Critical Exponents and Dimensions for Elliptic Equations

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1 Introduction

In these notes we outline some variational approaches to nonexistence results for elliptic equations. The first lecture will focus on the k-Hessian equation

$$\begin{cases} S_k(D^2u) = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

while the second lecture will concern equations of the form

$$\begin{cases} r^{-\gamma}(r^{\alpha}|u'|^{\beta}u')' = f(r,u), & r \in (0,R), \\ u > 0, & r \in (0,R), \\ u'(0) = u(R) = 0, \end{cases} \tag{2}$$

for certain values of α, β, and γ. Equation (2) represents the radial form of positive solutions to a large class of nonlinear PDE's, including (1) and the quasilinear elliptic equation

$$\begin{cases} \Delta_p u + f(x,u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

where $\Delta_p = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator. For reference we note the values of α, β and γ for these examples:

Operator	α	β	γ
Laplacian	$n-1$	0	$n-1$
p-Laplacian ($p > 1$)	$n-1$	$p-2$	$n-1$
k-Hessian	$n-k$	$k-1$	$n-1$

We will always assume Ω is a "nice" bounded domain in $\mathbb{R}^n$. What "nice" will mean will depend on the equation at hand.

2 Critical Exponents

In 1965 Pohozaev [6] discovered that solutions of the Dirichlet problem

$$\begin{cases} \Delta u + f(u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

must satisfy the identity

$$-\frac{1}{2}\int_{\partial\Omega} |Du|^2 (x \cdot \nu)\, ds = \int_{\Omega} \left[\frac{n-2}{2} u f(u) - nF(u)\right] dx, \tag{5}$$

where $F(u) = \int_0^u f(s)\, ds$. If Ω is a star-shaped with respect to the origin[1], then the left-hand side of (5) is nonpositive. Therefore, if

$$(n-2) u f(u) - 2nF(u) > 0, \qquad \text{for } u \neq 0, \tag{6}$$

then (4) has no nontrivial solutions. For example, if $f(u) = |u|^{p-1}u$, then (6) becomes

$$(n-2)|u|^{p+1} - \frac{2n}{p+1}|u|^{p+1} > 0$$

and it follows that the semilinear elliptic equation

$$\begin{cases} \Delta u + |u|^{p-1}u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{7}$$

has no nontrivial solutions when

$$p > \frac{n+2}{n-2}.$$

On the other hand, for $p < (n+2)/(n-2)$ one may use the Mountain Pass Theorem or constrained minimization (e.g., minimize the Dirichlet integral over the unit sphere in L^{p+1}) to obtain a nontrivial solution to (7). For this reason, the exponent $(n+2)/(n-2)$ is called the *critical exponent* for the Laplace operator. It corresponds to the loss of compactness of the continuous Sobolev embedding $H_0^1(\Omega) \subset L^q(\Omega)$ which is compact only for $1 \leq q < 2^* = (2n)/(n-2)$, the Sobolev exponent Note that $(n+2)/(n-2) = 2^* - 1$.

In 1985 P. Pucci and J. Serrin [7] extended Pohozaev's identity (5) to a larger class of variational equations. Let $L = L(p, z, x)$ denote a Lagrangian which is C^2 on the domain $\mathbb{R}^n \times \mathbb{R} \times \overline{\Omega}$. Smooth critical points of the associated "energy" functional satisfy the Euler-Lagrange equation

$$-\sum_{i=1}^{n} (L_{p_i}(Du, u, x))_{x_i} + L_z(Du, u, x) = 0, \qquad \text{in } \Omega. \tag{8}$$

We assume without loss of generality that $L(0, 0, x) = 0$ in Ω. The main identity of Pucci-Serrin is due to the following proposition

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[1] Ω is star-shaped if there exists $x_0 \in \Omega$ such that $(x - x_0) \cdot \nu \geq 0$ for all $x \in \partial\Omega$.

