^{-1,q}(\Omega)$ to denote the dual of V, where p and q are conjugate exponents, $1/p + 1/q = 1$. Letting $\langle\cdot\,,\cdot\rangle$ denote the pairing between these spaces, we define the operator

$$A_p : W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega)$$

by

$$\langle A_p u\, ,\, v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx \tag{2.18}$$

for $u, v \in W_0^{1,p}(\Omega)$. The operator A_p is defined by the p–Laplacian Δ_p,

$$\Delta_p(u) = -\nabla \cdot \left(|\nabla u|^{p-2} \nabla u\right). \tag{2.19}$$

As in the previous section, $\mathcal{V}$ denotes the space $L^2\,(0,T;V)$, and $\mathcal{K}$ is the set of functions $v \in \mathcal{V}$ such that $v(t) \in K$ for a.e. $t \in (0,T)$, where $K \subset V$ is a closed convex set to be specified below.

With this setup, we consider the problem of finding $u \in \mathcal{K}$ with the prescribed initial value

$$u(0) = u_0 \in L^2(\Omega) \tag{2.20}$$

and such that the inequality

$$\int_\Omega u'(t)\,(v(t) - u(t)) + |\nabla u(t)|^{p-2} \nabla u(t) \cdot \nabla\,(v(t) - u(t))\;dx \\ \geq \int_\Omega f(t)\,(v(t) - u(t))\;dx \tag{2.21}$$

holds for a.e. $t \in (0,T)$ and for all $v \in \mathcal{K}$, where

$$K = \{\, v \in V \mid v \geq \psi \,\}, \tag{2.22}$$

for a given $\psi \in W^{1,p}(\Omega)$ satisfying $\psi \leq 0$ on Γ. The closed convex set $\mathcal{K}$ represents an imposed constraint determined by the *obstacle* ψ. The existence results to follow guarantee a solution $u \in \mathcal{K}$ of the parabolic obstacle problem (2.21) for the p–Laplacian; we devote the remainder of this section to a description of the solution.

From the definition of the constraint set $\mathcal{K}$, we see that, at any time $t \in (0,T)$, $u(t)$ partitions Ω into the two sets

$$\Omega^+(t) = \{\, x \in \Omega \mid u(x,t) > \psi(x) \,\}$$

and

$$\Omega^0(t) = \{\, x \in \Omega \mid u(x,t) = \psi(x) \,\}.$$

For $\varepsilon \neq 0$ and any test function $\zeta \in C_0^\infty(\Omega^+(t))$, we follow the argument given earlier and substitute $v(t) = u(t) + \varepsilon\,\zeta$ into (2.21) to obtain

$$\int_\Omega u'(t)\zeta + |\nabla u(t)|^{p-2} \nabla u(t) \cdot \nabla \zeta - f(t)\zeta \, dx = 0, \tag{2.23}$$

which means that the equation

$$\frac{\partial u}{\partial t} - \Delta_p u = f \tag{2.24}$$

holds in the sense of distributions, where Δ_p is the p–Laplacian defined above (2.19). The solution u of the parabolic variational inequality (2.21) therefore satisfies the partial differential equation (2.24) on

$$\Omega^+ = \bigcup_{t \in (0,T)} \Omega^+(t)$$

and equals the obstacle ψ on

$$\Omega^0 = \bigcup_{t \in (0,T)} \Omega^0(t).$$

We emphasize, however, that the boundary of Ω^0, the *free boundary* for this problem, is unknown *a priori*. In contrast to the example in Section 2.2, this problem cannot be recast as a classical boundary value problem. This example indicates the role of variational inequalities in the study of free boundary problems arising from constraints.

2.4 A nonlinear evolution equation

Using the indicator functional ϕ_K of the constraint set K defined by (2.22), we can formulate the obstacle problem of the previous section as a parabolic variational inequality over the entire space $\mathcal{V}$. Specifically, $u \in \mathcal{V}$ solves inequality (2.21) if and only if it solves the inequality

$$\begin{aligned} \int_\Omega u'(t)\,(v(t) - u(t)) + |\nabla u(t)|^{p-2} \nabla u(t) \cdot \nabla\,(v(t) - u(t))\; dx \\ + \phi_K(v(t)) - \phi_K(u(t)) \;\geq \int_\Omega f(t)\,(v(t) - u(t))\; dx, \\ \forall\, v \in \mathcal{V},\; a.e.\; t \in (0,T), \end{aligned} \tag{2.25}$$

where $\phi_K : V \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\phi_K(v) = \begin{cases} 0, & \text{if} \quad v \in K, \\ +\infty, & \text{if} \quad v \notin K. \end{cases} \tag{2.26}$$

As K is convex and closed, the functional ϕ_K is convex and lower semicontinuous ([8], [22]). It is then natural to consider replacing ϕ_K

in (2.25) with more general convex lower semicontinuous functionals. To explore this idea, we define the functional $\phi : V \to \mathbb{R} \cup \{+\infty\}$ by

$$\phi(v) = \alpha^{-1} \int_\Omega |v|^\alpha \, dx, \tag{2.27}$$

where we choose the exponent α in accordance with the convexity requirement and with the Rellich–Kondrachov theorem ([1], [8]):

- for $p < N$, $\alpha \in (1, p^*)$, where $p^* = \frac{Np}{N-p}$ is the Sobolev conjugate of p ;
- for $p \geq N$, $\alpha \in (1, \infty)$.

The functional ϕ is lower semicontinuous by Fatou's lemma and, for $\alpha > 1$, is Fréchet differentiable, with derivative $D\phi : V \to V^*$ given by

$$\langle \, D\phi(u) \, , \, v \, \rangle = \int_\Omega |u|^{\alpha-2} u v \, dx \quad \text{for} \quad u, v \in V, \tag{2.28}$$

where $\langle \, \cdot \, , \, \cdot \, \rangle$ denotes the pairing between V and its dual V^*.

By the existence results in the following sections, there exists a solution $u \in \mathcal{V}$ of the corresponding variational inequality

$$\begin{aligned} \int_\Omega u'(t) \, (v(t) - u(t)) + |\nabla u(t)|^{p-2} \nabla u(t) \cdot \nabla \left(v(t) - u(t) \right) \, dx \\ + \phi(v(t)) - \phi(u(t)) \, \geq \int_\Omega f(t) \, (v(t) - u(t)) \, dx, \\ \forall \, v \in \mathcal{V}, \; a.e. \; t \in (0, T). \end{aligned} \tag{2.29}$$

As this holds for all $v \in \mathcal{V}$, we may substitute $v(t) = u(t) + \varepsilon \, \zeta$, for $\varepsilon > 0$, into (2.29) to find that $u(t)$ satisfies

$$\begin{aligned} \varepsilon \int_\Omega u'(t) \zeta + |\nabla u(t)|^{p-2} \nabla u(t) \cdot \nabla \zeta \; - f(t) \zeta \, dx \\ + \phi(u(t) + \varepsilon \zeta) - \phi(u(t)) \, \geq 0, \; \forall \, \zeta \in C_0^\infty(\Omega). \end{aligned} \tag{2.30}$$

Since ϕ is Fréchet differentiable, we have

$$\phi(u(t) + \varepsilon \zeta) - \phi(u(t)) = \langle \, D\phi(u(t)) \, , \, \varepsilon \zeta \, \rangle + o(\|\varepsilon \zeta\|).$$

Substituting this into (2.30) and dividing through by ε yields

$$\begin{aligned} \int_\Omega u'(t) \zeta + |\nabla u(t)|^{p-2} \nabla u(t) \cdot \nabla \zeta \; - f(t) \zeta \, dx \\ + \langle \, D\phi(u(t)) \, , \, \zeta \, \rangle + \frac{o(\|\varepsilon \zeta\|)}{\varepsilon} \geq 0, \; \forall \, \zeta \in C_0^\infty(\Omega). \end{aligned} \tag{2.31}$$

Letting ε tend to 0 and then repeating the argument for $\varepsilon < 0$ (which reverses inequalities), we obtain

$$\int_\Omega u'(t)\zeta + |\nabla u(t)|^{p-2}\nabla u(t)\cdot\nabla\zeta - f(t)\zeta\,dx \\ + \langle\, D\phi(u(t))\,,\,\zeta\,\rangle = 0,\ \forall\,\zeta \in C_0^\infty(\Omega). \quad (2.32)$$

Recalling (2.28), it follows that u is a solution of the nonlinear evolution equation

$$\frac{\partial u}{\partial t} - \nabla\cdot\big(|\nabla u|^{p-2}\nabla u\big) + |u|^{\alpha-2}u = f \quad \text{in} \quad \Omega\times(0,T), \qquad (2.33)$$

with initial and boundary conditions

$$u(x,0) = u_0(x) \quad \text{and} \qquad (2.34)$$

$$u(x,t) = 0 \quad \text{for} \quad x\in\Gamma. \qquad (2.35)$$

In case $\alpha = 1$, the above equation (2.33) will need to be replaced by the problem

$$\frac{\partial u}{\partial t} - \nabla\cdot\big(|\nabla u|^{p-2}\nabla u\big) + \partial(|u|) \ni f \quad \text{in} \quad \Omega\times(0,T), \qquad (2.36)$$

where

$$\partial|u| = \begin{cases} 1, & \text{if } u > 0 \\ [-1,1]\,, & \text{if } u = 0 \\ -1, & \text{if } u < 0, \end{cases}$$

(see subsequent discussion for such problems).

More general *slow-fast* inequality diffusion problems with differential operators of the form

$$\frac{\partial u}{\partial t} - \operatorname{div}\big(A(|\nabla u|^2)\nabla u\big) + \partial\phi(u),$$

with A a fast (or slow) growing function, arise naturally in many applications, as well. (See Section 3, where a static problem of this type is discussed.)

