Critical Point Theory and Applications to Nonlinear Differential Equations

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Chapter 1

Introduction

Critical point theory is a modern evolution of an old part of mathematical analysis which is called the *calculus of variations.* Problems of the calculus of variations can be traced to Greek mathematics, for example with Dido's problem of finding the closed curve of fixed length surrounding a maximal area. But, if we except some questions of optics treated by Hero of Alexandria et Pierre de Fermat, the first problems of the future calculus of variations where introduced at the very birth of the calculus, at the end of the seventeenth century. We can recall Newton's problem of the solid of revolution offering the least resistance in a fluid, or the problem of finding the shortest curve joining two fixed points in the plane, the space or on a surface (*geodesic*), or the problem of finding the curve along which a frictionless particle moves in a vertical plane, under the sole effect of gravity, from one point to another (*brachystrochrone*).

In contrast to classical problems of maximum and minimum, where one tries to localize the points at which a real function of one or several variables reaches a maximum or a minimum value, and which have constituted a strong motivation for the creation of the calculus, the above mentioned problems require finding a curve (i.e. one or several functions defining this curve) which minimizes or maximizes a given real-valued expression depending upon all the values of this or those functions. In modern terms (and it has taken centuries to realize it and make it efficient), the problems of the calculus of variations consist in finding maximums or minimums of real functions defined on a function space.

The typical problem of the calculus of variations, as shown by Euler in the middle of the eighteenth century, consists in finding the minimum (or the maximum) of an expression of the type

$$\int_a^b f(x, y(x), y'(x))\, dx, \tag{1.1}$$

over all real sufficiently smooth functions y over $[a, b]$ taking respective fixed values A and B at a and b. Euler showed in a quite indirect way (reduction to a finite-dimensional approximation, use of calculus and limit process), and then

Lagrange showed in a much more direct way that if such a function y exists, it must be solution of the second order ordinary differential equation

$$\frac{d}{dx}\partial_{y'} f(x, y(x), y'(x)) - \partial_y f(x, y(x), y'(x)) = 0. \tag{1.2}$$

This *Euler-Lagrange equation* corresponds, for the problem of calculus of variations, to the necessary condition

$$\nabla F(x) = 0$$

for the real function F to achieve a maximum or a minimum at x. Of course, when $y(x)$ is a function of n variables, and (1.1) a corresponding multiple integral over a n-dimensional subset Ω, then the associated Euler-Lagrange equation is a partial differential equation, and the conditions at a and b are replaced by conditions for y on the boundary $\partial\Omega$ (*boundary conditions*).

For a long time, the effort of mathematicians has been concentrated on solving explicitely whenever possible the Euler-Lagrange equation, and then finding a candidate for the minimization of (1.1) over the family of functions. Such an approach reaches rapidly its limits, when equation (1.2) is nonlinear and has no first integral. Notice that, for a long time, the existence of a minimum or a maximum was not questioned, being considered as evident on mathematical or physical grounds.

Near the half of the nineteenth century, a new viewpoint developed, motivated by the increasing care about existence theorems initiated by Gauss. This new viewpoint consisted in showing the existence of a solution to equation (1.2) satisfying the boundary conditions

$$y(a) = A, \quad y(b) = B \tag{1.3}$$

by proving directly that the expression (1.1) has a maximum or a minimum over the class of functions satisfying condition (1.3). Such an approach, called the *direct method of the calculus of variations,* first occured in an heuristic ways in uncomplete proofs by Gauss, Dirichlet, Kelvin and Riemann of the solvability of the so-called *Dirichlet problem*

$$\Delta y(x) = 0 \quad (x \in \Omega), \quad y(x) = f(x) \quad (x \in \partial\Omega), \tag{1.4}$$

where Δ is the Laplacian operator and f a given real function over $\partial\Omega$. Equation (1.4) is the Euler-Lagrange equation associated to the *Dirichlet integral*

$$\int_\Omega |\nabla y(x)|^2 \, dx, \tag{1.5}$$

and the existence of a minimum was considered as a trivial consequence of its positivity (*Dirichlet principle*), untill Weierstrass gave in 1870 a counterexample to such an evidence for a one dimensional problem, by showing that the integral

$$\int_{-1}^{1} [xy'(x)]^2 \, dx$$

has no minimum over the set of C^2 functions such that

$$y(-1) = a, \quad y(1) = b,$$

when $a \neq b$.

However, this was not a counter-example to the Dirichlet principle and, at the end of the nineteenth century, Arzelá tried to use his compactness theorem for a family of equibounded and equicontinuous functions to prove the existence of the minimum and came close to the conclusion in 1897. Apparently unaware of Arzela's contributions (at least not mentioning them), Hilbert [35] sketched a rigourous proof of the Dirichlet principle when $n = 2$ in 1900, under some conditions on f and Ω. It was the first step of a long sequence of papers realizing Hilbert's prophecy in his famous lecture at the 1900 Paris International Congress of Mathematicians [36] :

> I am convinced that it will be possible to prove existence theorems using a general principle whose nature is indicated by the Dirichlet principle.

In particular, the work of Levi, Fubini, Lebesgue, Zaremba, Lichtenstein, Courant, Tonelli and others has emphasized the important role played by a property of the integral (1.1), namely its *semicontinuity* with respect to a suitable type of convergence for y. The famous monograph "Fondamenti di calcolo delle variazioni" of Tonelli [77], published between 1921 and 1923, mostly devoted to this concept, has motivated the further work of mathematicians like Hahn, Lavrentiev, Bogoliubov, Krylov, Graves, Nagumo, Cinquini, McShane, Del Chiaro, Mania and others in the thirties. The situation at this moment is very well summarized by Volterra [79], who writes in 1932 :

> Instead of studying the problems of the calculus of variation by reducing them to differential equations, it is convenient to reduce problems which occur at first sight as differential equations problems to problems of the calculus of variations. This has an importance which can be said to be philosophical, and is also of practical importance. The new methods of the calculus of variations are not only important from the point of view of this science, but have also a great interest for the study of differential equations and of the many problems which are connected. They also give a new interest to the theory of functionals.

In the terminology of its founder Volterra, the *theory of functionals* is what we now call, after Paul Levy, *functional analysis* and, from its almost final formulation given in the thirties by Banach and the Polish school, this part of mathematics has played a unifying and generalizing role in the direct method of the calculus of variations. It is not surprizing that a short pioneering paper on *minimizing a functional in a Banach space*, due to the Polish mathematicians Mazur and Schauder [53], is presented at Oslo in 1936, after Golomb had defined the concept of *gradient of a functional* in a Hilbert space, i.e. of a real function defined over such a space. Some restrictive conditions have been removed successively by Lusternik, Sobolev, Tsitlanadze, Vainberg, Krasnosel'skii, Rothe and others. This is the type of results we describe in Chapter 2.

Minima or maxima are not the only critical points and solutions of differential problems with variational structure may be associated to critical points of *saddle point* type of the associated functional. A useful tool for the study of such critical points is a *deformation technique* introduced in 1934 by Lusternik and Schnirelman [46], which consists, in the case of a Hilbert space, in deforming a functional φ, outside the set of its critical points, through the solutions of its associated *gradient system* $\sigma' = -\nabla\phi(\sigma)$ or a qualitativaly equivalent one. Recent versions of this technique, in particular a *quantitative deformation lemma* introduced by Willem in 1983 [80], are described in Chapter 3, and used there to prove the existence of *minimizing sequences* made of *almost critical points* for some functionals which are bounded below, and to motivate the introduction of various *compactness conditions* (following the one introduced in 1965 by Palais and Smale [58]), leading to the existence of critical points. As emphasized in [48] and [81], such an approach clearly separates the geometrical and the compactness aspects of existence proofs.

But the deformation technique is also efficient to obtain critical points of saddle point type for functionals which need not to be bounded below, by characterizing the critical values through some *minimax arguments* over suitable families of sets. The simplest geometric situation is that of a *mountain pass.* If the value of φ at 0 and at some point e (valleys) are strictly smaller that the infimum of the values of φ on a sphere of centre 0 and radius $r < \|e\|$ (range of mountains), one can expect that, by taking the infimum over all paths joining 0 to e of the supremum of φ over such a path, one will obtain the value of φ at some critical point of mountain pass type. This is not always true, but gives at least the existence of a sequence of almost critical points. Suitable Palais-Smale conditions over φ lead then to the existence of a critical point having the mountain pass critical value. Special cases of this reasoning were used in finite dimensional settings in the study of the Plateau problem (see [21]), but the general *mountain pass lemma* in a Banach space was introduced by Ambrosetti and Rabinowitz in 1973 [3], and has been since one of the most used and fruitful tools of nonlinear functional analysis. Its study is the object of Chapter 4.

In Chapter 5, the very classical problem of the existence of *periodic solutions of forced pendulum-type equations* is analyzed by minimization and mountain-pass techniques. The solution associated to a minimum of the corresponding action functional was essentially discovered by Hamel in 1922 [33], and sixty years later a second solution of mountain pass type was obtained by Mawhin and Willem [50]. This pendulum case provides maybe the simplest situation of a functional which does not satisfy the classical Palais-Smale condition, i.e. of a problem with *lack of compactness.*

A more serious lack of compactness is associated to the problem of finding *solitary waves in an infinite lattice* of particles with only neighbor interactions, considered by Smets and Willem in 1997 [71]. The problem is reduced to finding suitable solutions defined over the real line of an advanced-retarded difference-differential equation. The mountain pass technique provides a sequence of almost critical points for this functional, and subtle tools due to P.L. Lions and Lieb have to be used to deduce from this information the existence of a critical point. This is the topics of Chapter 6.

The mountain pass geometry is the simplest one leading to a minimax characterization of a critical value. Other ones have been introduced in 1983 by Rabinowitz [61, 62] (*saddle point geometry* and *linking geometry*), and they are shortly analyzed in Chapter 7 in the light of the quantitative deformation lemma.

Besides the papers directly quoted in the text, the Bibliography is completed by a list of recent surveys and books devoted to the minimax approach in critical point theory.

Chapter 2

Minimization of weakly lower semi-continuous real functions in a reflexive Banach space

Of course, if φ is a real function defined on a subset of a Banach space E, and if we want to find necessary conditions for φ to have a minimum, we need a suitable generalization of the concept of derivative. The simplest one is undoubtly the *Gateaux derivative*. Let E^* denote the dual space to E and $\langle \cdot, \cdot \rangle$ the pairing between E^* and E. Let $U \subset E$ be non empty and open.

Definition 1 *If $\varphi : U \to \mathbb{R}$, we say that φ has a* Gateaux derivative $u^* \in E^*$ *at $y \in U$ if, for every $h \in E$ one has*

$$\lim_{t \to 0} \frac{\varphi(y + th) - \varphi(y) - \langle u^*, th \rangle}{t} = 0,$$

in which case we denote u^* by $\varphi'(y)$.