Proposition 2.1 (Pucci-Serrin [7]). *Let $u \in C^2(\Omega)$ be a solution of the Euler-Lagrange (8), and let a and $\vec{h}$ be, respectively, scalar and vector valued functions of class $C^1(\Omega)$. Then the following relation holds in Ω:*

$$\begin{aligned}
&\frac{\partial}{\partial x_i}\left[\vec{h}_i L(Du,u,x) - \vec{h}_j \frac{\partial u}{\partial x_j} L_{p_i}(Du,u,x) - auL_{p_i}(Du,u,x)\right] \\
&= \frac{\partial \vec{h}_i}{\partial x_i} L(Du,u,x) + \vec{h}_i L_{x_i}(Du,u,x) - \left(\frac{\partial u}{\partial x_j}\frac{\partial \vec{h}_j}{\partial x_i} + u\frac{\partial a}{\partial x_i}\right) L_{p_i}(Du,u,x) \\
&- a\left(\frac{\partial u}{\partial x_i} L_{p_i}(Du,u,x) + uL_z(Du,u,x)\right),
\end{aligned} \tag{9}$$

where repeated indices i and j are to be summed from 1 to n.

The proof is obtained by direct computation, using (8). If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solves (8) with $u = 0$ on $\partial\Omega$ then $u_{x_i} = (\partial u/\partial \nu)\nu_i$ on $\partial\Omega$ so

$$\vec{h}_j \frac{\partial u}{\partial x_j} L_{p_i}(Du,u,x)\nu_i = \frac{\partial u}{\partial x_i} L_{p_i}(Du,u,x)\vec{h}_j \nu_j \qquad \text{on } \partial\Omega. \tag{10}$$

Integrating (9) over Ω, applying (10), $u = 0$ on $\partial\Omega$, and the divergence theorem one obtains the fundamental identity

$$\begin{aligned}
&\int_{\partial\Omega}\left[L(Du,0,x) - \frac{\partial u}{\partial x_i} L_{p_i}(Du,0,x)\right](\vec{h}\cdot\nu)\,ds \\
&= \int_\Omega L(Du,u,x)\,\mathrm{div}\vec{h} + \vec{h}_i L_{x_i}(Du,u,x) - \left(\frac{\partial u}{\partial x_j}\frac{\partial \vec{h}_j}{\partial x_i} + u\frac{\partial a}{\partial x_i}\right) L_{p_i}(Du,u,x) \\
&- a\left(\frac{\partial u}{\partial x_i} L_{p_i}(Du,u,x) + uL_z(Du,u,x)\right)\,dx.
\end{aligned} \tag{11}$$

For example, if $L(p,z) = \frac{1}{2}|p|^2 - F(z), \vec{h} = x$, and a is constant, then (11) reduces to

$$-\int_{\partial\Omega}\frac{1}{2}|Du|^2(x\cdot\nu)\,ds = \int_\Omega\left[\frac{n}{2} - 1 - a\right]|Du|^2 - nF(u) + auf(u)\,dx. \tag{12}$$

The choice of $a(x) = (n-2)/2$ makes the $|Du|^2$ vanish and reduces (12) to the Pohozaev identity (5). However, identity (11) is applicable to a much larger class of equations. For instance, for the quasilinear equation (3) with associated Lagrangian $L(Du,u) = \frac{1}{p}|Du|^p - F(u)$, the choice of $\vec{h} = x$ and constant a yields

$$-\int_{\partial\Omega}\frac{1}{p}|Du|^p(x\cdot\nu)\,ds = \int_\Omega\left[\frac{n}{p} - 1 - a\right]|Du|^p - nF(u) + auf(u)\,dx. \tag{13}$$

Now we see the choice of $a = (n-p)/p$ implies

$$-\int_{\partial\Omega}\frac{1}{p}|Du|^p(x\cdot\nu)\,ds = \int_\Omega\left[\frac{n-p}{p}\right]uf(u) - nF(u)\,dx, \tag{14}$$

from which an appropriate nonexistence result can be stated. To determine the critical exponent we choose $f(u) = |u|^{q-1}u$ and find (3) has no nontrivial solutions when $p < n$ and

$$q > \frac{np}{n-p} - 1 = \frac{(p-1)n + p}{n-p}.$$

Note that $p^* = np/(n-p)$ is the Sobolev exponent, corresponding to the loss of compactness for the continuous embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$. Many further applications of (11) may be found in [7].

