Critical Exponents and Dimensions for Elliptic Equations

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1 Introduction

In these notes we outline some variational approaches to nonexistence results for elliptic equations. The first lecture will focus on the $k$-Hessian equation

$$\begin{cases}
S_k(D^2u) = f(x, u), & x \in \Omega, \\
u = 0, & x \in \partial\Omega,
\end{cases}$$

(1)

while the second lecture will concern equations of the form

$$\begin{cases}
r^{-\gamma}(r^\alpha |u'|^\beta u')' = f(r, u), & r \in (0, R), \\
u > 0, & r \in (0, R), \\
u'(0) = u(R) = 0,
\end{cases}$$

(2)

for certain values of $\alpha$, $\beta$, and $\gamma$. Equation (2) represents the radial form of positive solutions to a large class of nonlinear PDE’s, including (1) and the quasilinear elliptic equation

$$\begin{cases}
\Delta_p u + f(x, u) = 0, & x \in \Omega, \\
u = 0, & x \in \partial\Omega,
\end{cases}$$

(3)

where $\Delta_p = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplace operator. For reference we note the values of $\alpha$, $\beta$ and $\gamma$ for these examples:

<table>
<thead>
<tr>
<th>Operator</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laplacian</td>
<td>$n - 1$</td>
<td>0</td>
<td>$n - 1$</td>
</tr>
<tr>
<td>$p$-Laplacian ($p &gt; 1$)</td>
<td>$n - 1$</td>
<td>$p - 2$</td>
<td>$n - 1$</td>
</tr>
<tr>
<td>$k$-Hessian</td>
<td>$n - k$</td>
<td>$k - 1$</td>
<td>$n - 1$</td>
</tr>
</tbody>
</table>

We will always assume $\Omega$ is a “nice” bounded domain in $\mathbb{R}^n$. What “nice” will mean will depend on the equation at hand.
2 Critical Exponents


\[
\begin{cases}
\Delta u + f(u) = 0, & x \in \Omega, \\
u = 0, & x \in \partial\Omega,
\end{cases}
\]

must satisfy the identity

\[
-\frac{1}{2} \int_{\partial\Omega} |Du|^2 (x \cdot \nu) \, ds = \int_{\Omega} \left[ \frac{n - 2}{2} uf(u) - nF(u) \right] \, dx,
\]

where \( F(u) = \int_{0}^{u} f(s) \, ds \). If \( \Omega \) is a star-shaped with respect to the origin\(^1\), then the left-hand side of (5) is nonpositive. Therefore, if

\[
(n - 2)uf(u) - 2nF(u) > 0, \quad \text{for } u \neq 0,
\]

then (4) has no nontrivial solutions. For example, if \( f(u) = |u|^{p-1}u \), then (6) becomes

\[
(n - 2)|u|^{p+1} - \frac{2n}{p + 1} |u|^{p+1} > 0
\]

and it follows that the semilinear elliptic equation

\[
\begin{cases}
\Delta u + |u|^{p-1}u = 0, & x \in \Omega, \\
u = 0, & x \in \partial\Omega,
\end{cases}
\]

has no nontrivial solutions when

\[
p > \frac{n + 2}{n - 2}.
\]

On the other hand, for \( p < (n + 2)/(n - 2) \) one may use the Mountain Pass Theorem or constrained minimization (e.g., minimize the Dirichlet integral over the unit sphere in \( L^{p+1} \)) to obtain a nontrivial solution to (7). For this reason, the exponent \( (n + 2)/(n - 2) \) is called the critical exponent for the Laplace operator. It corresponds to the loss of compactness of the continuous Sobolev embedding \( H^1_0(\Omega) \subset L^q(\Omega) \) which is compact only for \( 1 \leq q < 2^* = (2n)/(n - 2) \), the Sobolev exponent Note that \( (n + 2)/(n - 2) = 2^* - 1 \).