A stronger requirement is *Fréchet derivability.*

Definition 2 *If $\varphi : U \to \mathbb{R}$, we say that φ has a* Fréchet derivative $u^* \in E^*$ *at $y \in U$ if*

$$\lim_{h \to 0} \frac{\varphi(y + h) - \varphi(y) - \langle u^*, h \rangle}{||h||} = 0.$$

Of course the Fréchet derivative at y is the Gateaux derivative at y and we keep the notation $\varphi'(y)$. We say that φ belongs to $C^1(U, \mathbb{R})$ if the Fréchet derivative exists and is continuous on U.

With those definitions, it is straightforward to generalize to the setting of Banach spaces Fermat's necessary condition for the existence of a local maximum or minimum of φ at $y \in U$,

$$\varphi'(y) = 0. \tag{2.1}$$

when φ is Gateaux differentiable at y. Any $y \in A$ satisfying equation (2.1) is called a *critical point* of φ, and the corresponding number $\varphi(y)$ is called a *critical value*.

Consequently, if a mapping $\Phi : E \to E^*$ can be written $\Phi = \varphi'$ for some Gateaux differentiable function $\varphi : E \to \mathbb{R}$, every critical point of φ provides a solution to the equation

$$\Phi(u) = 0.$$

It is in particular the case for any local minimum or local maximum of φ.

In classical problems of the calculus of variations, where

$$\varphi(y) = \int_a^b f(x, y(x), y'(x))\, dx,$$

it is a nontrivial interesting exercice to show that, under suitable assumptions upon f and by an appropriate choice of E, the equation (2.1) is nothing but the Euler-Lagrange equation (1.2).

The introduction by Calkin and Morrey, before the Second World War, of the *Sobolev spaces* in multidimensional variational problems (after a quite overlooked use of those spaces in one-dimensional problems by Lichtenstein as early as 1915 [41]), shows the role played by the *reflexivity* of the space in proving the existence of a minimum for a functional. A modern version of the Mazur-Schauder theorem, which subsumes the work we have described, can be stated as follows.

Theorem 1 *If E be a reflexive Banach space, $C \subset E$ is weakly closed, and $\varphi : C \to \mathbb{R}$ is weakly lower semi-continuous, then φ has a minimum over C if and only if it has a bounded minimizing sequence in C.*

We explain the terms in this theorem. E is *reflexive* if the canonical injection $J : E \to E^{**}$ between E and its bidual E^{**} is onto. For example, any Hilbert space is reflexive and all Lebesgue spaces $L^p(\Omega)$ or Sobolev spaces $W^{k,p}(\Omega)$, $k \geq 1$, $1 < p < \infty$ are reflexive. C *weakly closed* means that its contains the limit of any of its weakly convergent sequences, φ *weakly lower semi-continuous* (WLSC) at $y \in C$ means that

$$y_k \rightharpoonup y \quad \Longrightarrow \quad \liminf_{k \to \infty} \varphi(y_k) \geq \varphi(y).$$

Finally, a *minimizing sequence* for φ in C is a sequence $(y_k)_{k \in \mathbb{N}}$ in C such that

$$\varphi(y_k) \to \inf_C f.$$

The straightforward proof of Theorem 1 is left to the reader, and depends upon the following *weak compactness lemma* :

Lemma 1 *Any bounded sequence in a reflexive Banach space contains a weakly convergent subsequence.*

Some of the assumptions of Theorem 1 may not be easy to check, and the following facts may be helpful.

Lemma 2 *A convex set of a Banach space is weakly closed if and only if it is closed.*

Lemma 3 *A convex function $\varphi : E \to \mathbb{R}$ is WLSC at y if and only if it is lower semi-continuous (LSC) at y.*

The *lower semicontinuity* of a function consists in replacing the weak convergence by the strong converge in the definition of WLSC.

Of course, if C is bounded, the same is true for any minimizing sequence of φ in C. If C is unbounded, all the minimizing sequences of φ in C will be bounded if φ is *coercive* on C, i.e. if

$$\varphi(x) \to +\infty \quad \text{when} \quad \|x\| \to \infty.$$

By combining those results with Theorem 1, we obtain a more easily usable statement.

Corollary 1 *Let E be a reflexive Banach space, $C \subset E$ closed convex, and $\varphi : C \to \mathbb{R}$ WLSC (in particular LSC and convex) and coercive. Then φ has a minimum over C.*

As an application, already treated by Lichtenstein in 1915 [41], let us consider the *Dirichlet problem for a second order ordinary differential equation*

$$y'' + f(x,y) = 0, \quad y(0) = y(\pi) = 0, \tag{2.2}$$

where $f : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ is continuous and when there exists $A \geq 0$ such that its indefinite integral

$$F(x,y) := \int_0^y f(x,s)\, ds$$

satisfies the condition

$$F(x,y) \leq A \quad \text{for all} \quad (x,y) \in [0,\pi] \times \mathbb{R}. \tag{2.3}$$

The solutions of (2.2) are the critical points of the functional

$$\varphi : H_0^1(0,\pi) \to \mathbb{R}, \quad y \mapsto \int_0^\pi \left[\frac{y'^2(x)}{2} - F(x,y(x))\right] dx,$$

where $H_0^1(0,\pi)$ is the Hilbert space defined by

$$H_0^1(0,\pi) = \{y \in AC([0,\pi]) : y' \in L^2(0,\pi), \quad y(0) = y(\pi) = 0\}$$

with the inner product

$$(y,z)_1 = \int_0^\pi [y(x)z(x) + y'(x)z'(x)]\,dx$$

As *Poincaré's inequality*

$$\int_0^\pi y^2(x)\,dx \le \int_0^\pi y'^2(x)\,dx \tag{2.4}$$

holds for each $y \in H_0^1(0,\pi)$, the inner product $(\cdot,\cdot)_1$ can be replaced by the equivalent one

$$(y,z) = \int_0^\pi y'(x)z'(x)\,dx.$$

By condition (2.3),

$$\varphi(y) \ge \frac{1}{2}\|y\|^2 - \pi A, \tag{2.5}$$

and the functional φ is clearly coercive. We now show that it is WLSC. If $y_k \rightharpoonup y$, then, by the compact embedding of $H_0^1(0,\pi)$ into $C([0,\pi])$, $y_k \to y$ uniformly on $[0,\pi]$. Furthermore, from the trivial inequality

$$0 \le \int_0^\pi [y_k'(x) - y'(x)]^2\,dx,$$

we deduce

$$\int_0^\pi {y_k'}^2(x)\,dx \ge 2\int_0^\pi y_k'(x)y'(x) - \int_0^\pi y'^2(x)\,dx,$$

and hence

$$\liminf_{k\to\infty} \int_0^\pi {y_k'}^2(x)\,dx \ge \int_0^\pi y'^2(x)\,dx.$$

This easily implies that φ is WLSC on $H_0^1(0,\pi)$, and the existence of a minimum for φ follows from Corollary 1. Thus we have proved the following

Theorem 2 *Assume that $f : [0,\pi] \to \mathbb{R}$ is continuous and that its indefinite integral F satisfies condition (2.3). Then the Dirichlet problem (2.2) has at least one solution.*

A first application of Lichtenstein theorem is given by the Dirichlet problem

$$u'' - a\exp u = h(x), \quad u(0) = u(\pi) = 0, \tag{2.6}$$

where $a > 0$ and $h \in L^1(0,\pi)$. A simple computation shows that the linear problem

$$u'' = h(x), \quad u(0) = u(\pi) = 0,$$

has a unique solution $U(x)$. Letting $u = U + y$, problem (2.6) is reduced to the equivalent one

$$y'' - a\exp(U(x) + y) = 0, \quad y(0) = y(\pi) = 0,$$

and the corresponding function F given by

$$F(x,y) = -a\exp(U(x)+y) \leq 0$$

satisfies assumption (2.3). Thus *Problem (2.6) has at least one solution.*

A second application is given by the *Dirichlet problem for the forced pendulum equation*

$$u'' + a\sin u = h(x), \quad u(0) = u(\pi) = 0, \tag{2.7}$$

with a and h like above. With the same change of unknown function, one gets the equivalent problem

$$y'' + a\sin(U(x)+y) = 0, \quad y(0) = y(\pi) = 0, \tag{2.8}$$

and the corresponding function F given by

$$F(x,y) = a\cos(U(x)+y) \leq a$$

satisfies assumption (2.3). Thus *Problem (2.7) always has at least one solution.*

In 1930, using a complicated variational approach for the equivalent nonlinear integral equations, Hammerstein [34] has generalized Lichtenstein's theorem by replacing the assumption (2.3) by the more general one

$$F(x,y) \leq \lambda\frac{y^2}{2} + A, \tag{2.9}$$

for some $\lambda < 1$ and $A \geq 0$. It is easy to adapt, as done by Cinquini [19] in 1938, Lichtenstein's argument to cover this more general case, by using Poincaré's inequality (2.4) to replace inequality (2.5) by

$$\varphi(y) \geq \frac{1-\lambda}{2}\|y\|^2 - \pi A.$$

Chapter 3

Deformation of real functions on a Banach space and Ekeland's variational principle

Of course, a coercive function is bounded from below, but the converse is not true, and a function which is bounded from below needs not to have a minimum, as shown by the exponential. However, Ekeland [23] has shown in 1974 that, under quite general conditions, functions which are bounded below admit *almost critical points* to any degree of precision. There exist various proofs and variants of Ekeland's result. One approach can be based on a *quantitative deformation lemma* due to Willem [80], which is one of tha many variants of a deformation lemma introduced in 1966 by R. Palais [56]. Given a real function φ on a Banach space E, the aim is to construct a differential equation

$$\sigma'(t) = g(\sigma(t))$$

in E such that φ decreases along its solutions $\sigma(t; y)$ with initial condition y at $t = 0$. In the case of a Hilbert space one has

$$\frac{d}{dt}\varphi(\sigma(t;y)) = (\nabla\varphi(\sigma(t;y)), \sigma'(t;y)) = (\nabla\varphi(\sigma(t;y)), g(\sigma(t;y)))$$

and this expression will be negative (outside of the critical points of φ) if one chooses

$$g(\sigma) = -h(\sigma)\nabla\varphi(\sigma),$$

where h is a suitable scalar positive function that we can freely choose in order to facilitate the study of the differential equation.

We now make those ideas more precise. If φ is of class C^1, φ' is continuous only and solutions may not exist or need not te be unique for the associated gradient differential equation. This is the reason of the introduction of the concept of *pseudogradient vector field,* made by Palais in 1966 [56]. This pseudogradient is a locally Lipschitzian mapping which preserves the geometry of φ'.

Definition 3 *Let M be a metric space, E a normed space and $h : M \to E^* \setminus \{0\}$ a continuous mapping. A* pseudogradient vector field *for h on M is a locally Lipschitz continuous vector field $g : M \to E$ such that, for every $y \in M$,*

$$\|g(y)\| \leq 2\|h(y)\|, \quad \langle h(y), g(y)\rangle \geq \|h(y)\|^2.$$

The following existence lemma is due to Palais.