We seek to apply this idea to the k-Hessian equation (1). Equation (1) is of variational form, with solutions corresponding to critical points of the functional

$$I_k[u] = -\frac{1}{k+1}\int_\Omega u\ S_k(D^2u)\,dx + \int_\Omega F(x,u)\,dx, \tag{15}$$

where $F(x,u) = \int_0^u f(x,s)\,ds$ (see §4.1). However, Proposition 9 does not directly apply to (15) since the Lagrangian contains higher order terms, and one needs to derive an appropriate higher order analog of (9).

The Euler-Lagrange equation associated with the Lagrangian $L = L(D^2u, Du, u, x) = L(r_{ij}, p_i, z, x)$, where $r_{ij} = r_{ji}$ is

$$\sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} L_{r_{ij}}(D^2u, Du, u, x) - \sum_{i=1}^n (L_{p_i}(D^2u, Du, u, x))_{x_i} + L_z(D^2u, Du, u, x) = 0. \tag{16}$$

In our case L is independent of p and the fundamental identity (simplified for our purposes) takes the form (see equation (29) in [7])

Proposition 2.2 (Pucci-Serrin [7]). *Let $u \in C^4(\Omega)$ be a solution to the Euler-Lagrange equation (16) with $L_{p_i} = 0$ and $a \in C^2(\Omega)$ a scalar function. Then*

$$\begin{aligned}&\frac{\partial}{\partial x_i}\left[x_i L + \left(x_l \frac{\partial u}{\partial x_l} + au\right)\frac{\partial L_{r_{ij}}}{\partial x_j} - \frac{\partial}{\partial x_j}\left(x_l \frac{\partial u}{\partial x_l} + au\right) L_{r_{ij}}\right] \\ &= nL + x_i L_{x_i} - auL_z - (a+2)\frac{\partial^2 u}{\partial x_i \partial x_j} L_{r_{ij}}.\end{aligned} \tag{17}$$

Following Tso [10], we employ this identity to determine the critical exponent associated to the operator S_k. For simplicity we assume $F = F(z)$ (e.g., $f(u) = |u|^p$).

Theorem 2.3 (Tso [10]). *Let Ω be a smooth domain which is star-shaped with respect to the origin. Assume $f : (-\infty, 0] \to [0, \infty)$ is smooth, with $f(s) > 0$ for $s < 0$ and $f(0) = 0$. Then*

$$\begin{cases} S_k(D^2u) = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{18}$$

has no nontrivial solutions in $\Phi_0^k(\Omega)\cap C^4(\Omega)\cap C^1(\overline{\Omega})$ *when*

$$nF(u)-\frac{n-2k}{k+1}uf(u)>0, \qquad \textit{for } u<0. \tag{19}$$

Proof. Applying (2.2) to the Lagrangian $L=\frac{-zS_k(r_{ij})}{k+1}+F(z)$ one obtains

$$\frac{\partial}{\partial x_i}\left[x_i\left(\frac{-uS_k(D^2u)}{k+1}+F(u)\right)-\left(x_l\frac{\partial u}{\partial x_l}+au\right)\frac{u_{x_j}S^{ij}(D^2u)}{k+1}+\frac{\partial}{\partial x_j}\left(x_l\frac{\partial u}{\partial x_l}+au\right)\frac{uS^{ij}(D^2u)}{k+1}\right]$$
$$=[k(a+2)+a-n]\frac{uS_k(D^2u)}{k+1}+nF-auf. \tag{20}$$

Choosing $a=(n-2k)/(k+1)$ and integrating (20) we obtain

$$-\frac{1}{k+1}\int_{\partial\Omega}\left[x_lu_{x_l}u_{x_j}S^{ij}(D^2u)\right]\nu_i\,ds=\int_\Omega\left(nF(u)-\frac{n-2k}{k+1}uf(u)\right)dx, \tag{21}$$

which simplifies to

$$-\frac{1}{k+1}\int_{\partial\Omega}(x\cdot\nu)|Du|^2S^{ij}(D^2u)\nu_i\nu_j\,ds=\int_\Omega\left(nF(u)-\frac{n-2k}{k+1}uf(u)\right)dx. \tag{22}$$