In 1985 P. Pucci and J. Serrin [7] extended Pohozaev’s identity (5) to a larger class of variational equations. Let \( L = L(p, z, x) \) denote a Lagrangian which is \( C^2 \) on the domain \( \mathbb{R}^n \times \mathbb{R} \times \overline{\Omega} \). Smooth critical points of the associated “energy” functional satisfy the Euler-Lagrange equation

\[
-\sum_{i=1}^{n} (L_{p_i}(Du, u, x))_{x_i} + L_z(Du, u, x) = 0, \quad \text{in } \Omega.
\]

We assume without loss of generality that \( L(0, 0, x) = 0 \) in \( \Omega \). The main identity of Pucci-Serrin is due to the following proposition

\(^1\Omega \) is star-shaped if there exists \( x_0 \in \Omega \) such that \( (x - x_0) \cdot \nu \geq 0 \) for all \( x \in \partial\Omega \).
Proposition 2.1 (Pucci-Serrin [7]). Let \( u \in C^2(\Omega) \) be a solution of the Euler-Lagrange (8), and let \( a \) and \( \vec{h} \) be, respectively, scalar and vector valued functions of class \( C^1(\Omega) \). Then the following relation holds in \( \Omega \):

\[
\frac{\partial}{\partial x_i} \left[ \vec{h}_i L(Du, u, x) - \vec{h}_j \frac{\partial u}{\partial x_j} L_{p_i}(Du, u, x) - au L_{p_i}(Du, u, x) \right] = \frac{\partial \vec{h}_i}{\partial x_i} L(Du, u, x) + \vec{h}_i L_{x_i}(Du, u, x) - \left( \frac{\partial u}{\partial x_j} \frac{\partial \vec{h}_i}{\partial x_i} + u \frac{\partial a}{\partial x_i} \right) L_{p_i}(Du, u, x) - a \left( \frac{\partial u}{\partial x_i} L_{p_i}(Du, u, x) + u L_z(Du, u, x) \right),
\]

where repeated indices \( i \) and \( j \) are to be summed from 1 to \( n \).

The proof is obtained by direct computation, using (8). If \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) solves (8) with \( u = 0 \) on \( \partial \Omega \) then \( u_{x_i} = (\partial u/\partial \nu) \nu_i \) on \( \partial \Omega \) so

\[
\vec{h}_j \frac{\partial u}{\partial x_j} L_{p_i}(Du, u, x) \nu_i = \frac{\partial u}{\partial x_i} L_{p_i}(Du, u, x) \vec{h}_j \nu_j \quad \text{on} \ \partial \Omega.
\]

Integrating (9) over \( \Omega \), applying (10), \( u = 0 \) on \( \partial \Omega \), and the divergence theorem one obtains the fundamental identity

\[
\int_{\partial \Omega} \left[ L(Du, 0, x) - \frac{\partial u}{\partial x_i} L_{p_i}(Du, 0, x) \right] (\vec{h} \cdot \nu) \, ds = \int_{\Omega} L(Du, u, x) \text{div} \vec{h} + \vec{h}_i L_{x_i}(Du, u, x) - \left( \frac{\partial u}{\partial x_j} \frac{\partial \vec{h}_i}{\partial x_i} + u \frac{\partial a}{\partial x_i} \right) L_{p_i}(Du, u, x) \quad (11)
\]

\[
- a \left( \frac{\partial u}{\partial x_i} L_{p_i}(Du, u, x) + u L_z(Du, u, x) \right) \, dx.
\]

For example, if \( L(p, z) = \frac{1}{2} |p|^2 - F(z), \vec{h} = x \), and \( a \) is constant, then (11) reduces to

\[
- \int_{\partial \Omega} \frac{1}{2} |Du|^2 (x \cdot \nu) \, ds = \int_{\Omega} \left[ \frac{n}{2} - 1 - a \right] |Du|^2 - nF(u) + au f(u) \, dx. \tag{12}
\]

The choice of \( a(x) = (n - 2)/2 \) makes the \( |Du|^2 \) vanish and reduces (12) to the Pohozaev identity (5). However, identity (11) is applicable to a much larger class of equations. For instance, for the quasilinear equation (3) with associated Lagrangian \( L(Du, u) = \frac{1}{p} |Du|^p - F(u) \), the choice of \( \vec{h} = x \) and constant \( a \) yields

\[
- \int_{\partial \Omega} \frac{1}{p} |Du|^p (x \cdot \nu) \, ds = \int_{\Omega} \left[ \frac{n}{p} - 1 - a \right] |Du|^p - nF(u) + au f(u) \, dx. \tag{13}
\]