Another interesting set of applications of parabolic variational inequalities involving the p-Laplacian (or other nonlinear diffusion operators of the type just mentioned), i.e., equation (2.25), is the choice of the indicator functional ϕ_K, where the closed convex set K is given by

$$K = \{u \in W_0^{1,p}(\Omega) : |\nabla u| \le 1,\ a.e.\ x\in\Omega\}.$$

Such problems, particularly for large values of p, serve as approximate models for the formation of sandpiles, see e.g., [2], [24], [55].

These examples show that, by choosing different functionals ϕ, the formulation (2.29) captures a wide variety of problems. The next section exploits this observation.

3 The general problem

The progression of examples in Sections 2.2, 2.3, and 2.4 indicates a general formulation of parabolic variational inequalities that encompasses many different problems. Given a reflexive Banach space V and $T < \infty$, we let $\mathcal{V}$ denote the space

$$\mathcal{V} = L^2(0, T; V), \tag{3.1}$$

whose dual is the space

$$\mathcal{V}^* = L^2(0, T; V^*). \tag{3.2}$$

This identification of $\mathcal{V}^*$ is only possible because the underlying space V is reflexive ([12], [17]). These are standard function spaces in the treatment of evolution problems ([12], [16], [45], [59]). We require further that V be continuously embedded in some Hilbert space H, so that duality yields the pivot space structures

$$V \hookrightarrow H \hookrightarrow V^* \quad \text{and} \quad \mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*, \tag{3.3}$$

where $\mathcal{H} = L^2(0, T; H)$. Two consequences of (3.3) will be important for us ([59]) ; first, the embedding

$$W := \{v \in \mathcal{V} \mid v' \in \mathcal{V}^*\} \hookrightarrow C\left([0, T]; H\right) \tag{3.4}$$

holds, which shows that initial data in the Hilbert space H are appropriate for the problems that we discuss. Moreover, for functions $v \in W$, we have

$$\frac{d}{dt} \| v(t) \|_V^2 = 2 \int_\Omega v'(x, t)\, v(x, t)\, dx. \tag{3.5}$$

In addition to these spaces, we have an operator $A : V \to V^*$ that satisfies certain monotonicity and continuity conditions corresponding to the operators that arise in elliptic variational inequalities. To make the notation less cumbersome, we henceforth let $a(\cdot,\cdot)$ denote the form corresponding to A, i.e.,

$$a(u,v) := \langle\, Au\,,\, v\,\rangle, \quad \text{for} \quad u,v \in V,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the pairing between V^* and V. With this notation, we recall the definitions of the relevant properties of A ([45], [59]):

Definition 3.1
An operator $A : V \to V^*$ is

- ***monotone*** *if*

$$a(u-v,u-v)\;\geq\; 0,\;\forall\; u,v \in V, \tag{3.6}$$

  *and **strictly monotone** if equality forces $u = v$.*

- ***hemicontinuous*** *if the map*

$$t \mapsto a(u+tv,v)$$

  is continuous for each $u,v \in V$.

- ***pseudomonotone*** *if A is bounded and such that*

$$\begin{array}{c} u_n \rightharpoonup u \quad \textit{and} \quad \limsup a(u_n,u_n-u) \leq 0 \\ \textit{imply} \\ a(u,u-v) \leq \liminf a(u_n,u_n-v),\;\forall\; v \in V. \end{array} \tag{3.7}$$

As shown in [45], pseudomonotonicity ensures that A is a continuous map from V to V^*, where V is endowed with its norm topology and V^* is given the weak topology. Although we explicitly assume pseudomonotonicity of A in the problem (3.10) stated below, it suffices to verify monotonicity and hemicontinuity, as these two properties immediately imply that A is pseudomonotone ([45], [59]). As a specific example, simple calculations show that the operator A_p, induced by the p–Laplacian and introduced in Section 2.3, is monotone and hemicontinuous, so it fits the framework outlined here.

Finally, we are given a functional $\phi : V \to \mathbb{R} \cup \{+\infty\}$ which is convex, lower semicontinuous with respect to the topology of V, and whose *effective domain* $D(\phi)$,

$$D(\phi) := \{v \in V \mid \phi(v) < +\infty\,\}, \tag{3.8}$$

is nonempty.

Note that the integrals occurring in the preceding parabolic variational inequalities gave the explicit action of V^* on V. For conciseness, as in the definition of the form $a(\cdot,\cdot)$ corresponding to A, we therefore use $\langle \cdot\, , \cdot \rangle$ to denote the pairing between V^* and V, so that the following is the generalization of the problems considered in Sections 2.2, 2.3, and 2.4:

Problem 3.2

Let the spaces V, H, $\mathcal{V}$, and $\mathcal{H}$ be as described above. Suppose that the pseudomonotone operator $A : V \to V^*$ and the convex lower semicontinuous functional $\phi : V \to \mathbb{R} \cup \{+\infty\}$, with $D(\phi)$ nonempty, satisfy the coercivity condition

$$\lim_{\|v\| \to \infty} \frac{a(v, v - v_0) + \phi(v)}{\|v\|} = \infty, \tag{3.9}$$

for some $v_0 \in D(\phi)$. We seek $u \in \mathcal{V}$ such that the parabolic variational inequality

$$\begin{aligned}\langle\, u'(t) - f(t)\,,\, v(t) - u(t)\,\rangle + a(u(t), v(t) - u(t))& \\ + \phi(v(t)) - \phi(u(t)) \,\geq 0,\ \forall\, v \in \mathcal{V},\ a.e.\ t \in (0,T)&\end{aligned} \tag{3.10}$$

holds and such that u has the prescribed initial value

$$u(x,0) = u_0(x) \in H. \tag{3.11}$$

A solution u of (3.10) necessarily belongs to the effective domain of the functional ϕ. Although we did not mention the coercivity condition (3.9) in the previous examples, such conditions arise naturally in minimization problems. They are typically used to guarantee that certain approximate solutions form a bounded set; for spaces in which bounded sets are precompact, we may then extract a convergent subsequence (in the relevant topology) and try to show that the corresponding limit solves the problem in question.

4 Elliptic variational inequalities

4.1 Minimization of Functionals

4.2 Notation and assumptions

Let E be a real reflexive Banach space and let E^* be its topological dual.

Let us assume that $f\colon E \to \mathbb{R}\cup\{\infty\}$ a weakly lower semicontinuous functional which is coercive, i.e.,

$$f(v) \to \infty, \text{ as } v \to \infty,$$

and let K be a weakly closed set.

4.3 A minimization result

Theorem 4.1
Assume the above, then there exists $u \in K$ such that

$$f(u) = \min_{u\in K} f(v). \tag{4.1}$$

Proof: Since f coercive, f is bounded below on K. Hence $\alpha = \inf_{u\in K} f(v) > -\infty$. Choose a minimizing sequence $\{v_n\} \subset K$, i.e., $f(v_n) \to \alpha$. Again, since f is coercive, we obtain $\{v_n\}$ is bounded. Further, since E reflexive, we have that $\{v_n\}$ has a weakly convergent subsequence, say, after relabeling, $v_n \rightharpoonup u$. Since K weakly closed, we find that $u \in K$. Now f being weakly lower semicontinuous implies

$$f(u) \le \liminf_{n\to\infty} f(v_n),$$

and therefore

$$f(u) = \min_{v\in K} f(v),$$

proving the theorem.

Remark 4.1
Note that in the above theorem we have used the reflexivity of E to produce a convergent subsequence from a bounded sequence. Thus, if we use a different topology on E which has the property that bounded sequences have convergent subsequences (with respect to that topology) and that f is lower semicontiunous with respect to that toplogy, then the above theorem will hold in this different setting. We shall later have occasion to employ this fact.

4.3.1 Consequences

- If $K = E$ and $f \in C^1$, then

  $$f'(u) = 0,$$

  and u is a critical point for f.

- If K is convex, then

$$\langle f'(u), v-u\rangle \geq 0, \quad \forall v \in K, \tag{4.2}$$

i.e., u is a solution of a variational inequality.

4.3.2 On bilinear forms

Let

$$a\colon E \times E \to \mathbb{R}$$

be a continuous, symmetric, coercive bilinear form, i.e.,

$$|a(u,v)| \leq c_1\|u\|\|v\|, \; a(u,u) \geq c_2\|u\|^2,$$

where c_1 and c_2 are positive constants. Let $b \in E^*$ and K a weakly closed set. Let us consider the functional

$$f(u) = \frac{1}{2}a(u,u) - \langle b, u\rangle,$$

then f is coercive, weakly lower semicontinuous, and C^1, hence $\exists u \in K$ such that

$$f(u) = \min_{v\in K} f(v).$$

If K is convex, then

$$a(u, v-u) \geq \langle b, v-u\rangle, \quad \forall v \in K. \tag{4.3}$$

One immediately sees that problem (4.3) has a unique solution and that problem (4.3) and (4.1) are equivalent problems.

Let $T\colon E^* \to E$ be defined by $Tb = u$, where u is the unique solution of (4.3), then

$$\|Tb_1 - Tb_2\|_E \leq \frac{1}{c_2}\|b_1 - b_2\|_{E^*}. \tag{4.4}$$

Therefore, if

$$F : E \to E^*,$$

the variational inequality

$$a(u, v-u) \geq \langle F(u), v-u\rangle, \quad \forall v \in K, \tag{4.5}$$

is equivalent to the fixed point problem

$$u = TF(u). \tag{4.6}$$

4.3.3 Convex functionals

A functional

$$f \colon E \to \mathbb{R}$$

is strictly convex, if for all $u, v \in K$, $u \neq v$, $0 < t < 1$,

$$f(tu + (1-t)v) < tf(u) + (1-t)f(v). \tag{4.7}$$

It is convex if

$$f(tu + (1-t)v) \leq tf(u) + (1-t)f(v), \quad 0 \leq t \leq 1. \tag{4.8}$$

If f is convex and C^1, then

$$f(u) \geq f(v) + \langle f'(v), u - v \rangle, \quad \forall u, v \in K. \tag{4.9}$$

From this we deduce that if f is convex and C^1 and $v \in K$ is such that

$$\langle f'(v), u - v \rangle \geq 0, \quad \forall u \in K, \tag{4.10}$$

then

$$f(v) = \min_{w \in K} f(w). \tag{4.11}$$

And therefore if f is convex and C^1, then (4.10) and (4.11) are equivalent problems, which are uniquely solvable if f is strictly convex and if f is convex, the solution set is convex.