Lemma 4 *With the notations and under the assumptions of Definition 3, there exists a pseudogradient vector field for h on M*

Proof. For every $v \in M$, there exists $x \in E$ such that $\|x\| = 1$ and

$$\langle h(v), x\rangle > \frac{2}{3}\|h(v)\|.$$

Define $u := \frac{3}{2}\|h(v)\|$ so that

$$\|u\| < 2\|h(v)\|, \quad \langle h(v), u\rangle > \|h(v)\|^2.$$

Since h is continuous, there exists an open neighborhood N_v of v such that

$$\|u\| \leq 2\|h(y)\|, \quad \langle h(y), u\rangle \geq \|h(y)\|^2, \tag{3.1}$$

for every $y \in N_v$. The family $\mathfrak{N} := \{N_v : v \in M\}$ is an open covering of the paracompact space M. Hence there exists a locally finite open covering $\mathfrak{M} := \{M_i : i \in I\}$ of M finer than $\mathfrak{N}$. For each $i \in I$, there exists $v \in M$ such that $M_i \subset N_v$. Hence there exists $u = u_i$ such that (3.1) holds for every $y \in M_i$. If we define, on M,

$$\rho_i(y) := \operatorname{dist}(y, X \setminus M_i), \quad g(y) := \sum_{i \in I} \frac{\rho_i(y)}{\sum_{j \in I} \rho_j(y)} u_i,$$

then, as easily verified, g is a pseudogradient vector field for h on M.

For $S \subset E$, $\alpha > 0$ and $c \in \mathbb{R}$, we use the notations

$$S_\alpha := \{y \in E : \operatorname{dist}(y, S) \leq \alpha\}, \quad \varphi^c := \varphi^{-1}(]-\infty, c]).$$

We now state and prove the *quantitative deformation lemma.*

Lemma 5 *Let E be a Banach space, $\varphi \in C^1(E, \mathbb{R})$, $S \subset E$, $c \in \mathbb{R}$, ε, $\delta > 0$ be such that for all $y \in \varphi^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap S_{2\delta})$,*

$$\|\varphi'(y)\| \geq \frac{8\varepsilon}{\delta}. \tag{3.2}$$

Then there exists $\eta \in C([0,1] \times E, E)$ such that

1. *$\eta(t,y)=y$, if $t=0$ or if $u\notin\varphi^{-1}([c-2\varepsilon,c+2\varepsilon])\cap S_{2\delta}$;*

2. *$\eta(1,\varphi^{c+\varepsilon}\cap S)\subset\varphi^{c-\varepsilon}$;*

3. *$\eta(t,\cdot)$ is an homeomorphism of E, for all $t\in[0,1]$;*

4. *$\|\eta(t,y)-y\|\le\delta$, for all $y\in E$ and all $t\in[0,1]$;*

5. *$\varphi(\eta(\cdot,y))$ is non increasing for each $y\in E$,*

6. *$\varphi(\eta(t,y))<c$, for all $u\in\varphi^c\cap S_\delta$, and all $t\in]0,1]$.*

Proof. By Lemma 4, there exists a pseudogradient vector field g for φ' on $M:=\{y\in E:\varphi'(y)\neq 0\}$. Let us define

$$A:=\varphi^{-1}([c-2\varepsilon,c+2\varepsilon])\cap S_{2\delta},$$

$$B:=\varphi^{-1}([c-\varepsilon,c+\varepsilon])\cap S_{\delta},$$

$$\psi(y):=\frac{\operatorname{dist}(y,E\setminus A)}{\operatorname{dist}(y,E\setminus A)+\operatorname{dist}(y,B)},$$

so that ψ is locally Lipschitz continuous, $\psi=1$ on B, $\psi=0$ on A and $0\le\psi\le 1$. Define also the locally Lipschitz continuous vector field

$$f(y):=-\psi(y)\|g(y)\|^{-2}g(y),\quad y\in A;\quad f(y):=0,\quad y\in E\setminus A.$$

By Definition 3 and Assumption (3.2), $\|f(y)\|\le\delta/8\varepsilon$ for all $y\in E$. Hence, for each $y\in E$, the Cauchy problem

$$\sigma'(t)=f(\sigma(t)),\quad \sigma(0)=y$$

has a unique solution $\sigma(t,y)$ defined for all $t\in\mathbb{R}$ and continuous on $\mathbb{R}\times E$. Define $\eta:[0,1]\times E$ by $\eta(t,y):=\sigma(8\varepsilon t,y)$. Again Definition 3 and Assumption (3.2) imply that, for $t\ge 0$,

$$\|\sigma(t,y)-y\|=\left\|\int_0^t f(\sigma(\tau,y))\,d\tau\right\|\le\frac{\delta t}{8\varepsilon} \tag{3.3}$$

and

$$\frac{d}{dt}\varphi(\sigma(t,y))=\langle\varphi'(\sigma(t,y)),\frac{d}{dt}\sigma(t,y)\rangle$$
$$=\langle\varphi'(\sigma(t,y)),f(\sigma(t,y))\rangle\le-\frac{\psi(\sigma(t,y))}{4}. \tag{3.4}$$

The easy verifications of conclusions 1, 3, 4, 5 and 6 are left to the reader. For conclusion 2, let $y\in\varphi^{c+\varepsilon}\cap S$. If there is some $t\in[0,8\varepsilon]$ such that $\varphi(\sigma(t,y))<c-\varepsilon$, then $\varphi(\sigma(8\varepsilon,y))<c-\varepsilon$, and conclusion 2 is proved. If

$$\sigma(t,y)\in\varphi^{-1}([c-\varepsilon,c+\varepsilon])$$

for all $t \in [0, 8\varepsilon]$, we deduce from (3.3) and (3.4) that

$$\varphi(\sigma(8\varepsilon, y)) = \varphi(y) + \int_0^{8\varepsilon} \frac{d}{dt}\varphi(\sigma(t,y))\, dt \le \varphi(y) - \frac{1}{4}\int_0^{8\varepsilon} \psi(\sigma(t,y))\, dt$$
$$\le c + \varepsilon - 2\varepsilon = c - \varepsilon,$$

and conclusion 2 also holds.

The following variant of *Ekeland's principle* is an easy consequence of the above Lemma 5.

Corollary 2 *Let E be a Banach space, $\varphi \in C^1(E, \mathbb{R})$, bounded from below, $\varepsilon > 0$, $\delta > 0$, and $z \in E$ such that*

$$\varphi(z) \le \inf_E \varphi + \varepsilon.$$

Then there exists $y \in E$ with the following properties :

1. $\varphi(y) \le \inf_E \varphi + 2\varepsilon$;
2. $\|y - z\| \le 2\delta$;
3. $\|\varphi'(y)\| \le \frac{8\varepsilon}{\delta}$.

Proof. It suffices to apply Lemma 5 with $S := \{z\}$ and $c := \inf_E \varphi$. Assuming that for every $y \in \varphi^{-1}([c, c + 2\varepsilon]) \cap S_{2\delta}$, one has $\|\varphi'(y)\| \ge 8\varepsilon/\delta$, one gets $\varphi(\eta(1, z)) \le c - \varepsilon$, a contradiction.

By taking $\varepsilon = 1/k^2$, $\delta = 1/k$, $(k = 1, 2, \ldots)$ in Corollary 2, we obtain the existence of a minimizing sequence made of almost critical points.

Corollary 3 *Let E be a Banach space and $\varphi \in C^1(E, \mathbb{R})$, bounded from below. Then there exists a sequence (y_k) such that*

$$\varphi(y_k) \to \inf_E \varphi, \quad \varphi'(y_k) \to 0, \quad \textit{if} \quad k \to \infty.$$

Another consequence of Lemma 5 is a result proved by Brezis and Nirenberg in 1991 [15].

Corollary 4 *Let E be a Banach space and $\varphi \in C^1(E, \mathbb{R})$ be such chat*

$$c := \liminf_{\|y\| \to \infty} \varphi(y) \in \mathbb{R}.$$

Then, for every ε, $\delta > 0$, $R > 2\delta$, there exists $y \in E$ such that

1. $c - 2\varepsilon \le \varphi(y) \le c + 2\varepsilon$;
2. $\|y\| \ge R - 2\delta$;

3. $\|\varphi'(y)\| < 8\varepsilon/\delta$.

Proof. Assume the thesis false and let $S = E \setminus B(0,R)$. From the definition of c, $\varphi^{c+\varepsilon} \cap S$ is unbounded and $\varphi^{c-\varepsilon} \subset B(0,r)$ for large $r > 0$. By Lemma 5, $\eta(1, \varphi^{c+\varepsilon} \cap S) \subset \varphi^{c-\varepsilon}$ and

$$\varphi^{c+\varepsilon} \cap S \subset (\varphi^{c-\varepsilon})_\delta \subset B(0, r+\delta),$$

a contradiction.

Corollary 4 gives a proof a relation between the existence of sequences of almost critical points and coercivity for functions bounded below, obtained by Shujie Li [40] in 1986. See also [16, 20]

Corollary 5 *Let E be a Banach space and $\varphi \in C^1(E, \mathbb{R})$, bounded from below. If, for each $c \in \mathbb{R}$, every sequence (y_n) in E such that*

$$\varphi(y_n) \to c, \quad \varphi'(y_n) \to 0, \quad \text{if} \quad n \to \infty,$$

is bounded, then φ is coercive.

Proof. If the thesis is false, $c := \liminf_{\|y\| \to \infty} \varphi(y) \in \mathbb{R}$. By Corollary 4 with $\varepsilon = 1/k^2$, $\delta = 1/k$ and $R = k$, $(k = 1, 2, \ldots)$, there exists a sequence (y_k) in E such that

$$\varphi(y_k) \to c, \quad \varphi'(y_k) \to 0, \quad \|y_k\| \to \infty,$$

if $k \to \infty$, a contradiction.

To obtain the existence of a critical point for a function bounded from below, we now introduce variants of a compactness condition first introduced by Palais and Smale [58]. Let E be a Banach space and $\varphi : E \to \mathbb{R}$ be Gateaux differentiable. The original *Palais-Smale condition* is the following one.

Definition 4 *A* Palais-Smale sequence *for φ is a sequence (y_n) in E such that $(\varphi(y_n))$ is bounded and $(\varphi'(y_n))$ converges 0. The function φ satisfies the* PS-condition *if each Palais-Smale sequence contains a convergent subsequence.*

A weaker condition is the following one.

Definition 5 *Let E be a Banach space and $\varphi : E \to \mathbb{R}$ be Gateaux differentiable. A* Palais-Smale sequence at level $c \in \mathbb{R}$ *for φ is a sequence (y_n) in E such that*

$$\varphi(y_n) \to c, \quad \varphi'(y_n) \to 0, \quad \text{if} \quad n \to \infty.$$

The function φ satisfies the $(PS)_c$-condition *if the existence of a Palais-Smale sequence at level c for φ implies that c is a critical value for φ.*

Notice that, by Corollary 5, a function $\varphi \in C^1(E,\mathbb{R})$ bounded below, which has, for each $c \in \mathbb{R}$, its $(PS)_c$ sequences bounded, is coercive. Conversely, it is trivial that a coercive function has all its $(PS)_c$ sequences bounded.