Now we see the choice of \( a = (n - p)/p \) implies

\[
- \int_{\partial \Omega} \frac{1}{p} |Du|^p (x \cdot \nu) \, ds = \int_{\Omega} \left[ \frac{n - p}{p} \right] uf(u) - nF(u) \, dx, \tag{14}
\]
from which an appropriate nonexistence result can be stated. To determine the critical exponent we choose $f(u) = |u|^{q-1}u$ and find (3) has no nontrivial solutions when $p < n$ and

$$q > \frac{np}{n-p} - 1 = \frac{(p-1)n + p}{n-p}.$$ 

Note that $p^* = np/(n-p)$ is the Sobolev exponent, corresponding to the loss of compactness for the continuous embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$. Many further applications of (11) may be found in [7].

We seek to apply this idea to the $k$-Hessian equation (1). Equation (1) is of variational form, with solutions corresponding to critical points of the functional

$$I_k[u] = -\frac{1}{k+1} \int_\Omega u S_k(D^2u) \, dx + \int_\Omega F(x,u) \, dx,$$  

(15)

where $F(x,u) = \int_0^u f(x,s) \, ds$ (see §4.1). However, Proposition 9 does not directly apply to (15) since the Lagrangian contains higher order terms, and one needs to derive an appropriate higher order analog of (9).

The Euler-Lagrange equation associated with the Lagrangian $L = L(D^2u, Du, u, x) = L(r_{ij}, p_i, z, x)$, where $r_{ij} = r_{ji}$ is

$$\sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} L_{r_{ij}}(D^2u, Du, u, x) - \sum_{i=1}^n (L_{p_i}(D^2u, Du, u, x))_{x_i} + L_z(D^2u, Du, u, x) = 0.$$  

(16)

In our case $L$ is independent of $p$ and the fundamental identity (simplified for our purposes) takes the form (see equation (29) in [7])

**Proposition 2.2 (Pucci-Serrin [7]).** Let $u \in C^4(\Omega)$ be a solution to the Euler-Lagrange equation (16) with $L_{p_i} = 0$ and $a \in C^2(\Omega)$ a scalar function. Then

$$\frac{\partial}{\partial x_i} \left[ x_i L + \left( x_l \frac{\partial u}{\partial x_l} + au \right) \frac{\partial L_{r_{ij}}}{\partial x_j} - \frac{\partial}{\partial x_j} \left( x_l \frac{\partial u}{\partial x_l} + au \right) L_{r_{ij}} \right]$$

$$= nL + x_i L_{x_i} - auL_z - (a + 2) \frac{\partial^2 u}{\partial x_i \partial x_j} L_{r_{ij}}.$$  

(17)

Following Tso [10], we employ this identity to determine the critical exponent associated to the operator $S_k$. For simplicity we assume $F = F(z)$ (e.g., $f(u) = |u|^p$).

**Theorem 2.3 (Tso [10]).** Let $\Omega$ be a smooth domain which is star-shaped with respect to the origin. Assume $f : (-\infty,0] \to [0, \infty)$ is smooth, with $f(s) > 0$ for $s < 0$ and $f(0) = 0$. Then

$$\begin{cases} S_k(D^2u) = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$  

(18)
has no nontrivial solutions in $\Phi_0^k(\Omega) \cap C^4(\Omega) \cap C^1(\Omega)$ when
\[ nF(u) - \frac{n - 2k}{k + 1} uf(u) > 0, \quad \text{for } u < 0. \quad (19) \]