4.3.4 Cones

If K is a subspace of E, then (4.10) becomes

$$\langle f'(v), u \rangle = 0, \quad \forall u \in K, \tag{4.12}$$

i.e., $f'(v) \in K^{\perp} = \{ b \in E^* \mid \langle b, u \rangle = 0 \, \forall u \in K \}$.

If K is a cone , i.e.

$$u + v \in K \,,\; tu \in K \,,\; \forall u, v \in K \,,\; t \geq 0,$$

then (4.10) becomes

$$\langle f'(v), u \rangle \geq 0, \quad \forall u \in K \tag{4.13}$$

$$\langle f'(v), v \rangle = 0. \tag{4.14}$$

4.3.5 An obstacle problem

Let Ω be a bounded domain in $\mathbb{R}^N$ and let $E = L^2(\Omega)$. Let $\psi \in E$ be given and let

$$K = \{u \in E \mid u(x) \geq \psi(x), \text{ a.e. in } \Omega\}.$$

Then K is a closed, convex subset of E.

Let $y \in E$ and define $f\colon E \to \mathbb{R}$ as $f(u) = \frac{1}{2}\|u\|^2 - \int_\Omega yu$. Then there exists a unique $u \in K$ such that $f(u) = \min_{v \in K} f(v)$ and furthermore u solves the variational inequality

$$\langle f'(u), v - u \rangle \geq 0, \quad \forall v \in K,$$

i.e.,

$$\int_\Omega (u - y)(v - u) \geq 0, \quad \forall v \in K, \tag{4.15}$$

and the latter must have a unique solution. The natural candidate for this solution is $u = \max(\psi, y)$, as one easily verifies by substituting into (4.15).

4.3.6 Another example

Let $E = L^2(0,1)$, $K = \{u \mid \int_0^1 u = 1\}$. Then K is closed and convex (hence weakly closed). Let

$$f(u) = \int_0^1 u^2 = \|u\|^2,$$

then f is weakly lower semicontinuous, coercive and C^1, hence there exists a unique $u \in K$ such that $f(u) = \min_K f(v)$, which holds, if and only if, $((\cdot,\cdot)$ is the L^2 inner product)

$$(f'(u), v - u) \geq 0, \quad \forall v \in K,$$

i.e.

$$(u, v - u) \geq 0, \quad \forall v \in K,$$

or

$$(u, v) \geq (u, u), \quad \forall v \in K,$$

i.e.

$$\int_0^1 uv \geq \int_0^1 u^2, \quad \forall v \in K.$$

On the other hand

$$\left|\int_0^1 u\right| \leq \left(\int_0^1 u^2\right)^{\frac{1}{2}},$$

and hence

$$\int_0^1 uv \geq 1, \quad \forall v \in K.$$

Clearly $u = 1$ solves the inequality.

4.3.7 Some references

For additional and more detailed examples see the following: [6], [13], [14], [21], [26], [35], [47].

4.4 General variational inequalities

Let

$$a \colon E \times E \to \mathbb{R}$$

be a continuous, coercive, bilinear form, $b \in E^*$, and K a closed convex set.

4.4.1 The problem

We pose the following problem: Find (prove the existence of) $u \in K$ such that

$$a(u, v-u) \geq \langle b, v-u \rangle, \quad \forall v \in K. \tag{4.16}$$

In case a is symmetric, this problem has been solved above, thus, what is of interest here, is the case that a is not symmetric.

The development in this section follows closely the development in [48] and [35].

4.4.2 Uniqueness of the solution

Using properties of bilinear forms, one concludes that for all $b \in E^*$, problem (4.16) has at most one solution and if $b_1, b_2 \in E^*$ and solutions u_1, u_2 exist, then

$$\|u_1 - u_2\|_E \le \frac{1}{c_2}\|b_1 - b_2\|_{E^*},$$

where c_2 is a coercivity constant of a (cf. (4.4)).

4.4.3 Existence

Since the existence of a solution has been proved if a is symmetric, we write

$$a = a_e + a_o,$$

where

$$a_e(u,v) = \frac{1}{2}(a(u,v) + a(v,u))$$
$$a_o(u,v) = \frac{1}{2}(a(u,v) - a(v,u)),$$

then a_e is a continuous, symmetric, coercive, bilinear form and a_o is continuous and bilinear.

Consider the family of problems

$$a_e(u, v-u) + t a_o(u, v-u) \ge \langle b, v-u \rangle, \quad \forall v \in K, \quad 0 \le t \le 1, \tag{4.17}$$

or equivalently

$$a_e(u, v-u) \ge \langle b, v-u \rangle - t a_o(u, v-u), \quad \forall v \in K, \quad 0 \le t \le 1. \tag{4.18}$$

For $w \in K$ consider

$$a_e(u, v-u) \ge \langle b, v-u \rangle - t a_o(w, v-u), \quad \forall v \in K, \quad 0 \le t \le 1 \tag{4.19}$$

Note that for fixed $w \in K$,

$$b_w = b - t a_o(w, \cdot) \in E^*,$$

hence there exists a unique $u = Tw$ solving (4.19) and

$$\|Tw_1 - Tw_2\|_E \le \frac{1}{c_2}\|b_{w_1} - b_{w_2}\|_{E^*}.$$

On the other hand

$$\|b_{w_1} - b_{w_2}\|_{E^*} = \sup_{\|u\|=1} t|a_o(w_1, u) - a_o(w_2, u)| \le tc_1\|w_1 - w_2\|_E,$$

and hence

$$\|Tw_1 - Tw_2\|_E \le \frac{tc_1}{c_2}\|w_1 - w_2\|_E,$$

and $T\colon K \to K$ is a contraction mapping provided $\frac{tc_1}{c_2} < 1$. We hence have a unique solution of (4.18) as long as $t < \frac{c_2}{c_1}$.

Let $a_{t_0} = a_e + t_0 a_o$, where $t_0 = \frac{c_2}{2c_1}$, then a_{t_0} is coercive, continuous and bilinear, and the problem

$$a_{t_0}(u, v - u) \ge \langle d, v - u\rangle, \quad \forall v \in K \tag{4.20}$$

has a unique solution for all $d \in E^*$.

Note that the coercivity and continuity constants of a_{t_0} may be chosen the same as those of a_o. Hence by the uniqueness result in section 4.4.2 above, we have for $d_1, d_2 \in E^*$ and u_1, u_2 solutions of (4.20) that

$$\|u_1 - u_2\|_E \le \frac{1}{c_2}\|d_1 - d_2\|_{E^*}.$$

For fixed $w \in K$ consider

$$a_{t_0}(u, v - u) \ge \langle d, v - u\rangle - \tau a_o(w, v - u), \quad \forall v \in K, \tag{4.21}$$

and apply earlier reasoning to conclude that (4.21) has a unique solution $Tw \in K$, and, as long as, $0 < \tau < \frac{c_2}{c_1}$, then

$$\|Tw_1 - Tw_2\|_E \le \frac{\tau c_1}{c_2}\|w_1 - w_2\|_E.$$

Thus, we may apply the contraction mapping principle to get a unique solution of (4.19) for $0 \le t \le 2t_0$. Continuing this way we arrive at the unique solvability for $t = 1$.

4.4.4 A second order boundary value problem

Let Ω be a bounded open set in $\mathbb{R}^N$, let $\{a_{ij}(x)\}_{i,j=1}^N \subset L^\infty(\Omega)$ be such that

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \ge c_0|\xi|^2, \quad \xi \in \mathbb{R}^N, \quad c_0 > 0 \text{ a constant.}$$

Let $E = H_0^1(\Omega)$ with $\|u\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2$, and let $a(u,v)$ be given by

$$a(u,v) = \sum_{i,j} \int_\Omega a_{ij}(x)\partial_i u \partial_j v = \int_\Omega \nabla v^\top \{a_{ij}\} \nabla u.$$

Then

$$|a(u,v)| \leq c_1 \|\nabla v\|_{L^2} \|\nabla u\|_{L^2}, \quad c_1 = \max_{i,j} \|a_{ij}\|_{L^\infty(\Omega)}$$

and

$$|a(u,u)| \geq c_0 \|\nabla u\|^2_{L^2}.$$

For $b \in L^2(\Omega) \subset H_0^1(\Omega)^*$ we obtain the existence of a unique $u \in H_0^1(\Omega)$ such that

$$a(u, v-u) \geq \int_\Omega b(v-u), \quad \forall v \in H_0^1(\Omega),$$

hence

$$a(u,v) = \int_\Omega bv, \quad \forall v \in H_0^1(\Omega),$$

or in a distributional sense the partial differential equation

$$-\sum_{i,j} \partial_j(a_{ij}\partial_i u) = b$$

will have a unique solution $u \in H_0^1(\Omega)$ (see also [8], [23], [29], [47]).