The following result is an immediate consequence of the above Definition 5 and Corollary 3.

Corollary 6 *Let E be a Banach space and $\varphi \in C^1(E,\mathbb{R})$, bounded from below. If φ satisfies the $(PS)_c$ condition for $c = \inf_E \varphi$, then φ has a minimum and c is a critical value of φ.*

As an easy application, let us come back to the *Dirichlet problem for the forced pendulum equation* (2.7). We already know that the action functional of the equivalent problem (2.8) given by

$$\varphi(y) = \int_0^\pi \left[\frac{1}{2}y'^2 + a\cos(U+y)\right] \tag{3.5}$$

is bounded below on $H_0^1(0,\pi)$. We now show that φ *satisfies the PS-condition.* If (y_n) is a sequence such that $(\varphi(y_n))$ is bounded and $\varphi'(y_n) \to 0$, then it is immediate that (y_n) is bounded in $H_0^1(0,\pi)$ and hence, up to a subsequence, weakly converges to some $y \in H^1(0,\pi)$ and converges uniformly to y on $[0,\pi]$. Hence,

$$\lim_{n\to\infty} \langle \varphi'(y_n) - \varphi'(y), y_n - y\rangle = 0.$$

But

$$\langle \varphi'(y_n) - \varphi'(y), y_n - y\rangle$$
$$= \int_0^\pi \{|y_n' - y'|^2 + -[a\sin(U+y_n) - a\sin(U+y)](y_n - y)\}.$$

As the uniform convergence of (y_n) implies

$$\lim_{n\to\infty} \int_0^\pi \{[a\sin(U+y_n) - a\sin(U+y)](y_n - y)\} = 0,$$

it follows that $y_n \to y$ in $H_0^1(0,\pi)$, and PS-condition is proved. Thus $(PS)_c$ holds for all c and the existence of a critical point follows from Corollary 6.

Chapter 4

The mountain pass geometry

We now deduce from the deformation lemma an existence result for almost-critical points under the geometry of a *mountain pass* introduced by Ambrosetti and Rabinowitz [3].

Theorem 3 *Let E be a Banach space and $\varphi \in C^1(E, \mathbb{R})$, $e \in E$, $r > 0$ be such that $\|e\| > r$ and*

$$a := \max\{\varphi(e), \varphi(0)\} < b := \inf_{\|y\|=r} \varphi(y).$$

Let

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \gamma(1) = e\},$$
$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \varphi(\gamma(t)).$$

Then,

$$b \leq c < +\infty, \tag{4.1}$$

and for each $\varepsilon > 0$, $\delta > 0$ and $\gamma \in \Gamma$ such that

$$\sup_{t \in [0,t]} \varphi(\gamma(t)) \leq c + \varepsilon, \tag{4.2}$$

there exists $y \in E$ such that

1. $c - 2\varepsilon \leq \varphi(y) \leq c + 2\varepsilon$
2. $dist(y, \gamma([0,1])) \leq 2\delta$
3. $\|\varphi'(y)\| \leq 8\varepsilon/\delta$.

Proof. Notice first that, by the fact that $\|\gamma(0) < r\gamma(e)\|$ and connexity, for every $\gamma \in \Gamma$, there exists $\tau \in [0,1]$ such that $\|\gamma(\tau)\| = r$, and hence, by (4.3), we have

$$\sup_{t\in[0,1]} \varphi(\gamma(t)) \geq \varphi(\gamma(\tau)) \geq b,$$

and hence $c \geq b$. Without loss of generality we can assume that $\varepsilon < \frac{b-a}{2}$, i.e. that $a < b-2\varepsilon \leq c-2\varepsilon$. Now, assume that the last conclusion is not true. Then, by Lemma 5 with $S = \gamma([0,1])$, there exists $\eta \in C([0,1] \times E, E)$ satisfying the conditions of the lemma. In particular, if $\beta(t) = \eta(1, \gamma(t))$, we have

$$\beta(0) = \eta(1,0) = 0, \quad \beta(1) = \eta(1,e) = e.$$

as $\varphi(e), \varphi(0) \leq a < c - 2\varepsilon$. Consequently $\beta \in \Gamma$. From (4.2), we deduce that

$$\varphi(\eta(1,\gamma(t))) \leq c - \varepsilon, \quad t \in [0,1],$$

a contradiction with the definition of c.

Remark 1 *The assumptions of Theorem 3 need not to imply the existence of a critical value, as shown by the example of $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$\varphi(x,y) = x^2 + (1-x)^3 y^2$$

which satisfies these assumptions and has only 0 *as critical value (Brezis-Nirenberg [15]).*

We will need a slight variant of this result due to Brezis-Nirenberg [15].

Theorem 4 *Let E be a Banach space and $\varphi \in C^1(E,\mathbb{R})$, $e \in E$, $r > 0$ be such that $\|e\| > r$ and*

$$a := \max\{\varphi(e), \varphi(0)\} < b := \inf_{\|y\|=r} \varphi(y).$$

Let $P : E \to E$ be a continuous mapping such that, for all $y \in E$,

$$\varphi(P(y)) \leq \varphi(y), \quad P(0) = 0, \quad P(e) = e,$$

$$\Gamma = \{\gamma \in C([0,1],E) : \gamma(0) = 0,\ \gamma(1) = e\},$$

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \varphi(\gamma(t)).$$

Then,

$$b \leq c < +\infty,$$

and for each $\varepsilon > 0$, $\delta > 0$ and $\gamma \in \Gamma$ such that

$$\sup_{t\in[0,t]} \varphi(\gamma(t)) \leq c + \varepsilon,$$

there exists $y \in E$ such that

1. $c - 2\varepsilon \leq \varphi(y) \leq c + 2\varepsilon$
2. $dist(y, P(E)) \leq 2\delta$
3. $\|\varphi'(y)\| \leq 8\varepsilon/\delta$.

Proof. It suffices to mimick the above proof with $S = P(E)$ and $\beta(t) = \eta(1, P\gamma(t))$.

An immediate consequence of Theorem 3 is the version of the Ambrosetti-Rabinowitz *mountain pass lemma* obtained by Brezis-Coron-Nirenberg [14] in 1980 (Ambrosetti and Rabinowitz [3] required the classical PS-condition).

Theorem 5 *Let E be a Banach space and $\varphi \in C^1(E, \mathbb{R})$, $e \in E$, $r > 0$ be such that $\|e\| > r$, and*

$$a := \max\{\varphi(e), \varphi(0)\} < b := \inf_{\|y\|=r} \varphi(y).\varphi(e)\}. \tag{4.3}$$

Let

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \gamma(1) = e\},$$
$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \varphi(\gamma(t)).$$

and assume that φ satisfies the $(PS)_c$ condition. Then $c \geq b$ is a critical value for φ.

Proof. For $\varepsilon = 1/n^2$ and $\delta = 1/n$ ($n = 1, 2, ..$) we find, by Theorem 3 a sequence (y_n) in E such that $\varphi(y_n) \to c$ and $\varphi'(y_n) \to 0$ as $n \to \infty$. Thus, c is a critical value for φ.

We introduce another compactness condition, allowing a weakening of one assumption in the mountain pass lemma.

Definition 6 *Let E be a Banach space and $\varphi : E \to \mathbb{R}$ be Gateaux differentiable. The function φ satisfies the* BPS-condition *if any bounded sequence (y_n) such that $(\varphi(y_n))$ is bounded and $\varphi'(y_n) \to 0$ has a convergent subsequence.*

The following result is a version of the mountain pass lemma essentially obtained by Pucci and Serrin in the middle eighties [59], motivated by a result of Mawhin and Willem [50] for the forced pendulum equation described in the next Chapter.

Theorem 6 *Let E be a Banach space and $\varphi \in C^1(E, \mathbb{R})$, $e \in E$, $R > r > 0$, such that $\|e\| > R$, and*

$$a := \max\{\varphi(e), \varphi(0)\} \leq b := \inf_{r\leq\|y\|\leq R} \varphi(y). \tag{4.4}$$

Let

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \gamma(1) = e\},$$

$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \varphi(\gamma(t)).$$

and assume that φ satisfy the $(PS)_c$ and BPS conditions. Then $c \geq b$ is a critical value for φ. Moreover, if $b = c$, there is a critical point z such that $\varphi(z) = b$ and $\|z\| = (R + r)/2$.

Proof. If $c > b$, we can repeat the proof of Theorem 5, so let us assume that $c = b$. For each positive integer n such that

$$\frac{1}{\sqrt{n}} \leq \frac{R - r}{2}, \tag{4.5}$$

there exists $\gamma_n \in \Gamma$ such that

$$\max_{t \in [0,1]} \varphi(\gamma_n(t)) \leq b + \frac{1}{n},$$

and, because of (4.4), there exists $t_n \in [0, 1]$ such that

$$\|\gamma_n(t_n)\| = \frac{r + R}{2}.$$

If $w_n := \gamma_n(t_n)$, then

$$\varphi(w_n) \leq b + \frac{1}{n}, \quad \|w_n\| = \frac{r + R}{2}, \quad n = 1, 2, \ldots.$$

Let us apply Lemma 5 with $S = \{w_n\}$, $c = b$, $\varepsilon = 1/n$ and $\delta = 1/\sqrt{n}$. If, for every $y \in \varphi^{-1}([b - (2/n), b + (2/n)]) \cap S_{2/\sqrt{n}}$, we have $\|\varphi'(y)\| \geq 8/\sqrt{n}$, then $v_n = \eta(1, w_n) \in \varphi^{b-(1/n)} \cap S_{1/\sqrt{n}}$. Consequently, using (4.5),

$$\|v_n\| \leq \|v_n - w_n\| + \|w_n\| \leq \frac{R - r}{2} + \frac{R + r}{2} = R, \quad \varphi(v_n) \leq b - \frac{1}{n},$$

a contradiction to the definition of b. Thus, there exists some y_n such that

$$b - \frac{2}{n}\varphi(y_n) \leq b + \frac{2}{n}, \quad \|y_n - w_n\| \leq \frac{2}{\sqrt{n}}, \quad \|\varphi'(y_n)\| < \frac{8}{\sqrt{n}}.$$

By our compactness assumption, (y_n) contains a subsequence converging to some y such that $\varphi(y) = c$, $\|y\| = (r + R)/2$ and $\varphi'(y) = 0$.