Proof. Applying (2.2) to the Lagrangian $L = \frac{-zS_k(r_{ij})}{k+1} + F(z)$ one obtains
\[
\frac{\partial}{\partial x_i} \left[ x_i \left( -\frac{uS_k(D^2u)}{k + 1} + F(u) \right) - \left( x_l \frac{\partial u}{\partial x_l} + au \right) \frac{u x_j S^{ij}(D^2u)}{k + 1} + \frac{\partial}{\partial x_j} \left( x_l \frac{\partial u}{\partial x_l} + au \right) \frac{u S^{ij}(D^2u)}{k + 1} \right] = \left[ k(a + 2) + a - n \right] \frac{uS_k(D^2u)}{k + 1} + nF(uf).
\]
Choosing $a = (n - 2k)/(k + 1)$ and integrating (20) we obtain
\[
- \frac{1}{k + 1} \int_{\partial \Omega} \left[ x_l u_{x_l} u_{x_j} S^{ij}(D^2u) \right] \nu_i \, ds = \int_{\Omega} \left( nF(u) - \frac{n - 2k}{k + 1} uf(u) \right) \, dx, \quad (21)
\]
which simplifies to
\[
- \frac{1}{k + 1} \int_{\partial \Omega} (x \cdot \nu)|Dv|^2 S^{ij}(D^2u) \nu_i \nu_j \, ds = \int_{\Omega} \left( nF(u) - \frac{n - 2k}{k + 1} uf(u) \right) \, dx. \quad (22)
\]
For $u \in \Phi_0^k(\Omega)$ the operator $S_k$ is elliptic, thus $S^{ij}(D^2u) \nu_i \nu_j > 0$. Hence the left-hand side of (22) is nonpositive and the result follows. \qed

Note that when $k = 1$, (19) is equivalent to the Pohozaev criterion (6). If $f(u) = (-u)^{p}$ then (19) reduces to
\[
\frac{n - 2k}{k + 1} > \frac{n}{p + 1}. \quad (23)
\]
If $k \geq n/2$, then (23) can not hold and we obtain no a priori obstructions to solution from this method. On the other hand, when $k < n/2$, then (23) is true when $p \geq \frac{(n+2)k}{n-2k}$. Thus when $k < n/2$ the critical exponent $\gamma(k)$ for $S_k$ is defined by
\[
\gamma(k) = \frac{(n + 2)k}{n - 2k}. \quad (24)
\]

Tso also provides complementary existence results for radially symmetric solutions for subcritical exponents (and for all exponents when $k \geq n/2$), thus we can extend $\gamma(k)$ to all $k$ via
\[
\gamma(k) = \begin{cases} 
\infty & k > n/2 \\
\frac{(n+2)k}{n-2k} & k < n/2.
\end{cases} \quad (25)
\]
In particular, there is no critical exponent for the Monge-Ampère operator. Heuristically, operators “closer” to the Laplace operator have critical exponents, while operators “closer” to Monge-Ampère do not. Note that when $p = k$ one has an eigenvalue problem (see e.g., [3, 11, 2]).
3 Critical Dimension

In 1983 Brezis and Nirenberg observed that lower order perturbations to elliptic equations involving critical exponents recovered the lost compactness. More precisely, they proved the equation

\[
\begin{aligned}
\Delta u + u^{\frac{n+2}{n-2}} + \lambda u &= 0, & x \in \Omega, \\
u &= 0, & x \in \partial \Omega,
\end{aligned}
\]

has a positive solution if \(0 < \lambda < \lambda_1\) and \(n \geq 4\), where \(\lambda_1\) is the principal eigenvalue for \(-\Delta\) on \(H_0^1(\Omega)\). Surprisingly, for the case \(n = 3\) they observed that there exists \(\lambda^* > 0\) such that (26) has a solution for \(\lambda \in (\lambda^*, \lambda_1)\) and no solution for \(\lambda \in (0, \lambda^*)\). If \(\Omega\) is a ball, then \(\lambda^* = \lambda_1/4\). In this context the dimension \(n = 3\) is called a critical dimension.

From Section 1 we know that both \(\Delta_p\) and \(S_k\) have critical exponents (when \(p < n\) and \(k < n/2\), respectively). Thus it is natural to ask if results similar to the Brezis-Nirenberg result exist for these operators. Several authors have answered this question affirmatively. Rather that treat \(\Delta_p\) and \(S_k\) separately, we adopt the approach of Clément-DeFigueiredo-Mitidieri [1] and consider the equation

\[
\begin{cases}
(r^\alpha |u'|^\beta u')' = r^\gamma |u|^{q-2}u, & r \in (0, R), \\
u > 0, & r \in (0, R), \\
u'(0) = u(R) = 0,
\end{cases}
\]

and the perturbed form

\[
\begin{cases}
(r^\alpha |u'|^\beta u')' = r^\gamma |u|^{q-2}u + \lambda r^\delta |u|^{\beta}u, & r \in (0, R), \\
u > 0, & r \in (0, R), \\
u'(0) = u(R) = 0,
\end{cases}
\]

for various values of exponents \(\alpha, \beta, \delta\) and \(\gamma\). See the table on page 1 for the relevant values of constants for (1) or (3).