4.4.5 A unilateral problem

Let $b_0, c_0 \in L^\infty(0,1)$, $h \in L^2(0,1)$, $\psi_0, \psi_1 \in \mathbb{R}$. Consider the unilateral problem

$$\begin{aligned} &-u'' + b_0 u' + c_0 u = h \\ &u(0) \geq \psi_0,\ u'(0) \leq 0 \\ &u(1) \geq \psi_1,\ u'(1) \geq 0 \\ &(u(0) - \psi_0)u'(0) = 0 = (u(1) - \psi_1)u'(1). \end{aligned} \tag{4.22}$$

We let $E = H^1(0,1)$, $\|u\|^2 = \int_0^1 u^2 + \int_0^1 (u')^2$,

$$K = \{u \in E \mid u(0) \geq \psi_0\,,\ u(1) \geq \psi_0\}.$$

On $E \times E$ define the continuous bilinear form

$$a(u,v) = \int_0^1 u'v' + \int_0^1 b_0 u'v + \int_0^1 c_0 uv.$$

Then (4.22) is equivalent to

$$a(u, v-u) \geq \int_0^1 h(v-u), \ \forall v \in K. \tag{4.23}$$

If u solves (4.22) then $u'' \in L^2(0,1)$, hence the boundary conditions make sense. Thus multiplying the differential equation in (4.22) by $v-u$ and integrating by parts one obtains

$$\int_0^1 u'(u-v)' + \int_0^1 b_0 u'(v-u) + \int_0^1 c_0 u(v-u) = \int_0^1 h(v-u) + (v-u)u'|_0^1,$$
$\forall v \in K$,

or

$$a(u, v-u) - \int_0^1 h(v-u) = (v(1)-u(1))u'(1) - (v(0)-u(0))u'(0).$$

If $u(0) > \psi_0$ and $u(1) > \psi_1$, then the right hand side equals 0. If, on the other hand, $u(0) = \psi_0$, then $u'(0) \leq 0$ and $v(0) \geq u(0)$ or $u(1) = \psi_1$, then $u'(1) \geq 0$ and $v(1) \geq u(1)$. Thus, in any case,

$$a(u, v-u) \geq \int_0^1 h(v-u), \ \forall v \in K.$$

Conversely, if $u \in H^1(0,1)$ is a solution of (4.21), we may choose $v = u + \phi$, $\phi \in C_0^\infty(0,1)$, and obtain

$$a(u, \phi) \geq \int_\Omega h\phi, \quad \forall \phi \in C_0^\infty(0,1),$$

and hence the differential equation (4.22) holds in a weak sense implying that $u'' \in L^2(0,1)$, and thus u' has a trace.

Since $u \in K$, we automatically have $u(0) \geq \psi_0$, $u(1) \geq \psi_1$. Again, since the differential equation is satisfied, we may multiply it by $(v-u)$ and integrate by parts and obtain that

$$(v(1) - u(1))u'(1) - (v(0) - u(0))u'(0) \geq 0,$$

$\forall v \in K$. Choosing $v(0) = u(0)$ and $v(1) > u(1)$, we obtain $u'(1) \geq 0$ and similarly we obtain $u'(0) \leq 0$. Further, choosing $v(1) = \psi_1$, $v(0) = \psi_0$, we obtain

$$0 \geq (u(0) - \psi_0)u'(0) \geq (u(1) - \psi_1)u'(1) \geq 0,$$

and hence

$$(u(0) - \psi_0)u'(0) = (u(1) - \psi_1)u'(1) = 0.$$

A partial differential equations analogue of this problem is

$$-\sum_{i,j=1}^{N} \partial_j(a_{ij}\partial_i u) + \sum_{i=1}^{N} a_i \partial_i u + a_0 u = h, \quad \text{in } \Omega,$$

subject to the unilateral constraints

$$\begin{aligned} u &\geq \psi, \\ \frac{\partial_L u}{\partial \nu} &\geq 0, \\ (u-\psi)\frac{\partial_L u}{\partial \nu} &= 0, \\ \frac{\partial_L u}{\partial \nu} &= \sum_{i,j} a_{ij} n_j \partial_i u, \qquad \text{on } \partial\Omega, \end{aligned}$$

where $\nu = (n_1, n_2, \cdots, n_N)$ is the unit outward normal vector field to Ω.

For further material on unilateral problems see [18], [19], [38], [37], [44].

4.5 Quasilinear inequalities

4.5.1 Set-up

We now consider the case of more general inequalities of the form

$$\langle A(u) - f, v - u \rangle + j(v) - j(u) \geq 0, \quad \forall v \in E, \tag{4.24}$$

here

$$A\colon E \to E^*, \quad f \in E^*$$

and

$$j\colon E \to \mathbb{R} \cup \{\infty\}$$

is a convex lower semicontinuous functional which is bounded below and such that

$$D(j) = \{v \in E \mid j(v) < \infty\} \neq \emptyset,$$

and which is such that (there is no loss in generality in making this assumption)

$$j(0) = 0, \; j : E \to [0, \infty].$$

Concerning A we shall assume:

- A is monotone, i.e.

$$\langle Au - Av, v - u\rangle \geq 0 \quad \forall v, u \in E.$$

- A is continuous on finite dimensional subspaces, i.e., for all finite dimensional subspaces M of E, the mapping

$$\begin{array}{rcl} u & \mapsto & \langle Au, x\rangle \\ M & \to & \mathbb{R} \end{array}$$

  is continuous $\forall x \in E$.

- A is a bounded mapping.
- $\exists u_0 \in D(j)$ such that

$$\lim_{||u|| \to \infty} \frac{\langle A(u), u - u_0\rangle + j(u)}{\|u\|} = \infty$$

  (a coercivity assumption).

We have the following theorem (see [7], [9], [51]):

Theorem 4.2
$\forall f$, $\forall j$ (4.24) has a solution. If A is strictly monotone, i.e.

$$\langle A(u) - A(v), u - v\rangle > 0, \quad v \neq u,$$

only one solution exists.

This will allow us to define

$$T_{A,j}\colon E^* \to E$$

as

$$T_{A,j}f = u,$$

where u is the unique solution of (4.24), and, if we are given $F\colon E \to E^*$, to find the solutions of the inequality

$$\langle A(u) - F(u), v - u\rangle + j(v) - j(u) \geq 0, \quad \forall v \in E, \tag{4.25}$$

is equivalent to finding solutions to

$$u = T_{A,j}F(u). \tag{4.26}$$

We sketch a proof of the existence of a solution to the equation (4.24) in the case $j = I_K$, where

$$I_k(v) = \begin{cases} \infty, & v \notin K \\ 0, & v \in K \end{cases}$$

is the indicator functional of a closed, convex set K in E.

Let K_R denote the following set

$$K_R = \{u \in K \mid \|u\| \leq R\},$$

and consider the problems

$$\langle A(u), v - u\rangle \geq 0, \quad \forall v \in K, \tag{4.27}$$

and

$$\langle A(u), v - u\rangle \geq 0, \quad \forall v \in K_R. \tag{4.28}$$

We have the following lemma:

Lemma 4.3
A necessary and sufficient condition that (4.27) have a solution is that there exists $R > 0$ such that a solution u_R of (4.28) exists with $\|u_R\| < R$.

Proof: If a solution of (4.27) exists with $\|u\| < R$, then u solves (4.28).

Conversely, if u_R solves (4.28) with $\|u_R\| < R$, then given $y \in K$,

$$w = u_R + \epsilon(y - u_R) \in K_R,$$

for $\epsilon > 0$, small. Consequently

$$u_R \in K_R \subset K,$$

and

$$0 \leq \langle A(u_R), w - u_R \rangle = \epsilon \langle A(u_R), y - u_R \rangle, \quad \forall y \in K.$$

Hence, since $\epsilon > 0$, the lemma is proved.

4.5.2 Finite dimensional considerations

We next consider the case that the problem be finite dimensional and K is a bounded convex set. Then the problem

$$\langle A(u), v - u \rangle \geq 0, \quad \forall v \in K, \tag{4.29}$$

is equivalent to the problem

$$\langle -A(u), v - u \rangle \leq 0, \quad \forall v \in K,$$

or

$$\langle u, v - u \rangle \geq \langle u - A(u), v - u \rangle, \quad \forall v \in K,$$

and hence by earlier considerations, it is equivalent to the fixed point problem (here we have identified pairing with the inner product)

$$u = T(I - A)(u),$$

where T is the solution operator defined by the bilinear form $a(u, v) = \langle u, v \rangle$ (the nearest point projection, cf. Section 4.3.2). On the other hand, since

$$T(I - A) \colon K \to K,$$

it has a fixed point.

If K is not bounded, we consider the problem

$$\langle A(u), v - u \rangle \geq 0, \quad \forall v \in K_R. \tag{4.30}$$

This problem has a solution $u_R \in K_R$. We now let $R \to \infty$ and use the coercivity condition, which reads that for some $u_0 \in K$,

$$\lim_{\|u\| \to \infty} \frac{\langle A(u), u - u_0 \rangle}{\|u\|} = \infty. \tag{4.31}$$

Let us then consider $\|u\| >> 1$, $u \in K$. We then conclude from (4.31) that

$$\langle A(u), u - u_0 \rangle > 0.$$

Choosing R so that $u_0 \in K_R$, we have

$$\langle A(u), u_0 - u \rangle < 0 \quad \text{for } \|u\| >> R,$$

hence $\|u_R\| < R$ and we obtain the result.

To complete the proof we shall need the following result (cf. [49]).

THEOREM 4.4 (Minty)
Let $A \colon E \to E^*$ be monotone, then $u \in K$ satisfies

$$\langle A(u), v - u \rangle \geq 0, \quad \forall v \in K,$$

if and only if

$$\langle A(v), v - u \rangle \geq 0, \quad \forall v \in K.$$

Proof: If

$$0 \leq \langle A(v) - A(u), v - u \rangle = \langle A(v), v - u \rangle - \langle A(u), v - u \rangle,$$

then

$$0 \leq \langle A(u), v - u \rangle \leq \langle A(v), v - u \rangle.$$

Conversely, let $w \in K$ and set $v = u + t(w - u)$, $0 \leq t \leq 1$. Then

$$0 \leq \langle A(u + t(w - u)), v - u \rangle,$$

and thus

$$0 \leq \langle A(u), w - u \rangle, \quad \forall w \in K.$$

Let E be infinite dimensional and K be bounded. Then for $v \in K$,

$$S(v) = \{u \in K \mid \langle A(v), v - u \rangle \geq 0\}$$

is weakly closed, hence compact with respect to the weak topology. Therefore

$$\bigcap_{v \in K} S(v)$$

is a weakly compact set.