Remark 2 *When $b = c$, Theorem 6 implies the existence of a critical point on each sphere centered at* 0 *and of radius $\rho \in]r, R[$.*

Chapter 5

Periodic solutions of the forced pendulum

We consider the problem of *periodic solutions for the periodically forced pendulum-type equation*

$$y'' + g(y) = f(t), \quad y(0) = y(T), \quad y'(0) = y'(T), \tag{5.1}$$

when $g : \mathbb{R} \to \mathbb{R}$ is continuous, 2π-periodic, has zero mean value, namely

$$\overline{g} := \frac{1}{2\pi} \int_0^{2\pi} g(s)\, ds = 0,$$

and $f \in L^1(0, T)$ has zero mean value, namely

$$\overline{f} := \frac{1}{T} \int_0^T f(t)\, dt = 0.$$

If

$$E = H_T^1 = \{y \in AC([0, T]) : y' \in L^2(0, T),\ y(0) = y(T)\}$$

with the inner product

$$(y, z) = \frac{1}{T} \int_0^T y(t)z(t)\, dt + \frac{1}{T} \int_0^T y'(t)z'(t)\, dt,$$

and corresponding norm $\|y\|$, then E is a Hilbert space and the solutions of problem (5.1) correspond to the critical points of the action functional $\varphi : E \to \mathbb{R}$ defined by

$$\varphi(y) = \int_0^T \left[\frac{y'^2(t)}{2} - G(y(t)) + f(t)y(t) \right] dt,$$

where

$$G(y) = \int_0^y g(s)\, ds.$$

Notice that G is also 2π-periodic, as g has zero mean value. Notice also that if y is a solution of (5.1), then $y + 2k\pi$ is also a solution for any $k \in \mathbb{Z}$. Two solutions of (5.1) will be called *geometrically distinct* if they do not differ by a multiple of 2π.

It is easy to show that $\varphi \in C^1(E, \mathbb{R})$. Now, letting

$$y(t) = \overline{y} + \widetilde{y}(t),$$

with

$$\int_0^T \widetilde{y}(t)\, dt = 0,$$

we have *Wirtinger's inequality*

$$\left(\frac{2\pi}{T}\right)^2 \int_0^T \widetilde{y}^2(t)\, dt \leq \int_0^T y'^2(t)\, dt,$$

and *Sobolev's inequality*

$$\max_{t\in[0,T]} |\widetilde{y}(t)| \leq \frac{T^{1/2}}{2\sqrt{3}} \left(\int_0^T y'^2(t)\, dt\right)^{1/2}.$$

See [52] for more details.

Lemma 6 *φ is 2π-periodic, bounded from below, satisfies the BPS-condition, and satisfies the $(PS)_c$-condition for each $c \in \mathbb{R}$.*

Proof. We have, as $\overline{f} = 0$,

$$\varphi(y) = \int_0^T \left[\frac{y'^2(t)}{2} - G(y(t)) + \widetilde{f}(t)\widetilde{y}(t)\right] dt,$$

and hence

$$\varphi(y + 2\pi) = \varphi(y)$$

(i.e. φ is 2π-periodic), and

$$\varphi(y) \geq \int_0^T \frac{y'^2(t)}{2}\, dt - MT - \left(\int_0^T |\widetilde{f}(t)|\, dt\right) \max_{t\in[0,T]} |\widetilde{y}(t)|$$

$$\geq \int_0^T \frac{y'^2(t)}{2}\, dt - MT - \left(\int_0^T |\widetilde{f}(t)|\, dt\right) \frac{T^{1/2}}{2\sqrt{3}} \left(\int_0^T y'^2(t)\, dt\right)^{1/2}, \tag{5.2}$$

which shows that φ is bounded from below. Now, if $(\varphi'(y_n))$ converges to 0 and (y_n) is bounded in H_T^1, then, up to a subsequence, (y_n) converges weakly in H_T^1 and uniformly on $[0, T]$ to some $y \in H_T^1$. Consequently

$$\langle \varphi'(y_n) - \varphi'(y), y - y_n \rangle \to 0,$$

as $n \to \infty$. But

$$\langle \varphi'(y_n) - \varphi'(y), y - y_n \rangle$$

$$= \int_0^T [y_n'(t) - y'(t)]^2 \, dt - \int_0^T [g(y_n(t)) - g(y(t))][y_n(t) - y(t)] \, dt,$$

and

$$\int_0^T [g(y_n(t)) - g(y(t))][y_n(t) - y(t)] \, dt \to 0$$

as $n \to \infty$. Consequently,

$$\int_0^T [y_n'(t) - y'(t)]^2 \, dt \to 0,$$

as $n \to \infty$, and hence $y_n \to y$ in H_T^1. Thus φ satisfies the BPS-condition. Finally, let (y_n) be such that $\varphi(y_n) \to c$ and $\varphi'(y_n) \to 0$ as $n \to \infty$. Thus $(\varphi(y_n))$ is bounded and, using (5.2) we see that $(\|y_n'\|_{L^2})$ is bounded. Now, by 2π-periodicity of φ, there exists $z_n \in [0, 2\pi[$ such that $z_n = \overline{y}_n \pmod{2\pi}$. Letting $w_n(t) = z_n + \widetilde{y}_n(t)$, we have $\varphi(y_n) = \varphi(w_n)$, $\varphi'(y_n) = \varphi'(w_n)$ and (w_n) is bounded in H_T^1. By the reasoning for BPS applied to this sequence, we see that, up to a subsequence, $w_n \to w \in H_T^1$ and hence

$$\varphi(w) = \lim_{n\to\infty} \varphi(w_n) = c, \quad \varphi'(w) = \lim_{n\to\infty} \varphi'(w_n) = 0.$$

Thus c is a critical value of φ, and φ satisfies $(PS)_c$.

Remark 3 *Notice the difference between the case of Dirichlet and of periodic boundary conditions for the action functional φ associated to the forced pendulum. It was shown in Section 3 that φ satisfied the Palais-Smale condition on the space H_0^1 of functions satisfying Dirichlet conditions. The above result shows that it satisfies only the weaker $(PS)_c$ and BPS-condition on the space H_T^1 of periodic functions. From a Palais-Smale sequence at level c, one has constructed a* bounded *sequence containing a convergent sequence with limit giving the minimum and the mountain pass level to the action functional.*

Theorem 7 *For each $f \in L^1(0,T)$ such that $\overline{f} = 0$, problem (5.1) has at least two geometrically distinct solutions.*

Proof. It follows from Lemma 6 that φ satisfies the conditions of Corollary 6. Thus problem (5.1) has a solution y^* which minimizes φ over H_T^1. Let now $\psi(y) := \varphi(y^* + y)$. Clearly, ψ satisfies the properties given by Lemma 6, and, furthermore,

$$\psi(0) = \psi(2\pi) = \min_{H_T^1} \varphi.$$

Let $0 < r < R < 2\pi$, $\quad e = 2\pi$. Then, all conditions of Theorem 6 are satisfied. If the critical value $c > \min_{H_T^1} \varphi$, then the corresponding critical point is geometrically distinct from y^*. If $c = \min_{H_T^1} \varphi$, then ψ has a critical point z such that $r \leq \|z\| \leq R$. Consequently, $y^{**} = y^* + z$ is a critical point of φ such that $0 < r \leq \|y^{**} - y^*\| \leq R < 2\pi$, so that y^* and y^{**} are two geometrically distinct solutions of problem (5.1).

The first solution was essentially discovered by Hamel [33] in 1922, and the second one by Mawhin and Willem [50] in 1983.

Remark 4 *Theorem 7 can also be proved by considering the functional analogous to (3.5). As* $\overline{f} = 0$, *the problem*

$$u'' = f, \quad u(0) = u(T), \quad u'(0) = u'(T), \quad \overline{u} = 0$$

has a unique solution U. *Letting* $y = U + u$, *problem (5.1) is equivalent to*

$$u'' + g(U + u) = 0, \quad u(0) = u(T), \quad u'(0) = u'(T),$$

associated to the functional

$$\varphi(u) = \int_0^T \left\{ \frac{[u'2(t)]^2}{2} - G\left[U(t) + u(t)\right] \right\} dt.$$

Remark 5 *Other proofs of this theorem have been given since. One uses essentially Lusternik-Schnirelman's theory and provides the existence of at least* $N + 1$ *geometrically distinct solutions for N-dimensional systems of the form*

$$y'' = F'_y(t, y)$$

with F periodic in y (Mawhin [49], Fournier-Willem[28], Rabinowitz [64]). For this system, the mountain pass lemma only gives two solutions (see [51, 47]). Another one is a variant of the Lyapunov-Schmidt method and covers nonlinearities g which are almost periodic in y (Serra-Tarallo [70]).

Chapter 6

Solitary waves with prescribed speed on infinite lattices

Finite lattices with nearest neighbor interactions were considered numerically by Fermi, Pasta and Ulam [27] in 1955. Toda [73] proved in 1967 that infinite lattices were integrable for some exponential interaction potential (*Toda lattices*), which gives explicit formulas for the solitary waves. The first applications of the calculus of variations to the study of various solutions for lattices were made in the middle eighties by Friesecke and Wattis [30] (solitary waves), Ruf and Srikanth [67], Arioli and Gazzola [4, 5] (periodic solutions), Arioli, Gazzola and Terraccini [6] (multibumps periodic solutions).

An infinite lattice of particles with nearest neigbor interaction is described by an infinite system of ordinary differential equations

$$q_k''(t) = V'[q_{k+1}(t) - q_k(t)] - V'[q_k(t) - q_{k-1}(t)], \quad k \in \mathbb{Z}, \tag{6.1}$$

where $V : \mathbb{R} \to \mathbb{R}$ is of class C^1, $V(0) = V'(0) = 0$ and q_k denotes the displacement of the k^{th} particle with respect to its equilibrium position.

A *solitary wave* is a solution of (6.1) of the form

$$q_k(t) = y(k - ct), \quad k \in \mathbb{Z},$$

where $y \in C^2(\mathbb{R}, \mathbb{R})$ and $c \in \mathbb{R}$ (*speed* of the solitary wave). By substitution, we see that for such a solitary wave, u must be solution of the forward-backward difference-differential equation

$$c^2 y''(x) = V'[y(x+1) - y(x)] - V'[y(x) - y(x-1)]. \tag{6.2}$$

Friesecke and Wattis[30] have proved the existence of solitary waves with prescribed averaged potential energy

$$\int_{\mathbb{R}} V[y(x-1) - y(x)]\,dx = K$$

by minimizing the averaged kinetic energy

$$\frac{1}{2}\int_{\mathbb{R}} [y'(x)]^2\,dx$$

under such a constraint, using P.L. Lions concentration-compactness method [44, 45]. The speed c is the *unknown* corresponding Lagrange multiplier. Smets and Willem [71] have considered in 1997 the existence of solitary waves with *prescribed* speed by using a version of the mountain pass theorem, a weak convergence argument and Lieb's lemma [42]. We describe their results in this Chapter.

The natural action functional associated to equation (6.2) is given by

$$\varphi(y) := \int_{\mathbb{R}} \left\{ \frac{c^2}{2}[y'(x)]^2 - V[y(x+1) - y(x)] \right\} dx. \tag{6.3}$$

Because of the form of this action, the underlying function must have derivatives in $L^2(\mathbb{R})$ but, as y itself only occurs through the difference of its values at two values differing from one, y needs not to be in $L^2(\mathbb{R})$ and its value at zero is arbitrary. So its seems natural to work in the function space

$$E = \{y \in H^1_{loc}(\mathbb{R}) : y' \in L^2(\mathbb{R}), \quad y(0) = 0\}, \tag{6.4}$$

equipped with the inner product

$$(y, z) = \int_{\mathbb{R}} y'z' \tag{6.5}$$

and the corresponding norm $\|\cdot\|$.