The critical exponent associated with (27) is

\[
q^* = \frac{(\gamma + 1)(\beta + 2)}{\alpha - \beta - 1}.
\]

For the \(p\)-Laplacian, \(q^* = \frac{np}{n+p}\) and for \(S_k\), \(q^* = \frac{n(k+1)}{n-2k}\), agreeing with our previous observations in Section 1.\(^2\)

Throughout this section we will assume the following inequalities hold

\[
\begin{align*}
q - 1 &> \beta + 1 > 0, \quad \gamma + 1 > \alpha - \beta - 1, \quad \text{and} \quad \delta + 1 \geq \alpha - \beta - 1 \\
\alpha - \beta - 1 &> 0 \\
\gamma, \delta &> \alpha - 1 \\
\alpha - \beta - 2 &< \delta,
\end{align*}
\]

\(^2\)Note that the exponent in (27) is \(q - 2\). This notational convenience has the advantage that the “critical exponent” agrees with the Sobolev exponent.
When applied to $S_k$ (resp. $\Delta_p$) these inequalities simply imply $q > k + 1$ (resp. $q > p$) and $k < n/2$ (resp. $n < p$), the realm of critical exponents.

The goal of this section will be to prove the following two nonexistence results:

**Theorem 3.1 ([1]).** Assume (30), (31), (32), (33) hold. If $\lambda \leq 0$ and $q = q^*$, then (28) has no solution.

**Theorem 3.2.** Assume (30), (31), (32), (33), $\beta \geq 0$ and $q = q^*$. If
\[ (\beta + 1)(\delta + 1) - (\alpha - \beta - 1)(\beta + 2) > 0, \]
then there exists $\lambda^* > 0$ such that (28) has no solution for $\lambda \in (0, \lambda^*)$.

For the model operators $S_k$ and $\Delta_p$, their parameters satisfying (34) correspond to certain values of the dimension $n$, called critical dimensions by Pucci and Serrin [8]. For the $p$-Laplace operator, (34) corresponds to $n < p^2$, thus the critical dimensions for $\Delta_p$ are those $n$ with $p < n < p^2$. Note that for the Laplacian $p = 2$ and we obtain $2 < n < 4$, thus the only critical dimension is $n = 3$, as observed by Brezis and Nirenberg. For the $k$-Hessian the critical dimensions are those $n$ with $2k < n < 2k(k + 1)$.

The proofs are based on the following identity of Pohozaev-Pucci-Serrin type:

**Proposition 3.3 ([1]).** Let $a, b \in C^1[0, \infty)$. If $u \in C^2(0, \infty) \cap C^1[0, \infty)$ solves
\[ -(r^\alpha |u'|^\beta u')' = f(r, u) \quad \text{in } (0, \infty), \]
then for $R > 0$ we have
\[ \left[ -r^\alpha u'|u'|^\beta \left( au + \frac{\beta + 1}{\beta + 2} bu' \right) \right]_{r=R} + \int_0^R r^\alpha a' u'u'|u'|^\beta \]
\[ + \int_0^R r^\alpha \left( a + \frac{\beta + 1}{\beta + 2} b' - \frac{\alpha}{\beta + 2} \frac{b'}{r} \right) |u'|^{\beta + 2} \]
\[ = [bF(r, u)]_{r=R} + \int_0^R a f(r, u) - b F_r(r, u) - b' F(r, u). \]

**Proof.** The proof is a nice application of the “abc-method”.

Now we prove Theorem 3.1:

**Proof.** Without loss of generality assume $R = 1$ and let $u$ solve (28). Using (36) with $b(r) = r$, a constant, and
\[ f(r, u) = r^\gamma |u|^{q-2} u + \lambda r^\delta |u|^\beta u \]
\[ \lambda^* > 0 \text{ such that (28) has no solution for } \lambda \in (0, \lambda^*). \]

\[ \text{Proof.} \text{ The proof is a nice application of the “abc-method”.} \]