That this set is nonempty will follow from the finite intersection property. Thus let $\{v_1, ..., v_m\} \subset K$. Then we claim

$$S(v_1) \cap S(v_2) \cap ... \cap S(v_m) \neq \emptyset.$$

Let M be the finite dimensional subspace spanned by $\{v_1, ..., v_m\}$ and let $K_M = K \cap M$. Then, as before, there exists $u_M \in K_M$ such that

$$\langle A(u_M), v - u_M \rangle \geq 0, \quad \forall v \in K_M,$$

and by Theorem 4.4,

$$\langle A(v), v - u_M \rangle \geq 0, \quad \forall v \in K_M,$$

in particular

$$\langle A(v_i), v_i - u_M \rangle \geq 0, \quad \forall i = 1, ..., m,$$

so

$$u_M \in S(v_i), \quad \forall i = 1, ..., m.$$

Hence, there exists $u \in S(v)$ such that

$$\langle A(v), v - u \rangle \geq 0, \quad \forall v \in K_M$$

and using Theorem 4.4 once more, we obtain the result.

If K is unbounded, we employ the coercivity condition imposed on A and argue as before.

4.5.3 Fixed points for non expansive operators

Let E be a Hilbert space and let K be a bounded, closed, convex subset of E.

A mapping

$$F \colon K \to K$$

is called non expansive (see [11]), whenever

$$\|Fu - Fv\| \leq \|u - v\|, \ \forall u, v \in K.$$

Using the above result we establish the following theorem.

Theorem 4.5
Let E be a Hilbert space, K a closed, bounded, convex subset and

$$F: K \to K$$

be a non expansive mapping. Then the set $\operatorname{Fix} F = \{u \in K \mid F(u) = u\}$ *is a nonempty, closed, and convex subset of* K.

Proof: The following calculation shows that the mapping $I - F$ is monotone:

$$\begin{aligned}&\langle u - v - F(u) + F(v), u - v\rangle = \\ &\langle u - v, u - v\rangle - \langle F(u) + F(v), u - v\rangle = \\ &\|u - v\|^2 - \|F(u) - F(v)\|\|u - v\| \geq 0.\end{aligned}$$

Let P be the projection operator associated with K, then

$$I - FP$$

is monotone also, and for any u_0,

$$\lim_{\|u\|\to\infty} \frac{\langle u - FP(u), u - u_0\rangle}{\|u\|} = \infty.$$

Using the results in the previous section with $Au = u - FP(u)$, one concludes that there exists $u \in K$ such that

$$\langle u - FP(u), v - u\rangle \geq 0, \quad \forall v \in K,$$

and hence

$$\langle u - F(u), v - u\rangle \geq 0, \quad \forall v \in K.$$

Letting $v = F(u)$, we have

$$\langle u - F(u), F(u) - u\rangle \geq 0,$$

or

$$u = F(u).$$

That Fix F is closed and convex follows easily.

4.5.4 Continuity of the solution operator

We return to inequality (4.24) and impose the following additional conditions on A :

- A is strictly monotone.
- $\exists c > 0$, $\exists p > 1$ such that

$$\langle Au, u\rangle \geq c\|u\|^p.$$

- A belongs to class (S), i.e., $\forall \{v_n\} \in E$ such that $v_n \rightharpoonup v$ and

$$\lim\langle A(v_n), v_n - v\rangle = 0,$$

  it follows that

$$v_n \to v.$$

Define the solution operator

$$P\colon E^* \to E$$

by

$$Pf = u$$

where u is the unique solution of

$$\langle A(u) - f, v - u\rangle + j(v) - j(u) \geq 0, \quad \forall v \in E. \tag{4.32}$$

We have the following theorem (see [44]):

Theorem 4.6
P is a continuous operator.

Proof: Let $\{f_n\}$ be such that

$$f_n \to f \text{ in } E^*$$

and let $u_n = Pf_n$. Then it follows from (4.32) and the properties of A that

$$c\|u_n\|^p \leq \langle f_n, u_n\rangle$$

and hence (since $p > 1$) that the sequence $\{u_n\}$ is a bounded sequence. It therefore has a weakly convergent subsequence, say, after relabeling,

$$u_n \rightharpoonup w.$$

A straightforward calculation shows (using Minty's theorem and the fact that solutions are unique) that $w = u$ must solve (4.32). I.e., all such subsequences must have the same limit and thus the whole sequence converges weakly to u.

Again, using the form of (4.32), we obtain

$$\langle f - f_n, u - u_n \rangle \geq \langle A(u_n) - A(u), u_n - u \rangle$$

and therefore by the monotonicity of A that

$$\lim \langle A(u_n) - A(u), u_n - u \rangle = 0.$$

And therefore

$$\lim \langle A(u_n), u_n - u \rangle = 0.$$

Since A belongs to class (S) we deduce that

$$u_n \to u,$$

proving the continuity.

4.5.5 On the p-Laplacian

Let Ω be a bounded open set, with smooth boundary $\phi\Omega$, $E = W_0^{1,p}(\Omega)$, $p \in (1, \infty)$, $E^* = W^{-1,q}(\Omega)$, where the dual space is given by

$$W^{-1,q}(\Omega) = \{f \mid f = \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} \phi^{\alpha} g_{\alpha} : g_{\alpha} \in L^q(\Omega)\},$$

and $\frac{1}{p} + \frac{1}{q} = 1$. The pairing $\langle f, u \rangle$ is given by

$$\langle f, u \rangle = \sum_{|\alpha| \leq 1} \int_{\Omega} g_{\alpha} \phi^{\alpha} u.$$

Let AȨ $\to E^*$ be defined by

$$\langle A(u), v \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v.$$

Then for $p \geq 2$, $N \geq 1$,

$$\langle Au - Av, v - u \rangle \geq c \|u - v\|^p.$$

Whereas for $N \geq 1$ and all $p \in (1, \infty)$,

$$\langle Au - Av, v - u \rangle \geq c(\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|).$$

These calculations show that A is a strictly monotone operator.

We may therefore conclude that for all $f \in W^{-1,q}(\Omega)$ (e.g. $f \in L^q(\Omega)$), $\exists! u \in E$ such that

$$-\mathrm{div}(|\nabla u|^{p-2} \nabla u) = f,$$

in the sense of distributions. On the other hand, if $v_n \rightharpoonup v$ and $\lim \langle Av_n, v_n - v \rangle = 0$, then

$$\langle Av_n - Av, v_n - v \rangle \geq c(\|v_n\|^{p-1} - \|v\|^{p-1})(\|v_n\| - \|v\|)$$

implies that (note $p > 1$)

$$\|v_n\| \to \|v\|,$$

and since $W_0^{1,p}(\Omega)$ is a uniformly convex space,

$$v_n \rightharpoonup v \qquad \|v_n\| \to \|v\|$$

implies

$$v_n \to v,$$

and therefore A belongs to class (S) which has as a consequence that the solution operator is a continuous mapping.

4.6 Existence results

Throughout we shall assume that V is a real Banach space with its topological dual denoted by V^*, and the pairing between V^* and V, by $\langle \cdot, \cdot \rangle$. Let

$$F : V \to \mathbb{R} \cup \pm\infty = [-\infty, \infty]$$

be a functional with *effective domain*

$$D(F) = \{u \in V \mid F(u) \neq \pm\infty\}.$$

A point $u^* \in V^*$ is called a *subgradient* for the functional F at the point u provided that $u \in D(F)$ and

$$F(v) \geq F(u) + \langle u^*, v - u \rangle, \ \forall v \in V. \tag{4.33}$$

The set of all subgradients at a point $u \in D(F)$ is denoted by $\partial F(u)$ and called the *subdifferential* of F at the point u. (Concerning the properties of the subdifferential for convex functions, we refer the reader to [54] and [62].)

We shall state and prove here, one of the basic results relating minimization problems with variational inequalities. To this end we shall assume that the functional F has the following properties:

$$F, J, j : V \to (-\infty, \infty]$$

where F has the form

$$F = J + j,$$

where J and j are functionals which are lower semicontinuous with respect to a topology τ, i.e., the sets

$$\{u \mid J(u) \leq a\}, \ \{u \mid j(u) \leq a\}$$

are closed with respect to τ for each $a \in \mathbb{R}$. Further we assume that F is coercive, i.e., that

$$F(u) \to \infty, \text{ as } \|u\| \to \infty,$$

and that bounded subsets of V are precompact with respect to the topology τ.

The topologies τ most frequently employed are the weak topology, in case V is a reflexive space, or the weak star topology, in case V is the dual of a separable space. In what is to follow, examples for both cases will be of interest.

We have the following result. We also give a brief sketch of a proof.