Lemma 7 *With the inner product (6.5), E is a Hilbert space.*

Proof. Let (y_n) be a Cauchy sequence in E. Then (y_n') is a Cauchy sequence in $L^2(\mathbb{R})$ and there exists $v \in L^2(\mathbb{R})$ such that (y_n') converges to v in $L^2(\mathbb{R})$. The function $y(x) := \int_0^x v(s)\,ds$ is continuous, hence in $L^2_{loc}(\mathbb{R})$, and such that

$$y(0) = 0, \quad y' = v.$$

Consequently $y \in E$ and (y_n) converges to y in E.

Let us define the linear operator $A : E \to L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ by

$$Ay(x) := y(x+1) - y(x),$$

so that the action functional (6.3) can be written

$$\varphi(y) := \int_{\mathbb{R}} \left\{ \frac{c^2}{2}[y'(x)]^2 - V[Ay(x)] \right\} dt.$$

Lemma 8 *A is continous and such that*

$$|Ay|_\infty \le \|y\|, \quad |Ay|_2 \le \|y\|.$$

Furthermore, $Ay \in H^1(\mathbb{R})$ *and*

$$\|Ay\|_{H^1} \le 3\|y\|.$$

Proof. By Cauchy-Schwarz inequality, we have

$$Ay(x) = |y(x+1) - y(x)| = \left| \int_x^{x+1} y'(s)\, ds \right| \le \left(\int_x^{x+1} |y'(x)|^2\, ds \right)^{1/2} \le \|y\|.$$

By Cauchy-Schwarz inequality and Fubini's theorem, we have

$$\int_{\mathbb{R}} |Ay(x)|^2\, dx = \int_{\mathbb{R}} \left| \int_x^{x+1} y'(s)\, ds \right|^2 dx \le \int_{\mathbb{R}} \int_x^{x+1} |y'(s)|^2\, ds\, dx$$

$$= \int_{\mathbb{R}} \int_{s-1}^{s} |y'(s)|^2\, dx\, ds = \int_{\mathbb{R}} |y'(s)|^2\, ds = \|y\|^2.$$

Finally,

$$\left(\int_{\mathbb{R}} |(Ay)'(x)|^2\, dx \right)^{1/2} = \left(\int_{\mathbb{R}} |y'(x+1) - y'(x)|^2\, dx \right)^{1/2} \le 2\|y\|,$$

and hence

$$\|Ay\|_{H^1} = |Ay|_2 + |(Ay)'|_2 \le 3\|y\|.$$

We now show that, under mild conditions upon V, the action functional φ is well defined and smooth on E.

Lemma 9 *If* $V \in C^1(\mathbb{R}, \mathbb{R})$, $V(0) = V'(0) = 0$ *and* $V''(0)$ *exists, then* φ *is well defined over* E, $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ *and*

$$\langle \varphi'(y), h \rangle = \int_{\mathbb{R}} [c^2 y' h' - V'(Ay)Ah].$$

Proof. To show that φ is well defined on E it suffices to show that $V(Ay) \in L^1(\mathbb{R})$ when $y \in E$. From the assumptions upon V we see that, for each $R > 0$ there exists $C_R > 0$ such that

$$\left| \frac{V'(u)}{u} \right| \le C_R$$

whenever $|u| \le R$. This implies immediately that

$$\left| \frac{V(u)}{u^2} \right| \le \frac{C_R}{2}$$

whenever $|u| \leq R$. As $Ay \in H^1(\mathbb{R})$ by Lemma 8, we know that Ay is continuous and has zero limits at $\pm\infty$, and hence, there exists $\rho_R > 0$ such that $|Ay(x)| \leq R$ whenever $|x| \geq \rho_R$. Consequently, $V(Ay)$ is continuous and

$$|V(Ay(x))| \leq \frac{C_R}{2}|Ay(x)|^2$$

for $|x| \geq \rho_R$. As $|Ay|^2$ is integrable over $\mathbb{R}$, the same is true for $V(Ay)$. The proof that $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ and that $\varphi'(y)$ has the given expression is left as an exercice.

Lemma 10 *Under the assumptions of Lemma 9, if y is a critical point of φ, then y is a solution of (6.2).*

Proof. If y is a critical point of φ, then, for each $h \in \mathfrak{D}(\mathbb{R})$ we have

$$0 = \int_{\mathbb{R}} \{c^2 y'(x)h'(x) - V'[y(x+1) - y(x)][h(x+1) - h(x)]\}\,dx$$

$$= \int_{\mathbb{R}} \{c^2 y'(x)h'(x) - [V'[y(x) - y(x-1)] - V'[y(x+1) - y(x))]h(x)\}\,dx,$$

so that u is a weak solution of (6.2), and is therefore of class C^2 by the smoothness of V.

Let us now make stronger assumptions on V in order that φ has the geometry of the mountain pass lemma.

(V_1) $V(u) = c_0^2 \frac{u^2}{2} + W(u)$, *where* $c_0 \geq 0$, $W \in C^1(\mathbb{R}, \mathbb{R})$, $W(0) = 0$, $W'(u) = o(|u|)$ *when* $u \to 0$.

(V_2^+) *There exists* $u_+ > 0$ *such that* $W(u_+) > 0$ *and* $\alpha > 2$ *such that, for all* $u \geq 0$, *one has*

$$0 \leq \alpha W(u) \leq uW'(u). \tag{6.6}$$

(V_2^-) *There exists* $u_- < 0$ *such that* $W(u_-) > 0$ *and* $\alpha > 2$ *such that, for all* $u \leq 0$, *one has*

$$0 \leq \alpha W(u) \leq uW'(u).$$

We shall be first interested in *monotone waves,* i.e. waves such that u is either nondecreasing or nonincreasing. As only Au occurs in φ, we can assume without loss of generality that $W(u) = 0$ for $u \leq 0$ when (V_2^+) holds and that $W(u) = 0$ for $u \geq 0$ when (V_2^-) is satisfied. We treat only explicitely the case of Assumption (V_2^+), the other one being similar and left to the reader.

Lemma 11 *If Assumptions (V_1) and (V_2^+) hold, then V and W are nonnegative and nondecreasing on $\mathbb{R}$, positive for $u \geq u_+$, and there exist $a_0 \geq 0$, $c > 0$ and $a_1 > 0$ such that*

$$W(u) \leq a_0|u|^\alpha \quad \textit{for} \quad |u| \leq 1, \quad W(u) \geq cu^\alpha \quad \textit{for} \quad |u| \geq u_+,$$

$$V(u) \geq a_1(u^\alpha - u^2) \quad \textit{for} \quad u \geq 0.$$

Proof. The first inequality follows from integrating the differential inequality (6.6) from $u < u_+$ to u_+ and the second one from integrating it from u_+ to $u > u_+$. The third one is a consequence of the preceding inequalities.

Define the mapping $P : E \to E$ by

$$Py(x) := \int_0^x |y'(s)|\, ds.$$

Notice that $Py = y$ if y is nondecreasing, so that $P(E)$ is made of the nondecreasing elements of E.

Lemma 12 *For each $y \in E$, we have $\varphi(Py) \leq \varphi(y)$.*

Proof. As $(Py)' = |y'|$ we have

$$\int_{\mathbb{R}} [(Py)']^2 = \int_{\mathbb{R}} |y'|^2.$$

Furthermore,

$$APy(x) = \int_0^{x+1} |y'| - \int_0^x |y'| = \int_x^{x+1} |y'| \geq \int_x^{x+1} y' = y(x+1) - y(x) = Ay(x),$$

and, as V is non decreasing, the result follows.

We know show that the assumptions of Theorem 4 are satisfied.

Proposition 1 *Under the assumptions (V_1), (V_2^+) and $c > c_0$, there exists $e \in P(E)$ and $r > 0$ such that $\|e\| > r$ and*

$$b := \inf_{\|y\|=r} \varphi(y) > \varphi(0) \geq \varphi(e).$$

Proof. If $y \in E$ and $\|y\| \leq 1$, then, by Lemma 8, $|Ay|_\infty \leq 1$ and hence, by Lemma 11, $W(Ay)(x) \leq a_0|Ay(x)|^\alpha$, so that

$$\varphi(y) \geq \int_{\mathbb{R}} \left[\frac{c^2}{2}y'^2 - \frac{c_0}{2}|Ay|^2 - a_0|Ay|^\alpha\right] \geq \frac{c^2 - c_0^2}{2}\|y\|^2 - a_0|Au|_\alpha^\alpha.$$

By interpolation and Lemma 8, $A : E \to L^\alpha(\mathbb{R}, \mathbb{R})$ is continous and hence, for some positive constant K we have

$$\varphi(y) \geq \frac{c^2 - c_0^2}{2}\|y\|^2 - a_0K\|u\|^\alpha,$$

which implies, as $\alpha > 2$, the existence of $r > 0$ such that $\inf_{\|y\|=r} \varphi(y) > 0$. Now, let $v \in P(E) \setminus \{0\}$ and $\lambda > 0$. We have, using Lemmas 8 and 11,

$$\varphi(\lambda v) \leq \lambda^2 \frac{c^2}{2} \|v\| + \lambda^2 a_1 |Av|_2^2 - \lambda^\alpha a_1 |Av|_\alpha^\alpha.$$

Since $\alpha > 2$, the right hand member is negative of λ sufficiently large and there exists $e := \lambda v \in P(E)$ such that $\|e\| > r$ and $\varphi(e) \leq 0$.

A consequence of Proposition 1 and Theorem 4 is the existence of a sequence of almost critical points.

Proposition 2 *Under the assumptions of Proposition 1, there exists a bounded sequence (y_n) in E such that*

$$\varphi(y_n) \to d, \quad \varphi(y_n') \to 0, \quad dist(y_n, P(E)) \to 0,$$

where d is defined by

$$d := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \varphi(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \gamma(1) = e\}.$$

Furthermore, Ay_n does not converge to 0 in measure.