For $u\in\Phi_0^k(\Omega)$ the operator S_k is elliptic, thus $S^{ij}(D^2u)\nu_i\nu_j>0$. Hence the left-hand side of (22) is nonpositive and the result follows. ☐

Note that when $k=1$, (19) is equivalent to the Pohozaev criterion (6). If $f(u)=(-u)^p$ then (19) reduces to

$$\frac{n-2k}{k+1}>\frac{n}{p+1}. \tag{23}$$

If $k\geq n/2$, then (23) can not hold and we obtain no a priori obstructions to solution from this method. On the other hand, when $k<n/2$, then (23) is true when $p\geq\frac{(n+2)k}{n-2k}$. Thus when $k<n/2$ the *critical exponent* $\gamma(k)$ for S_k is defined by

$$\gamma(k)=\frac{(n+2)k}{n-2k}. \tag{24}$$

Tso also provides complementary existence results for radially symmetric solutions for subcritical exponents (and for all exponents when $k\geq n/2$), thus we can extend $\gamma(k)$ to all k via

$$\gamma(k)=\begin{cases}\infty & k>n/2\\ \frac{(n+2)k}{n-2k} & k<n/2.\end{cases} \tag{25}$$

In particular, there is no critical exponent for the Monge-Ampère operator. Heuristically, operators "closer" to the Laplace operator have critical exponents, while operators "closer" to Monge-Ampère do not. Note that when $p=k$ one has an eigenvalue problem (see e.g., [3, 11, 2].)

3 Critical Dimension

In 1983 Brezis and Nirenberg observed that lower order perturbations to elliptic equations involving critical exponents recovered the lost compactness. More precisely, they proved the equation

$$\begin{cases} \Delta u + u^{\frac{n+2}{n-2}} + \lambda u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{26}$$

has a positive solution if $0 < \lambda < \lambda_1$ and $n \geq 4$, where λ_1 is the principal eigenvalue for $-\Delta$ on $H_0^1(\Omega)$. Surprisingly, for the case $n = 3$ they observed that there exists $\lambda^* > 0$ such that (26) has a solution for $\lambda \in (\lambda^*, \lambda_1)$ and no solution for $\lambda \in (0, \lambda^*)$. If Ω is a ball, then $\lambda^* = \lambda_1/4$. In this context the dimension $n = 3$ is called a *critical dimension.*

From Section 1 we know that both Δ_p and S_k have critical exponents (when $p < n$ and $k < n/2$, respectively). Thus it is natural to ask if results similar to the Brezis-Nirenberg result exist for these operators. Several authors have answered this question affirmatively. Rather that treat Δ_p and S_k separately, we adopt the approach of Clément-DeFigueiredo-Mitidieri [1] and consider the equation

$$\begin{cases} (r^\alpha |u'|^\beta u')' = r^\gamma |u|^{q-2} u, & r \in (0, R), \\ u > 0, & r \in (0, R), \\ u'(0) = u(R) = 0, \end{cases} \tag{27}$$

and the perturbed form

$$\begin{cases} (r^\alpha |u'|^\beta u')' = r^\gamma |u|^{q-2} u + \lambda r^\delta |u|^\beta u, & r \in (0, R), \\ u > 0, & r \in (0, R), \\ u'(0) = u(R) = 0, \end{cases} \tag{28}$$

for various values of exponents α, β, δ and γ. See the table on page 1 for the relevant values of constants for (1) or (3).

The critical exponent associated with (27) is

$$q^* = \frac{(\gamma + 1)(\beta + 2)}{\alpha - \beta - 1}. \tag{29}$$

For the p-Laplacian, $q^* = \frac{np}{n-p}$ and for S_k, $q^* = \frac{n(k+1)}{n-2k}$, agreeing with our previous observations in Section 1.[2]

Throughout this section we will assume the following inequalities hold

$$q - 1 > \beta + 1 > 0, \; \gamma + 1 > \alpha - \beta - 1, \text{ and } \delta + 1 \geq \alpha - \beta - 1 \tag{30}$$

$$\alpha - \beta - 1 > 0 \tag{31}$$

$$\gamma, \delta > \alpha - 1 \tag{32}$$

$$\alpha - \beta - 2 < \delta. \tag{33}$$

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[2] Note that the exponent in (27) is $q-2$. This notational convenience has the advantage that the "critical exponent" agrees with the Sobolev exponent.