THEOREM 4.7
Assume the above conditions and that J is convex and $D(J) \neq \emptyset$, $D(j) = V$, with j Gâteaux differentiable, with Gâteaux derivative $j'(u)$. Then there exists $u \in D(J)$ such that

$$F(u) = \min_{v \in V} F(v)$$

and

$$0 \in \partial J(u) + j'(u), \tag{4.34}$$

or equivalently that

$$J(v) - J(u) + \langle j'(u), v - u \rangle \geq 0, \ \forall v \in V. \tag{4.35}$$

It follows from the assumptions on F (particularly the assumption of lower semicontinuity and coercivity) that F is bounded below, say,

$$\infty < \alpha := \inf_{v \in V} F(v).$$

We thus obtain a bounded sequence $\{u_n\}$ with

$$F(u_n) \to \alpha,$$

and therefore a subsequence $\{u_{n_j}\}$ such that, with respect to the topology τ,

$$u_{n_j} \to u,$$

and

$$F(u) = \alpha.$$

Therefore

$$F(u) = J(u) + j(u) \le F(v) = J(v) + j(v), \;\; \forall v \in V.$$

Hence, for $t > 0$ and $v \in V <$ we obtain

$$0 \le \frac{1}{t}\left(J(u + t(v-u)) - J(u)\right) + \frac{1}{t}\left(j(u + t(v-u)) - j(u)\right),$$

and, using the convexity of J and the differentiability of j, we obtain

$$0 \le J(v) - J(u) + \langle j'(u), v - u \rangle + \frac{1}{t} o(t),$$

from which follows (4.35) and thus, by definition of the subdifferential,

$$-j'(u) \in \partial J(u),$$

i.e., we also have (4.34).

For monotone mappings we have another fundamental result ([45]), known as the Browder–Minty Theorem. It is the following:

Theorem 4.8
Let V be a reflexive Banach space, and let $A : V \to V^*$ be a monotone hemicontinuous mapping which is bounded. Let ϕ be a convex, lower semicontinuous functional from V to $\mathbb{R} \cup \{\infty\}$ with nonempty effective domain $D(\phi)$. Finally, suppose that A and ϕ satisfy the coercivity condition

$$\lim_{\|u\| \to \infty} \frac{\langle\, Au\,,\, u - u_0 \,\rangle + \phi(u)}{\|u\|} = \infty, \tag{4.36}$$

for some $u_0 \in D(\phi)$. *Then, for all* $f \in V^*$, *there exists a solution* $u \in V$ *of the variational inequality*

$$\langle\, Au - f \,,\, v - u \,\rangle + \phi(v) - \phi(u) \;\geq\; 0 \quad \forall v \in V. \tag{4.37}$$

The solution is unique, whenever A *is strictly monotone.*

We point out important special cases of the above theorems, when in the case of Theorem 4.7 the functional j and in the case of Theorem 4.8 the functional ϕ are the indicator functionals of a convex set K in V, i.e.,

$$\phi(u) = \begin{cases} 0, & \text{for} \quad u \in K \\ \infty, & \text{for} \quad u \notin K, \end{cases} \tag{4.38}$$

with the set K closed with respect to either the topology τ (Theorem 4.7) or the topology of V (Theorem 4.8).

In these cases, the solution u of the variational inequality (4.37) clearly belongs to the set K; such sets K typically correspond to obstacles, unilateral constraints, or certain boundary conditions.

For more information on static variational inequalities, we refer to [3], [13], [21], [26], [34], [35], [44], [43], [56], and the references which they provide.

4.7 An example

Let us consider the boundary value problem

$$\begin{cases} -\mathrm{div}(A(|\nabla u|^2)\nabla u) + F(x,u) &= 0, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \langle \Omega, , \rangle \end{cases} \tag{4.39}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary.

Let

$$\phi : \mathbb{R} \to \mathbb{R}, \;\; \phi(s) = A(s^2)s.$$

Then, if $\phi(s) = |s|^{p-1}s, \; p > 1$, problem (4.39) is the stationary equation corresponding to some of the problems indicated in the previous section, and is fairly well understood and a great variety of existence results are available. These results are usually obtained using variational methods, monotone operator methods or fixed point and degree theory arguments in the Sobolev space $W_0^{1,p}(\Omega)$. If, on the other hand, we assume that ϕ is an odd nondecreasing function such that:

$$\phi(0) = 0, \;\; \phi(t) > 0, \;\; t > 0,$$

$$\lim_{t \to \infty} \phi(t) = \infty,$$

and

$$\phi \text{ is right continuous,}$$

then a Sobolev space setting for the problem is not appropriate. The first general existence results using the theory of monotone operators in Orlicz-Sobolev spaces were obtained in [20] and in [30], [31]. Other recent work that puts the problem into this framework is contained in the papers [15] and [27].

We assume that $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that satisfies certain growth conditions to be specified later.

A natural way of formulating the boundary value problem is a variational inequality formulation of the problem in a suitable Orlicz-Sobolev space. In order to do this we shall have need of some facts about Orlicz-Sobolev spaces which we shall give now.

Let us put $\Phi(t) = \int_0^t \phi(s)ds,\ t \in \mathbb{R}$. Then Φ is a Young (or N-) function (cf. [1], [36], [39]). Also, following these references, we denote by $\overline{\Phi}$ the conjugate Young function of Φ, i.e.,

$$\overline{\Phi}(t) = \sup\{ts - \Phi(s) : s \in \mathbb{R}\},$$

and by Φ^* the Sobolev conjugate of Φ, i.e.,

$$(\Phi^*)^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)ds}{s^{\frac{N+1}{N}}} ds, \tag{4.40}$$

provided that

$$\int_1^\infty \frac{\Phi^{-1}(s)ds}{s^{\frac{N+1}{N}}} ds = \infty. \tag{4.41}$$

Let Φ be a Young function. The Orlicz class $\tilde{L}_\Phi := \tilde{L}_\Phi(\Omega)$ is the set of all (equivalence classes) of measurable functions u defined on Ω such that

$$\int_\Omega \Phi(|u(x)|)dx < \infty.$$

The Orlicz space $L_\Phi := L_\Phi(\Omega)$ is the linear hull of $\tilde{L}_\Phi$, i.e., the set of all measurable functions u on Ω such that

$$\int_\Omega \Phi\left(\frac{|u(x)|}{k}\right) dx < \infty, \quad \text{for some } k > 0.$$

Then L_Φ is a Banach space when equipped with the norm (the Luxemburg norm)

$$\|u\|_\Phi = \inf\left\{k > 0 : \int_\Omega \Phi\left(\frac{|u|}{k}\right) dx \le 1\right\},$$

or the equivalent norm (the Orlicz norm)

$$\|u\|_{(\Phi)} = \sup\left\{\left|\int_\Omega uvdx\right| : v \in \tilde{L}_{\overline{\Phi}},\ \int_\Omega \overline{\Phi}(|v|)dx \le 1\right\}.$$

If Φ_1 and Φ_2 are two Young functions, one writes

$$\Phi_1 \le \Phi_2,$$

provided there exist constants t_0 and k such that

$$\Phi_1(t) \le \Phi_2(kt),\ t \ge t_0,$$

and one says that Φ_1 and Φ_2 are equivalent whenever

$$\Phi_1 \le \Phi_2 \text{ and } \Phi_2 \le \Phi_1.$$

If Φ_1 and Φ_2 are equivalent, then they determine the same Orlicz space. Also, it is the case that the imbedding

$$L_{\Phi_2} \hookrightarrow L_{\Phi_1},$$

is continuous, whenever $\Phi_1 \le \Phi_2$.

The closure of L^∞ in L_Φ is denoted by E_Φ, which is a separable Banach space. The Orlicz-Sobolev space $W^1 L_\Phi := W^1 L_\Phi(\Omega)$ is the set of all $u \in L_\Phi$ such that the distributional derivatives $\langle , i\rangle u = \dfrac{\langle u, \rangle}{\langle x,\rangle\, i}$, $i = 1, \cdots, N$, are also in L_Φ. This is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} = \|u\|_{W^1 L_\Phi} = \|u\|_\Phi + \sum_{i=1}^{N} \|\langle , i\rangle u\|_\Phi.$$

It is known (cf. [36], [53]) that L_Φ is the dual space of $E_{\overline{\Phi}}$, i.e.,

$$L_\Phi = (E_{\overline{\Phi}})^*,\ \text{and } L_{\overline{\Phi}} = (E_\Phi)^*.$$

The space $W^1 E_\Phi$ is defined similarly.

We denote by $W_0^1 L_\Phi$ the closure of $C_0^\infty(\Omega)$ with respect to the weak* topology.

We also mention a notion of relative growth of Young functions which will play a role in our considerations (cf. [1], [36], [39]). A Young function Φ_1 is said to grow essentially more slowly than another Young function Φ_2, abbreviated by

$$\Phi_1 \ll \Phi_2,$$

if

$$\lim_{t\to\infty} \frac{\Phi_1(t)}{\Phi_2(kt)} = 0, \ \forall k > 0.$$

Now, we formulate and extend (4.39) as a variational inequality in a suitable Orlicz-Sobolev space.

Multiplying both sides of (4.39) by $v \in C_0^\infty(\Omega)$ and integrating by parts (provided these integrations may be performed), we see that the weak form of (4.39) is

$$\int_\Omega A(|\nabla u|^2)\nabla u \cdot \nabla v dx + \int_\Omega F(x,u)v dx = 0. \tag{4.42}$$

A natural choice of the space of test functions v is, of course, $W_0^1 L_\Phi$. However, the mapping $u \mapsto L(u)$, where

$$\langle L(u), v\rangle = \int_\Omega A(|\nabla u|^2)\nabla u \cdot \nabla v dx, \ v \in W_0^1 L_\Phi,$$

is not necessarily defined on the whole space, we, hence, formulate (4.42) as a variational inequality.