Proof. All we have to prove is that any sequence (y_n) obtained by taking $\epsilon = 1/n^2$ and $\delta = 1/n$ in Theorem 4 is bounded and that $Ay_n \not\to 0$ in measure. For n sufficiently large, we have

$$d + 1 + \|y_n\| \geq \varphi(y_n) - \frac{1}{\alpha}\langle \varphi'(y_n), y_n \rangle$$

$$= \left(\frac{1}{2} - \frac{1}{\alpha}\right) \left[c^2\|y_n\|^2 - c_0^2|Ay_n|_2^2\right] + \int_{\mathbb{R}} [\alpha^{-1} Ay_n W'(Ay_n) - W(Ay_n)]$$

$$\geq \left(\frac{1}{2} - \frac{1}{\alpha}\right) (c^2 - c_0^2)\|y_n\|^2,$$

because of assumption (V_2^+). Thus, (y_n) is bounded. Let $a_2 := \sup_{n \in \mathbb{N}} \|y_n\|$. It follows from Lemma 8 that the sequence (Ay_n) is bounded in $H^1(\mathbb{R})$. On the other hand, by Assumption (V_1),

$$\frac{1}{2} W'(u)u - W(u) = o(u^2) \quad \text{as} \quad u \to 0,$$

so that

$$a_3 := \sup_{|u| \leq a_2} \frac{\frac{1}{2}W'(u)u - W(u)}{u^2} < \infty.$$

Given $\epsilon > 0$, there exists $\delta > 0$ such that, for $|u| \leq \delta$, we have

$$\left| \frac{1}{2} W'(u)u - W(u) \right| \leq \epsilon u^2.$$

Consequently

$$\int_{\mathbb{R}} \left[\frac{1}{2}W'(Ay_n)Ay_n - W(Ay_n)\right]$$

$$= \left[\int_{|Ay_n(x)|>\delta} + \int_{|Ay_n(x)|\le\delta}\right] \left[\frac{1}{2}W'(Ay_n)Ay_n - W(Ay_n)\right]$$

$$\le \text{meas}\{|Ay_n(x)| > \delta\} a_3 |Ay_n|_\infty^2 + \epsilon |Ay_n|_2^2 \le \text{meas}\{|Ay_n(x)| > \delta\} a_3 a_2^2 + \epsilon a_2^2.$$

Hence, if $Ay_n \to 0$ in measure on $\mathbb{R}$, we obtain

$$0 < d = \varphi(y_n) - \frac{1}{2}\langle \varphi'(y_n), y_n \rangle + o(1)$$

$$= \int_{\mathbb{R}} \left[\frac{1}{2}W'(Ay_n)Ay_n - W(Ay_n)\right] + o(1) = o(1),$$

a contradiction.

To deduce the existence of a critical point from the information about the sequence (y_n), we need two lemmas of real analysis. The first one is a special case of a result of P.L. Lions [44], and tells essentially that a bounded sequence in $H^1(\mathbb{R})$ which converges to zero locally in $L^q(\mathbb{R})$ $(q \ge 2)$ converges to zero in $L^p(\mathbb{R})$ $(p > 2)$.

Lemma 13 *Let $r > 0$ and $q \ge 2$. If the sequence (u_n) is bounded in $H^1(\mathbb{R})$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} |u_n|^q \to 0,$$

then $y_n \to 0$ in $L^p(\mathbb{R})$ for any $p > 2$.

Proof. Let $q < s < t$ and $u \in H^1(\mathbb{R})$. Hölder inequality implies that, if $I_y = [y-r, y+r]$,

$$|u|_{L^s(I_y)} = \left(\int_{I_y} |u|^s\right)^{1/s} = \left(\int_{I_y} |u|^{s(1-\lambda)}|u|^{s\lambda}\right)^{1/s}$$

$$\le \left(\int_{I_y} |u|^q\right)^{\frac{1-\lambda}{q}} \left(\int_{I_y} |u|^t\right)^{\frac{\lambda}{t}} \le C \left(\int_{I_y} |u|^q\right)^{\frac{1-\lambda}{q}} |u|_{L^\infty}^{\lambda}(I_y),$$

where $0 < \lambda < 1$ and $\frac{s(1-\lambda)}{q} + \frac{s\lambda}{t} = 1$. Now, by Sobolev inequality,

$$|u|_{L^\infty(I_y)} \le C' \left[\int_{I_y} (|u|^2 + |u'|^2)\right]^{1/2},$$

where C' depends only upon $|I_y| = 2r$, so that

$$|u|^s_{L^s(I_y)} \leq C^s \left(\int_{I_y} |u|^q \right)^{\frac{(1-\lambda)s}{q}} C'^{\lambda s} \left[\int_{I_y} (|u|^2 + |u'|^2) \right]^{\lambda s/2}.$$

Choosing $\lambda s = 2$, i.e. $s = 2 + q\left(1 - \frac{2}{t}\right)$, which gives, as $t > s$ is arbitrary, $2 < s < q + 2$, we obtain

$$|u|^s_{L^s(I_y)} \leq C'' \left(\int_{I_y} |u|^q \right)^{\frac{(1-\lambda)s}{q}} \left[\int_{I_y} (|u|^2 + |u'|^2) \right].$$

Consequently,

$$\int_{\mathbb{R}} |u|^s = \sum_{n\in\mathbb{Z}} \int_{2rn}^{2r(n+1)} |u|^s$$

$$\leq C'' \sum_{n\in\mathbb{Z}} \left\{ \left(\int_{2rn}^{2r(n+1)} |u|^q \right)^{\frac{(1-\lambda)s}{q}} \left[\int_{2rn}^{2r(n+1)} (|u|^2 + |u'|^2) \right] \right\}$$

$$\leq \sup_{y\in\mathbb{R}} \left(\int_{y-r}^{y+r} |u|^q \right)^{\frac{(1-\lambda)s}{q}} \left[\int_{\mathbb{R}} (|u|^2 + |u'|^2 \right].$$

Applying this inequality to each u_n, we see that $u_n \to 0$ in $L^s(\mathbb{R})$ for $2 < s < q + 2$. As $u_n \in L^r(\mathbb{R})$ for each $r > 2$, it follows by interpolation that $u_n \to 0$ in $L^p(\mathbb{R})$ for each $p > 2$.

As kindly communicated to us by Willem, a consequence of Lemma 13is the following weak version of a result of Lieb [42].

Lemma 14 *Let (u_n) be a sequence in $H^1(\mathbb{R})$ satisfying the following conditions.*

(A) (u_n) is bounded in $H^1(\mathbb{R})$.

(B) There exist $\epsilon, \delta > 0$ such that, for each n, meas$\{|u_n(x)| > \epsilon\} > \delta$.

Then there exist a sequence (y_n) in $\mathbb{R}$ and a subsequence (u_{n_k}) of (u_n) such that

$$u_{n_k}(\cdot + y_{n_k}) \rightharpoonup u \neq 0 \quad in \quad H^1(\mathbb{R}).$$

Proof. If, for some $r > 0$ and $q \geq 2$, one has

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} |u_n|^q = 0,$$

then, by Lemma 13, $u_n \to 0$ in $L^p(\mathbb{R})$ for each $p > 2$, so that $u_n \to 0$ in measure on $\mathbb{R}$, a contradiction with assumption (B). Consequently, there exists $\rho > 0$ and a subsequence (n_k) of (n) such that

$$\sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} |u_{n_k}|^q > \rho, \quad k \in \mathbb{N},$$

and, consequently, for each $k \in \mathbb{N}$, there exists $y_{n_k} \in \mathbb{R}$ such that

$$\int_{y_{n_k}-r}^{y_{n_k}+r} |u_{n_k}|^q > \rho.$$

Let $v_{n_k} = u(\cdot + y_{n_k})$. By assumption (A), (v_{n_k}) is bounded in $H^1(\mathbb{R})$ and so, up to a subsequence, weakly converges in $H^1(\mathbb{R})$ to some $u \in H^1(\mathbb{R})$. So it also weakly converges to u in $H^1(-r,r)$ and, by Rellich compact imbedding theorem, strongly converges to u in $L^q(-r,r)$. But

$$\int_{-r}^{r} |v_{n_k}|^q = \int_{y_{n_k}-r}^{y_{n_k}+r} |u_{n_k}|^q > \rho,$$

so that

$$\int_{-r}^{r} |u|^q > \rho,$$

and $u \neq 0$.

We can now prove the existence of a nondecreasing wave.

Theorem 8 *Under assumptions (V_1) and (V_2^+), equation (6.2) has, for each $c > c_0$, a nontrivial nondecreasing solution $y \in E$.*

Proof. By Lemma 2, the exist a bounded sequence (y_n) in E such that $\varphi(y_n) \to d$, $\varphi(y_n') \to 0$, $\operatorname{dist}(y_n, P(E)) \to 0$, (Ay_n) is bounded in $H^1(\mathbb{R})$ and (Ay_n) does not converge to zero in measure on $\mathbb{R}$, where d is the mountain pass value. By Lemma 14, there exist a sequence (x_n) in $\mathbb{R}$ and a subsequence (n_k) of (n) such that

$$Ay_{n_k}(\cdot + x_{n_k}) \rightharpoonup f \neq 0$$

in $H^1(\mathbb{R})$. Furthermore, going if necessary to a subsequence, we can assume that the sequence in E

$$w_k := y_{n_k}(\cdot + x_{n_k}) - y_{n_k}(x_{n_k}) \rightharpoonup w$$

in E. Since

$$Aw_k = Ay_{n_k}(\cdot + x_{n_k}) \rightharpoonup f \neq 0,$$

it follows that $w \neq 0$. Furthermore,

$$\|\varphi'(w_k)\| = \|\varphi'(y_{n_k})\| \to 0 \quad \text{as} \quad k \to \infty.$$

Now, for each $h \in \mathfrak{D}(\mathbb{R})$ such that $h(0) = 0$, we have

$$\langle \varphi'(w_k), h \rangle = \int_{\mathbb{R}} \left[\frac{c^2}{2} w_k' h' - V'(Aw_k) Ah \right] \to 0$$

as $k \to \infty$. As $w_k \rightharpoonup w$ in $H^1(\mathbb{R})$, we have

$$\lim_{k\to\infty} \int_{\mathbb{R}} \frac{c^2}{2} w_k' h' = \int_{\mathbb{R}} \frac{c^2}{2} w' h'.$$

As $Aw_k \rightharpoonup Aw$ in $H^1(\mathbb{R})$, (Aw_k) is bounded in $H^1(\mathbb{R})$ and hence $|Aw_k|_\infty \leq C$ for some $C > 0$ and all $k \in \mathbb{N}$. Also (Aw_k) uniformly converges to Aw on $\mathrm{supp}(Ah)$ and, V' being uniformly continuous on $[-C, C]$, $(V'(Aw_k)Ah)$ converges uniformly to $V'(Aw)Ah$ on $\mathrm{supp}(Ah)$. By Lebesgue dominated convergence theorem, this implies that

$$\int_{\mathbb{R}} V'(Aw_k)Ah \to \int_{\mathbb{R}} V'(Aw)Ah,$$

and hence

$$\langle \varphi'(w), h \rangle = \lim_{k \to \infty} \langle \varphi'(w_k), h \rangle = 0.$$

Thus $\varphi'(w) = 0$, (being nul on a dense subspace) and w is a nontrivial solution of (6.2). Since

$$\mathrm{dist}(w_k, P(E)) = \mathrm{dist}(y_k, P(E)) \to 0$$

when $k \to \infty$, we can find a sequence (h_k) in $P(E)$ such that $h_k \rightharpoonup w$. As $P(E)$ is convex and closed, it is weakly closed and $w \in P(E)$, i.e. is nondecreasing.