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\[ f(r, u) = r^\gamma |u|^{q-2} u + \lambda r^\delta |u|^\beta u \]
\[ \lambda^* > 0 \text{ such that (28) has no solution for } \lambda \in (0, \lambda^*). \]
we obtain
\[
\left[ -\frac{\beta + 1}{\beta + 2} |u'|^{\beta + 2} \right]_{r=1} + \int_0^1 r^\alpha \left( a + \frac{\beta + 1 - \alpha}{\beta + 2} \right) |u'|^{\beta + 2} = \int_0^1 r^\gamma \left[ a - \frac{\delta + 1}{\beta + 2} \right] \lambda |u|^{\beta + 2} + \int_0^1 r^\gamma \left[ a - \frac{\gamma + 1}{q} \right] |u|^q.
\]
(38)

If we choose
\[
a = \frac{\alpha - \beta - 1}{\beta + 2},
\]
then the integral on the left-hand side of (38) vanishes. Since \( q = q^* \), the same is true for the last integral in (38). Moreover, the coefficient in the first integrand on the right-hand side of (38) becomes
\[
a - \frac{\delta + 1}{\beta + 2} = \frac{\alpha - \beta - \delta - 2}{\beta + 2}.
\]
Since \( \lambda \leq 0 \), the right-hand side of (38) is nonnegative. On the other hand, the left-hand side of (38) is negative. Note that from existence and uniqueness of the initial value problem we must have \( u'(1) \neq 0 \).

Finally, we prove the “critical dimension” Theorem 3.2:

**Proof.** We again apply (36) with \( R = 1 \) and \( f \) as in (37), now with
\[
a = a_1 + a_2 r^m \quad b = -r + r^{m+1},
\]
where \( a_1, a_2, m \) are constants to be determined. Since \( b(1) = 0 \) and \( u(1) = 0 \), all the boundary terms vanish. We choose \( a_1 \) and \( a_2 \) so that the integrals containing \( |u'|^{\beta + 2} \) vanish, i.e.,
\[
a_1 = -\frac{\alpha - \beta - 1}{\beta + 2} \quad a_2 = \frac{\alpha - (m + 1)(\beta + 1)}{\beta + 2}.
\]
With the free parameter \( m \) left we have:
\[
I_5 = \int_0^1 r^\alpha a' u'u' |u'|^{\beta} = I_1 + I_2 + I_3 + I_4,
\]
(39)
where
\[
I_1 = \lambda \int_0^1 \left[ a_1 + \frac{\delta + 1}{\beta + 2} \right] r^\delta |u|^{\beta + 2}
\]
(40)
\[
I_2 = \lambda \int_0^1 \left[ a_2 - \frac{\delta + m + 1}{\beta + 2} \right] r^\delta + m |u|^{\beta + 2}
\]
(41)
\[
I_3 = \int_0^1 a_1 \left[ \frac{\gamma + 1}{q} \right] r^\gamma |u|^q
\]
(42)
\[
I_4 = \int_0^1 a_2 \left[ \frac{\gamma + m + 1}{q} \right] r^\gamma + m |u|^q
\]
(43)
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From (33) it follows that $I_1 > 0$. Since $q = q^*$, $I_3 = 0$.

Let us examine $I_5$. From (28) we observe

$$-r^\alpha u'(r)|u'(r)|^\beta = \int_0^r \lambda r^\delta u|u|^\beta + r^\gamma u|u|^{q-2}dr > 0,$$

for positive solutions of (28). We conclude $u'(r) < 0$ for all $r \in (0, 1]$. If $a_2 < 0$ (the choice of $m$ will imply this!), then

$$I_5 = \int_0^1 r^{\alpha} a' uu'|u'|^\beta = m|a_2| \int_0^1 r^{\alpha+m-1}u|u'|^{\beta+1} = C \int_0^1 r^{\alpha+m-1} \left| \left( \frac{\beta+2}{\beta+1} \right)^{\beta+1} \right|, $$

where $C = C(m, |a_2|, \beta) > 0$. It follows from an embedding theorem (see §5) that

$$\int_0^1 r^{\alpha+m-1} \left| \left( \frac{\beta+2}{\beta+1} \right)^{\beta+1} \right| \geq c \int_0^1 r^\delta \left( \frac{\beta+2}{\beta+1} \right)^{\beta+1} = c \int_0^1 r^{\delta} u^{\beta+2},$$