Consider the functional

$$J : W_0^1 L_\Phi \to \mathbb{R} \cup \{\infty\}, \ J(u) := \int_\Omega \Phi(|\nabla u|) dx.$$

Since $\frac{\langle \Phi, \rangle}{\langle \xi, \rangle\, i}(|\xi|) = A(|\xi|^2)\xi_i, \ i = 1, \ldots, N$, we have, at least formally,

$$\langle J'(u), v\rangle = \int_\Omega \sum_{i=1}^N A(|\nabla u|^2)\, \langle, i\rangle\, u \langle, i\rangle v dx = \int_\Omega A(|\nabla u|^2)\nabla u \cdot \nabla v dx.$$

Let us now assume that F satisfies the growth condition

$$|F(x,s)| \le B(x) + |\Psi_0'(s)|, \ s \in \mathbb{R}, \ x \in \Omega, \tag{4.43}$$

where Ψ_0 is a differentiable Young function such that Ψ_0 is strictly convex,

$$\Psi_0 \ll \Phi^*, \tag{4.44}$$

and

$$B \in L_{\overline{\Psi}_0}. \tag{4.45}$$

We then may, for $u \in W^1L_\Phi$, define $k(u) \in (W^1L_\Phi)^*$ by

$$\langle k(u), v\rangle := \int_\Omega F(x,u)v\,dx, \ \forall v \in W^1L_\Phi.$$

The following lemma holds (see [43]):

Lemma 4.9
The mapping

$$k : W^1L_\Phi \to (W^1L_\Phi)^*$$

is continuous.

In many situations, it is more convenient to work in an Orlicz space which lies between L_{Φ^*} and L_{Ψ_0}. We choose a Young function Ψ such that

$$\Psi_0 \ll \Psi \ll \Phi^*. \tag{4.46}$$

We can replace, because of this ordering, Φ^* by Ψ in the proof of Lemma 4.9 and obtain:

Lemma 4.10
If $u \in L_\Psi$, then $\psi_0(u) \in L_{\overline{\Psi}_0}$ and

$$F(\cdot,u) \in L_{\overline{\Psi}_0} \subset L_{\overline{\Psi}}. \tag{4.47}$$

Moreover,

$$\|F(\cdot,u)\|_{\overline{\Psi}} \le C\|F(\cdot,u)\|_{\overline{\Psi}_0} \le \|B\|_{\overline{\Psi}_0} + \|\psi_0(|u|)\|_{\overline{\Psi}_0}.$$

We also have:

Lemma 4.11
If $u \in L_\Psi$, then $F(\cdot,u) \in L_{\overline{\Psi}}$. The mapping

$$k : u \mapsto k(u) = F(\cdot,u)$$

is continuous and bounded from L_Ψ to $L_{\overline{\Psi}}$.

Thus one may formulate (4.42) (at least formally) as the equation:

$$J'(u) + k(u) = 0. \tag{4.48}$$

However, J is not differentiable in general (J is not even defined on the whole space $W_0^1 L_\Phi$, since J assumes, in general, finite values only on a convex, nondense subset of $W_0^1 L_\Phi$). On the other hand, since J is convex and lower semicontinuous (as will be stated later), we replace (4.48) by the inclusion

$$0 \in \partial J(u) + k(u), \tag{4.49}$$

where ∂J is the subdifferential of J; this, in turn, is equivalent to the variational inequality

$$\left\{ \begin{array}{l} J(v) - J(u) + \langle k(u), v - u \rangle \geq 0, \ \forall v \in W_0^1 L_\Phi \\ u \in W_0^1 L_\Phi. \end{array} \right. \tag{4.50}$$

The advantage of this formulation is that solutions of (4.49) are included in the effective domain of the functional J,

$$D(J) = \{u \in W_0^1 L_\Phi : J(u) = \int_\Omega \Phi(|\nabla u|) dx < \infty\}.$$

We therefore may consider (4.50) as the variational inequality formulation of (4.42) (and hence (4.39)).

We now proceed to discuss the existence of solutions of (4.50) and more general inequalities. We first provide some properties of the functional J, (see again [43]).

Lemma 4.12
The functional J is convex and lower semicontinuous on $W^1 L_\Phi$. If Φ is strictly convex, then J is strictly convex on $W_0^1 L_\Phi$.

In what follows, we consider the following variational inequality associated with the boundary value problem (4.50):

$$\left\{ \begin{array}{l} J(v) - J(u) + \langle k(u), v - u \rangle \geq 0, \ \forall v \in K \\ u \in K, \end{array} \right. \tag{4.51}$$

where K is a convex subset of $W_0^1 L_\Phi$, closed with respect to the weak* topology and $0 \in K$ (in the case $K = W_0^1 L_\Phi$, (4.51) reduces to (4.50)).

We consider the problem that k is independent of u:

$$\left\{ \begin{array}{l} J(v) - J(u) + \langle k, v - u \rangle \geq 0, \ \forall v \in K \\ u \in K, \end{array} \right. \tag{4.52}$$

with $k \in L_{\overline{\Psi}}$, $\langle k, v \rangle = \int_\Omega kv dx, \ v \in W_0^1 L_\Phi$.

We may rewrite (4.52) as

$$\begin{cases} J(v) + \langle k, v \rangle \geq J(u) + \langle k, u \rangle, \ \forall v \in K \\ u \in K, \end{cases}$$

and see that u solves (4.52) if and only if u is a minimizer of the problem

$$\min_{v \in K} [J(v) + \langle k, v \rangle]. \tag{4.53}$$

We will indicate why (4.53) has a solution.

To accomplish this we shall, in what follows, make the additional assumption:

- $\overline{\Phi}$ satisfies a Δ_2 condition at infinity (cf. [36]), which has as a consequence that $L_{\overline{\Phi}} = E_{\overline{\Phi}}$.

It follows from the work in [30] that in the space $W_0^1 L_\Phi$ a Poincaré inequality holds and consequently that $\||\nabla u|\|_\Phi$ furnishes an equivalent norm for $W_0^1 L_\Phi$. Thus for $u \in W_0^1 L_\Phi$ we shall henceforth use

$$\|u\| := \||\nabla u|\|_\Phi.$$

We have the following lemma ([43]):

Lemma 4.13
The functional J is sequentially lower semicontinuous with respect to the weak topology of $W_0^1 L_\Phi$ and is coercive in the sense that*

$$\frac{J(u)}{\|u\|} \to \infty, \quad \textit{as} \ \|u\| \to \infty. \tag{4.54}$$

From this lemma, and Theorem 4.7 we obtain the following result:

Theorem 4.14
For each $k \in L_{\overline{\Psi}} (\subset (W_0^1 L_\Phi)^*)$, the set U_k of solutions of (4.53) (and thus of (4.52)) is nonempty, convex, and bounded in $W_0^1 L_\Phi$, and thus precompact in L_Ψ. The solution set is a singleton, whenever Φ is strictly convex.

In case k is dependent upon u, various assumptions may be imposed on k in order that Theorem 4.7 may be applied to deduce the existence of a solution of (4.51). We remark that conditions have been given in [15] guaranteeing that $k = j'$ for some functional j.

5 Parabolic Variational Inequalities

Rothe's Method ([33], [50], [63]), also known as the *method of lines* or the *method of semidiscretization*, is an extension of the backward Euler scheme for initial value problems for ordinary and parabolic parabolic partial differential equations and is a powerful tool in both the theoretical and numerical analyses of evolution problems. To illustrate the method, we use it to solve the sample parabolic variational inequality (2.13) discussed in Section 2.2.

5.1 The problem

Let us recall the problem posed: Let $H = L^2(\Omega)$ and

$$V \hookrightarrow H \hookrightarrow V^*,$$

where $V = H_0^1(\Omega)$. Furthermore let K be a closed convex set in V. Find $u \in \mathcal{K}$ with $u(0) = u_0 \in H$ and such that

$$\langle\, u'(t)\,,\, v(t) - u(t)\,\rangle + a(u(t), v(t) - u(t)) \;\geq 0, \qquad \forall\, v \in K,\;\; a.e.\; t \in (0,T), \tag{5.1}$$

where

$$K = \{\, v \in V \mid v(x) \geq 0 \quad \text{for} \quad x \in \Gamma\,\}, \tag{5.2}$$

and $\mathcal{K}$ consists of those $v \in \mathcal{V}$ such that $v(t) \in K$ for a.e. $t \in (0,T)$, and $a(\cdot,\cdot)$ is the form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \text{for} \quad u, v \in V.$$

As mentioned earlier, this problem models diffusion in a domain with a semipermeable boundary.

5.2 Rothe's method

The first step of Rothe's method is to partition the time interval $[0,T]$ into n equal subintervals $[t_{i-1}, t_i]$, where $i = 1, 2, \ldots, n, \;\; t_i = ih$, and h is the mesh width $\dfrac{T}{n}$. For each $i = 1, 2, \ldots$, we consider the problem of finding a solution $u_i \in K$ of

$$\langle\, \frac{u_i - u_{i-1}}{h}\,,\, v - u_i\,\rangle + a(u_i, v - u_i) \;\geq 0,\;\; \forall\, v \in K. \tag{5.3}$$

with

$$u_0 = u_0(x), \ \ x \in \Omega.$$

Rewriting inequality (5.3) in the form

$$\frac{1}{h}\langle\, u_i\,,\, v - u_i\,\rangle + a(u_i, v - u_i) \ \geq \frac{1}{h}\langle\, u_{i-1}\,,\, v(t) - u_i(t)\,\rangle, \ \forall\, v \in K, \quad (5.4)$$

and observing that u_{i-1} is known at each step, we see that (5.3) is an elliptic variational inequality for the bilinear form

$$u, v \longmapsto \frac{1}{h}\int_\Omega u\, v\, dx + \int_\Omega \nabla u \cdot \nabla v\, dx \qquad (5.5)$$

defined on $V \times V$. For n large, the coefficient $1/h$ is large, so the form (5.5) satisfies the coercivity condition (4.36) with $v_0 = 0$. Our basic existence result (4.8) therefore applies to the elliptic variational inequality (5.3) to guarantee a solution $u_i \in K$.