Remark 6 *Notice here that we were not able to prove the (PS_d)-condition. From the existence of a bounded sequence (y_n) of almost critical points related to the minimax value d, we have constructed another sequence (w_k) converging to a critical point w, shown to be nontrivial using Lieb's lemma, but at which φ does not need to take the mountain pass value d.*

One proves in an analogous way the dual result, as well as an existence result for nontrivial nonmonotone solitary waves.

Theorem 9 *Under Assumptions (V_1) and (V_2^-), equation (6.2) has, for each $c > c_0$, a nontrivial nonincreasing solution $y \in E$.*

Theorem 10 *Under Assumptions*

(V_1') $V(u) = \lambda \frac{u^2}{2} + W(u)$, $W \in C^1(\mathbb{R}, \mathbb{R})$, $W(0) = 0$, $W'(u) = o(|u|)$, $u \to 0$;

(V_2) $\sup_{\mathbb{R}} W > 0$ *and there exists $\alpha > 2$ such that, for all $u \in \mathbb{R}$, one has*

$$\alpha W(u) \leq u W'(u),$$

equation (6.2) has, for each c such that $c^2 > \max\{0, \lambda\}$ a nontrivial solution $y \in E$.

It is easy to check that the conditions of Theorem 8 are satisfied for the potential

$$V(u) := c_0^2 \frac{u^2}{2} + \frac{u^{2k+1}}{2k+1}$$

with $k \geq 1$ an integer, and that the conditions of Theorem 8 and 9 are satisfied for the potential

$$V(u) := c_0^2 \frac{u^2}{2} + \frac{u^{2k}}{2k}.$$

For $k = 2$, this last potential is the *Ferma-Pasta-Ulam potential.*

Let us consider now the *Toda potential*

$$V(u) := ab^{-1}(e^{-bu} + bu - 1), \quad ab > 0.$$

We have in this case $c_0 = ab/2$,

$$W(u) = ab^{-1} \sum_{k=3}^{\infty} (-b)^k \frac{u^k}{k!},$$

$$W'(u) = ab^{-1} \sum_{k=3}^{\infty} (-b)^k \frac{u^{k-1}}{(k-1)!},$$

and hence

$$uW'(u) = ab^{-1} \sum_{k=3}^{\infty} (-b)^k \frac{u^k}{(k-1)!}.$$

Thus $uW'(u) \geq 3W(u)$ for $u \geq 0$ if $b < 0$ (resp. for $u \leq 0$ if $b > 0$) and Assumption (V_2^+) (resp. (V_2^-)) holds for $b < 0$ (resp. for $b > 0$). Consequently, *the Toda lattice has a nontrivial nondecreasing solitary wave when $b < 0$ and $c > \sqrt{ab}$ and a nontrivial nonincreasing solitary wave when $b > 0$ and $c > \sqrt{ab}$.*

Chapter 7

A general minimax principle and applications

The mountain pass geometry is the simplest minimax geometry. The method of Section 4 can be used to prove a more general result (see e.g. Mawhin [48], Willem [81]).

Theorem 11 *Let E be a Banach space, M_0 a closed subspace of the metric space M and $\Gamma_0 \subset C(M_0, E)$. Assume that $\varphi \in C^1(E, \mathbb{R})$ is such that*

$$a := \sup_{\gamma_0 \in \Gamma_0} \sup_{y \in M_0} \varphi(\gamma_0(y)) < c := \inf_{\gamma \in \Gamma} \sup_{y \in M} \varphi(\gamma(y)) < \infty, \tag{7.1}$$

where

$$\Gamma := \{\gamma \in C(M, E) : \gamma|_{M_0} \in \Gamma_0\}.$$

Then, for each $\varepsilon \in]0, (c-a)/2[$, $\delta > 0$ and $\gamma \in \Gamma$ such that

$$\sup_{y \in M} \varphi(\gamma(y)) \leq c + \varepsilon, \tag{7.2}$$

there exists $y \in E$ such that

a) $c - 2\varepsilon \leq \varphi(y) \leq c + 2\varepsilon$,

b) $dist(y, \gamma(M)) \leq 2\delta$,

c) $\|\varphi'(y)\| \leq 8\varepsilon/\delta$.

Proof. We apply the deformation Lemma 5 with $S := \gamma(M)$. We have by assumption

$$c - 2\varepsilon > a, \tag{7.3}$$

and we define $\beta(y) := \eta(1, \gamma(y))$. For every $y \in M_0$, we deduce from (7.3) that

$$\beta(y) = \eta(1, \gamma_0(y)) = \gamma_0(y),$$

so that $\beta \in \Gamma$. Therefore, assumption (7.2) implies that

$$\sup_{y\in M} \varphi(\beta(y)) = \sup_{y\in M} \varphi(\eta(1,\gamma(y)) \leq c - \epsilon,$$

a contradiction with the definition of c.

Corollary 7 *Under assumption (7.1), and with the notations of Theorem 11, there exists a sequence (y_n) in E such that*

$$\varphi(y_n) \to c, \quad \varphi'(y_n) \to 0.$$

The difficulty in applying those results, besides finding M, M_0 and Γ_0, is to check condition (7.1). It requires the use of some topological argument. In the mountain pass situation of Theorem 3, which corresponds to $M = [0,1]$, $M_0 = \{0,1\}$, $\Gamma_0 = \{\gamma_0\}$ with $\gamma_0(0) = 0,\ \gamma_0(1) = e$, the assumption

$$a := \max\{\varphi(0), \varphi(e)\} < b := \int_{\|y\|=r} \varphi(y)$$

means that the values of φ on the sphere ∂B_r are strictly larger than its values on M_0 and, as $0 \in B_r$ and $e \notin B_r$, a connectivity argument implies that, for each $\gamma \in \Gamma$, $\gamma([0,1])$ intersects ∂B_r so that

$$c := \inf_{\gamma\in\Gamma} \sup_{y\in M} \varphi(\gamma(y)) \geq b := \int_{\|y\|=r} \varphi(y) > a := \max\{\varphi(0), \varphi(e)\}.$$

We now describe rapidly two other minimax geometries introduced by Rabinowitz in 1978 [61, 62].

Theorem 12 (Saddle point geometry). *Assume that $E = X \bigoplus Z$ is a Banach space with X and Z closed subspaces, dim $X < \infty$. For some $\rho > 0$, let*

$$M := \{y \in X : \|y\| \leq \rho\}, \quad M_0 = \{y \in X : \|y\| = \rho\}.$$

Define

$$\Gamma := \{\gamma \in C(M, E) : \gamma|_{M_0} = id\}.$$

Let $\varphi \in C^1(E, \mathbb{R})$ be such that

$$b := \inf_Z \varphi > a := \max_{M_0} \varphi.$$

Then, for each $\varepsilon > 0$, $\delta > 0$ and $\gamma \in \Gamma$ such that

$$\sup_{y\in M} \varphi(\gamma(y)) \leq c + \varepsilon, \tag{7.4}$$

there exists $y \in E$ such that

a) $c - 2\varepsilon \le \varphi(y) \le c + 2\varepsilon$,

b) $dist(y, \gamma(M)) \le 2\delta$,

c) $\|\varphi'(y)\| \le 8\varepsilon/\delta$.

Proof. To apply Theorem 11, we have only to verify that $c \ge b$, and this will be a consequence of the fact that for each $\gamma \in \Gamma$, $\gamma(M)$ intersects Z. If $\gamma(M) \cap Z = \emptyset$, and if $P : E \to E$ is the projector onto X with kernel Z, then the map ψ defined by $\psi(y) = \rho P\gamma(y)/\|\rho P\gamma(y)\|$ is a retraction from the ball M onto its boundary M_0, which is impossible as X has finite dimension. This intersection result implies that

$$c := \inf_{\gamma \in \Gamma} \sup_{y \in M} \varphi(\gamma(y)) \ge b := \inf_{Z} \varphi.$$

Theorem 13 (Linking geometry). *Assume that* $E = X \bigoplus Z$ *is a Banach space with* X *and* Z *closed subspaces, dim* $X < \infty$. *For some* $\rho > r > 0$, *and* $z \in Z$ *with* $\|z\| = r$, *let*

$$M := \{y = x + \lambda z : x \in X,\ \|x\| \le \rho,\ \lambda \ge 0\},$$

$$M_0 := \{y = x + \lambda z : x \in X,\ \|x\| = \rho \text{ and } \lambda \ge 0 \quad \text{or} \quad \|x\| \le \rho \text{ and } \lambda = 0\},$$

$$N := \{y \in Z : \|y\| = r\}.$$

Define

$$\Gamma := \{\gamma \in C(M, E) : \gamma|_{M_0} = id\}.$$

Let $\varphi \in C^1(E, \mathbb{R})$ *be such that*

$$b := \inf_N \varphi > a := \max_{M_0} \varphi.$$

Then, for each $\varepsilon > 0$, $\delta > 0$ *and* $\gamma \in \Gamma$ *such that*

$$\sup_{y \in M} \varphi(\gamma(y)) \le c + \varepsilon, \tag{7.5}$$

there exists $y \in E$ *such that*

a) $c - 2\varepsilon \le \varphi(y) \le c + 2\varepsilon$,

b) $dist(y, \gamma(M)) \le 2\delta$,

c) $\|\varphi'(y)\| \le 8\varepsilon/\delta$.

Proof. Again, we only have to check that $c \geq b$ and this will be, as above, a consequence of the fact that, for every $\gamma \in \Gamma$, $\gamma(M) \cap N \neq \emptyset$. If this intersection is empty, it is easy to show that the map ψ defined by

$$\psi(y) = R(P\gamma(y) + \|(I - P)\gamma(y)\| r^{-1} z)$$

is a retraction from M to M_0, where $P : E \to E$ is the projector onto X with kernel Z and R is a retraction from $X \bigoplus \mathbb{R}z \setminus \{z\}$ onto M_0. Again this is impossible, as M is homeomorphic to a finite-dimensional ball.

Remark 7 *One should notice the analogy in the topological situations occuring in the mountain pass, saddle point and linking geometries. In each case one has one "manifold" A and another one B which is a boundary, and the functional φ is assumed to take values on A strictly greater than its values on B. One must prove a* linking *between A and B, namely that some class of "manifolds" C having B as a boundary intersects A. In the mountain pass geometry, A is the sphere ∂B_r in X centered at 0 and $B = \{0, e\}$ with $\|e\| < r$, (the "manifolds" C will be curves in E joining 0 to e), in the saddle point geometry, A is the subspace Z, B is the sphere ∂B_ρ in X centered at 0, the C are the images of mappings from B_r into E equal to identity on ∂B_r, and in the linking geometry, A is the sphere ∂B_r in Z centered at 0, B is topologically the boundary of half a ball in X, and the C are the images of mappings from the corresponding half-ball in X into E equal to identity on B.*

To obtain from those theorems the existence of critical points, it is necessary to be able to deduce it from the sequence of almost critical points. If the problem has not enough compactness to have a Palais-Smale-type property, one has to use refined arguments like in Section 6 or heavier techniques like the *concentration-compactness* method of P.L. Lions [44, 45]. Various interesting problems for semi-linear elliptic partial differential equations are treated in this way in [81].

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