(44)

provided $m \leq \delta - \alpha + \beta + 2$. We then choose

$$m = \delta - \alpha + \beta + 2,$$

which is positive in view of the hypothesis of the theorem. From (44) it follows that

$$I_5 \geq \tilde{c}I_1,$$

for some $\tilde{c} > 0$. If our choice of $m$ renders $a_2 < 0$, then $I_2 < 0$ and $I_4 < 0$ and a sign analysis of (39) implies there must exist a $\lambda^* > 0$ such that there is no solution for $\lambda \leq \lambda^*$. Thus to complete the proof we need to show $a_2 < 0$, i.e.,

$$\alpha - (\delta - \alpha + \beta + 3)(\beta + 1) < 0.$$

But this is equivalent to our hypothesis (34) and the proof is complete. \(\square\)
4 Appendix

4.1 Variational form for \( S_k \)

Recall \( S_k(D^2u) \) is defined in terms of the elementary symmetric polynomials acting on the eigenvalues of \( D^2u \). In the two extreme cases \( k = 1 \) and \( k = n \), the fact that \( \sum \lambda_i \) and \( \Pi \lambda_i \) are, respectively, the trace and determinant of the matrix allows us to see immediately the partial differential operator defined by \( S_k \), i.e., \( S_1(D^2u) = \Delta u \) and \( S_n(D^2u) = \det D^2u \). In general, for a symmetric matrix \( r \), \( S_k(r) \) is the sum of all principal \( k \times k \) minors of \( r \), i.e.,

\[
S_k(r) = \frac{1}{k!} \sum \delta(i_1, \ldots, i_k, j_1, \ldots, j_k) r_{i_1j_1} \cdots r_{i_kj_k}, \tag{45}
\]

where \( \delta(i_1, \ldots, i_k) \) is 1 (resp. \(-1\)) if \( (i_1, \ldots, i_k) \) are distinct and \( (j_1, \ldots, j_k) \) is an even (resp. odd) permutation of \( (i_1, \ldots, i_k) \), otherwise it is 0. From this we can determine \( \frac{\partial S_k}{\partial r_{ij}} = S^{ij}(r) : \)

\[
S^{ij}(r) = \frac{1}{(k-1)!} \sum \delta(i_1, \ldots, i_{k-1}, i, j_1, \ldots, j_{k-1}, j) r_{i_1j_1} \cdots r_{i_{k-1}j_{k-1}}. \tag{46}
\]

In particular, this implies

\[
S_k(r) = \frac{1}{k} \sum_{i,j=1}^n r_{ij} S^{ij}(r), \tag{47}
\]

i.e.,

\[
S_k(D^2u) = \frac{1}{k} \sum_{i,j=1}^n u_{x_ix_j} S^{ij}(D^2u). \tag{48}
\]

A computation (see [9, 10]) shows

\[
\sum_{j=1}^n \frac{\partial}{\partial x_j} S^{ij}(D^2u) = 0 \quad \text{for each } i. \tag{49}
\]

Together with (48) this implies \( S_k \) has a divergence form:

\[
S_k(D^2u) = \frac{1}{k} \sum_{i,j=1}^n \frac{\partial}{\partial x_j} (u_{x_i} S^{ij}(D^2u)). \tag{50}
\]

The Euler-Lagrange equation associated with the Lagrangian \( L = \frac{-zS_k(r_{ij})}{k+1} + F(x, z) \) is

\[
-\frac{1}{k+1} \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (u S^{ij}(D^2u)) - \frac{S_k(D^2u)}{k+1} + f(x, u) = 0. \tag{51}
\]
From (49) we may rewrite the first term as

$$-\frac{1}{k+1} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(u_{x_i} S^{ij}(D^2u))$$

from which it follows (51) is equivalent to

$$-\frac{1}{k+1} k S_k(D^2u) - \frac{1}{k+1} S_k(D^2u) + f(x,u) = 0,$$

or

$$-S_k(D^2u) + f(x,u) = 0,$$

which is precisely (1).