We thus obtain n autonomous functions $u_i \in K$, $i = 1, \ldots, n$, that may be combined to form *Rothe's function*, a proposed approximate solution of the original parabolic variational inequality:

$$u_n(x,t) = u_{i-1}(x) + \frac{t - t_{i-1}}{h}\left(u_i(x) - u_{i-1}(x)\right), \quad t \in [t_{i-1}, t_i]. \quad (5.6)$$

Observe that Rothe's function $u_n(x,t)$ is linear in time over each subinterval $[t_{i-1}, t_i]$; the time variable plays the role of a homotopy parameter connecting u_{i-1} at time t_{i-1} to u_i at time t_i.

To show that $u_n(x,t)$ converges to a solution $u(x,t)$ of (5.1) as $n \to \infty$, we establish some necessary estimates and then apply the Arzelà–Ascoli Theorem. Thus, for $j \geq 2$, we take $v = u_j$ in the inequality (5.3) for $i = j-1$ and $v = u_{j-1}$ in the inequality for $i = j$ to produce

$$\langle\, \frac{u_{j-1} - u_{j-2}}{h}\,,\, u_j - u_{j-1}\,\rangle + a(u_{j-1}, u_j - u_{j-1}) \geq 0, \qquad (5.7)$$

$$\langle\, \frac{u_j - u_{j-1}}{h}\,,\, u_{j-1} - u_j\,\rangle + a(u_j, u_{j-1} - u_j) \geq 0. \qquad (5.8)$$

Adding inequalities (5.7) and (5.8) yields

$$\frac{1}{h}\|\, u_j - u_{j-1}\,\|^2 + \|\,\nabla(u_j - u_{j-1})\,\|^2 \leq \frac{1}{h}\langle\, u_{j-1} - u_{j-2}\,,\, u_j - u_{j-1}\,\rangle, \quad (5.9)$$

from which we obtain

$$\|\, u_j - u_{j-1}\,\|^2 + 2h\,\|\,\nabla(u_j - u_{j-1})\,\|^2 \leq \|\, u_{j-1} - u_{j-2}\,\|^2, \quad j \geq 2 \quad (5.10)$$

by applying the Cauchy–Schwarz inequality and using the elementary inequality $2ab \le a^2 + b^2$. The norm $\|\cdot\|$ being used here is the norm of $H = L^2(\Omega)$.

For the case $j = 1$, we choose $v = u_0$ in (5.3) to get

$$\frac{1}{h}\| u_1 - u_0 \|^2 + \| \nabla(u_1 - u_0) \|^2 \le |\langle \nabla u_0 \, , \, \nabla(u_1 - u_0) \rangle|.$$

If the initial datum u_0 belongs to $H^2(\Omega) \cap H_0^1(\Omega)$, integration by parts reveals that

$$|\langle \nabla u_0 \, , \, \nabla(u_1 - u_0) \rangle| \le |\langle \Delta u_0 \, , \, u_1 - u_0 \rangle| \le \| \Delta u_0 \| \, \| u_1 - u_0 \|,$$

so that we have the basic bound

$$\| \frac{u_1 - u_0}{h} \| \le \| \Delta u_0 \|. \tag{5.11}$$

Combining this estimate with inequality (5.10) shows that

$$\| \frac{u_i - u_{i-1}}{h} \| \le C, \quad i = 1, 2, \ldots, n, \tag{5.12}$$

for some constant C that is independent of n. Upon choosing $v = u_i$ in (5.3), a similar uniform bound involving the norm of V follows:

$$\| u_i \|_V \le C \quad i = 1, 2, \ldots, n. \tag{5.13}$$

These bounds provide a uniform estimate on the derivative u_n', since

$$u_n' = \frac{u_i - u_{i-1}}{h} \, .$$

Thus, (5.12) says that

$$\| u_n' \| \le C \quad \text{for} \quad t \in [0, T], \tag{5.14}$$

which immediately gives the equicontinuity result

$$|u_n(t) - u_n(\tau)| \le C \, |t - \tau|, \quad \text{for} \quad t, \tau \in [0, T]$$

for the family $\{u_n\}$, $n \in \mathbb{N}$. The Arzelà–Ascoli Theorem and the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ then guarantee that u_n converges to some function u in the space $C\big([0,T], L^2(\Omega)\big)$. In fact, u is Lipschitz continuous and therefore differentiable almost everywhere in $[0, T]$.

It remains to show that the limit u solves the parabolic variational inequality (5.1). To accomplish this, we next define $\overline{u}_n(t)$ to be the step function

$$\overline{u}_n(t) = u_i \quad \text{for} \quad t \in [t_{i-1}, t_i]. \tag{5.15}$$

It follows from (5.13) that $\{\overline{u}_n\}$ has a subsequence that converges weakly in $H_0^1(\Omega)$, which we relabel as $\{\overline{u}_n\}$. Moreover, (5.12) yields the bound

$$|\overline{u}_n(t) - u_n(t)| \leq \frac{C}{n},$$

from which it follows that the weak limit of this sequence is u. By exploiting the bound (5.14) in a similar fashion, we see that u_n' converges weakly to u' in $L^2\left(0, T; L^2(\Omega)\right)$.

In terms of u_n and $\overline{u}_n$, the elliptic variational inequality (5.3) is

$$\langle\, u_n'(t)\,,\, v(t) - \overline{u}_n(t)\,\rangle + a(\overline{u}_n(t), v(t) - \overline{u}_n(t)) \geq 0, \ \forall v \in \mathcal{K}, \tag{5.16}$$

which holds almost everywhere in $[0, T]$. For arbitrary points τ_1 and τ_2 in $[0, T]$, integrating (5.16) from τ_1 to τ_2 gives

$$\int_{\tau_1}^{\tau_2} \langle\, u_n'(t)\,,\, v(t) - \overline{u}_n(t)\,\rangle + a(\overline{u}_n(t), v(t) - \overline{u}_n(t))\, dt \geq 0, \ \forall v \in \mathcal{K}. \tag{5.17}$$

Taking $\liminf$ as $n \to \infty$ in this inequality, we have

$$\int_{\tau_1}^{\tau_2} \langle\, u'(t)\,,\, v(t) - u(t)\,\rangle + a(u(t), v(t) - u(t))\, dt \geq 0, \ \forall v \in \mathcal{K}, \tag{5.18}$$

since $\overline{u}_n \to u$ and $u_n' \rightharpoonup u$ in $L^2\left(0, T; L^2(\Omega)\right)$ and the bilinear form $a(\cdot, \cdot)$ is weakly lower semicontinuous. As (5.18) holds for any τ_1 and τ_2, the parabolic variational inequality (5.1) follows, proving that u is the desired solution.

Rothe's method for parabolic variational inequalities (3.10) may thus be summarized as the following algorithm:

- For a given integer n, divide the time interval $[0, T]$ into equal intervals of width $h = \dfrac{T}{n}$.
- For each $i = 1, \ldots, n$, obtain a solution $u_i \in K$ of the *elliptic* variational inequality

$$\langle\, \frac{u_i - u_{i-1}}{h} - f\,,\, v - u_i\,\rangle + a(u_i, v - u_i) \geq 0 \quad \forall v \in K, \tag{5.19}$$

  where $u_{i-1} \in K$ is known.

- Construct Rothe's function $u_n(x,t)$ (5.6) and prove that $\{u_n\}$ converges to a solution u of (3.10) as $n \to \infty$.

The first two steps of this procedure are simple, as long as there are existence results for elliptic variational inequalities involving the particular operator A and the function spaces in question. The details of the third step, however, will depend heavily on the particular problem under consideration. A general result that follows from an application of Rothe's method is the following ([33]):

THEOREM 5.1
Let the spaces V, H, $\mathcal{V}$, and $\mathcal{H}$ be as described before. Suppose that the pseudomonotone operator $A : V \to V^*$ and the convex lower semicontinuous functional $\phi : V \to \mathbb{R} \cup \{+\infty\}$, with $D(\phi)$ nonempty, satisfy the coercivity condition

$$\lim_{\|v\| \to \infty} \frac{a(v, v - v_0) + \phi(v)}{\|v\|} = \infty \tag{5.20}$$

for some $v_0 \in D(\phi)$, and suppose that there exists $z_0 \in H$ satisfying

$$\langle z_0 , v \rangle + a(u_0, v) + \phi(v) - \phi(u_0) \ \geq \langle f(0) , v - u_0 \rangle \ \forall v \in V \tag{5.21}$$

for the initial datum $u_0 \in H$. Finally, suppose that $f : [0,T] \times H \to H$ is Lipschitz. Then there exists a unique $u \in L^\infty(0,T;V) \cap C([0,T];H)$ with $u' \in L^\infty(0,T;H)$ such that

$$\begin{aligned} &\langle u'(t) - f(t, u(t) , v(t) - u(t) \rangle + a(u(t), v(t) - u(t)) \\ &\qquad + \phi(v(t)) - \phi(u(t)) \ \geq \ 0, \ \forall v \in \mathcal{V}, \ a.e. \ t \in (0,T) \end{aligned} \tag{5.22}$$

and $u(0) = u_0 \in H$.

We remark that Kačur,[33], actually proves this theorem for the more general case of a maximal monotone operator A. Operators of this type arise in many evolution problems and are much more general than the pseudomonotone operators considered here. For instance, maximal monotone operators, such as the fundamental example provided by the subdifferential of a convex function, are generally multi-valued, whereas we have only considered single–valued operators from V to V^*. We refer to [7] for a thorough treatment of such operators and their fundamental role in evolution problems on Hilbert spaces.

Although this technique and the translation method of the previous section both employ a difference approximation of u', we emphasize that the two approaches are quite different. Rothe's method produces strong solutions $u \in C([0,T];H)$, whereas the translation method only provides weak solutions, whose regularity must then be investigated. In addition, the constructive nature of Rothe's method renders it effective in numerical analysis and computation. For more on this aspect of the method, as well as applications to a wide variety of evolution problems, we refer the reader to [33], [63].

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