### 4.2 Radial form of $S_k$

If $u : \Omega \to \mathbb{R}$ is radially symmetric then a calculation show

$$\frac{\partial u}{\partial x_i} = u'(r) \frac{x_i}{r} \quad \text{and} \quad \frac{\partial^2 u}{\partial x_i^2} = u''(r) \frac{x_i x_j}{r^2} + u'(r) \left[ \frac{r^2 \delta_{ij} - x_i x_j}{r^3} \right],$$

for $i,j = 1, \ldots, n$. At the point $x = (r,0,\ldots,0)$ the Hessian matrix $D^2u$ is diagonal with $u_{11} = u''(r)$ and $u_{ii} = u'(r)/r$, for $i > 1$. Since the operator $S_k$ is invariant with respect to rotations, it follows that

$$S_k(D^2u) = u''(n-1) \left( \frac{u'}{r} \right)^{k-1} + \frac{n-1}{k} \left( \frac{u'}{r} \right)^k = \frac{1}{k} \binom{n-1}{k-1} r^{1-n} \left( r^{-k} (u')^k \right),$$

where $\binom{n}{k}$ is the binomial coefficient.

### 5 An embedding theorem

We quote an embedding theorem needed in §3:

**Proposition 5.1 ([5]).** Let $u : (0,R] \to \mathbb{R}$ be absolutely continuous. If $u(R) = 0$ and

(i) for $1 \leq \beta + 2 \leq q < \infty$ one has

(a) $\alpha > \beta + 1, \gamma \geq \alpha \frac{q}{\beta+2} - q \frac{\beta+1}{\beta+2} - 1$, or

(b) $\alpha \leq \beta + 1, \gamma > -1$,

(i) for $1 \leq q < \beta + 2 < \infty$ one has
(c) $\alpha > \beta + 1, \gamma > \alpha \frac{q}{\beta + 2} - q \frac{\beta + 1}{\beta + 2} - 1$, or
(d) $\alpha \leq \beta + 1, \gamma > -1$,

then
\[
\left( \int_0^R x^\gamma |u(x)|^q \, dx \right)^{1/q} \leq c \left( \int_0^R x^\alpha |u'(x)|^{\beta + 2} \, dx \right)^{1/(\beta + 2)}.
\]

This proposition corresponds to a continuous embedding $X_R \subset L^q_\gamma(0, R)$, where $L^q_\gamma(0, R)$ is the Banach space of measurable functions $u : [0, R] \to \mathbb{R}$ with finite weighted norm:
\[
\|u\|_{L^q_\gamma} = \left( \int_0^R x^\gamma |u(x)|^q \, dx \right)^{1/q},
\]
and $X_R$ is defined as follows. For $0 < R < \infty, \alpha > 0$, and $\beta > -1$ let $\tilde{X}_R$ denote the set of real valued $L^1_{\text{loc}}$ functions defined on $(0, R)$ with distributional derivatives in $L^1_{\text{loc}}$ such that
\[
\int_0^R x^\alpha |u(x)|^{\beta + 2} \, dx < \infty \quad \text{and} \quad \int_0^R x^\alpha |u'(x)|^{\beta + 2} \, dx < \infty.
\]
Then $\tilde{X}_R$ is a Banach space with norm $\|\cdot\|_{\tilde{X}_R}$ defined by
\[
\|u\|_{\tilde{X}_R}^{\beta + 2} = \int_0^R x^\alpha |u|^\beta \, dx + \int_0^R x^\alpha |u'|^{\beta + 2} \, dx.
\]

It follows that $u \in \tilde{X}_R$ is absolutely continuous in $(0, R]$ and, thus, we can consider the subspace $X_R$ of those $u \in \tilde{X}_R$ such that $u(R) = 0$. By Proposition 5.1 above it follows that for $u \in X_R$:
\[
\int_0^R x^\alpha |u|^\beta \, dx \leq C \int_0^R x^\alpha |u'|^{\beta + 2} \, dx.
\]

Thus $\|\cdot\|_{\tilde{X}_R}$ and $\|\cdot\|_{X_R}$ are equivalent norms on $X_R$. For different values of $\alpha$ and $\beta$, the spaces $X_R$ are “weighted Sobolev spaces” [5]. In this way we can understand the critical exponent results above in terms of loss of compactness of the embedding of the weighted Sobolev space $X_R$ into the weighted $L^q_\gamma$ space.

References