When applied to S_k (resp. Δ_p) these inequalities simply imply $q > k+1$ (resp. $q > p$) and $k < n/2$ (resp. $n < p$), the realm of critical exponents.

The goal of this section will be to prove the following two nonexistence results:

Theorem 3.1 ([1]). *Assume (30),(31),(32),(33) hold. If $\lambda \le 0$ and $q = q^*$, then (28) has no solution.*

Theorem 3.2. *Assume (30),(31),(32),(33), $\beta \ge 0$ and $q = q^*$. If*

$$(\beta+1)(\delta+1) - (\alpha-\beta-1)(\beta+2) > 0, \tag{34}$$

then there exists $\lambda^* > 0$ such that (28) has no solution for $\lambda \in (0, \lambda^*)$.

For the model operators S_k and Δ_p, their parameters satisfying (34) correspond to certain values of the dimension n, called *critical dimensions* by Pucci and Serrin [8]. For the p-Laplace operator, (34) corresponds to $n < p^2$, thus the critical dimensions for Δ_p are those n with $p < n < p^2$. Note that for the Laplacian $p = 2$ and we obtain $2 < n < 4$, thus the only critical dimension is $n = 3$, as observed by Brezis and Nirenberg. For the k-Hessian the critical dimensions are those n with $2k < n < 2k(k+1)$.

The proofs are based on the following identity of Pohozaev-Pucci-Serrin type:

Proposition 3.3 ([1]). *Let $a, b \in C^1[0,\infty)$. If $u \in C^2(0,\infty) \cap C^1[0,\infty)$ solves*

$$-(r^\alpha |u'|^\beta u')' = f(r,u) \qquad \text{in } (0,\infty), \tag{35}$$

then for $R > 0$ we have

$$\begin{aligned} \left[-r^\alpha u'|u'|^\beta \left(au + \frac{\beta+1}{\beta+2} bu'\right)\right]_{r=R} &+ \int_0^R r^\alpha a' u u' |u'|^\beta \\ &+ \int_0^R r^\alpha \left(a + \frac{\beta+1}{\beta+2} b' - \frac{\alpha}{\beta+2}\frac{b}{r}\right) |u'|^{\beta+2} \\ = [bF(r,u)]_{r=R} &+ \int_0^R auf(r,u) - bF_r(r,u) - b'F(r,u). \end{aligned} \tag{36}$$

Proof. The proof is a nice application of the "abc-method".[3]

□

Now we prove Theorem 3.1:

Proof. Without loss of generality assume $R = 1$ and let u solve (28). Using (36) with $b(r) = r$, a constant, and

$$f(r,u) = r^\gamma |u|^{q-2} u + \lambda r^\delta |u|^\beta u \tag{37}$$

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[3] Friedrichs' moniker for the "energy" method of multiplying a PDE by $au + bu_x + cu_y$ and integrating.

we obtain

$$\begin{aligned}&\left[-\frac{\beta+1}{\beta+2}|u'|^{\beta+2}\right]_{r=1}+\int_0^1 r^\alpha\left(a+\frac{\beta+1-\alpha}{\beta+2}\right)|u'|^{\beta+2}\\ &=\int_0^1 r^\gamma\left[a-\frac{\delta+1}{\beta+2}\right]\lambda|u|^{\beta+2}+\int_0^1 r^\gamma\left[a-\frac{\gamma+1}{q}\right]|u|^q.\end{aligned}\tag{38}$$

If we choose

$$a=\frac{\alpha-\beta-1}{\beta+2},$$

then the integral on the left-hand side of (38) vanishes. Since $q=q^*$, the same is true for the last integral in (38). Moreover, the coefficient in the first integrand on the right-hand side of (38) becomes

$$a-\frac{\delta+1}{\beta+2}=\frac{\alpha-\beta-\delta-2}{\beta+2}.$$

Since $\lambda\le 0$, the right-hand side of (38) is nonnegative. On the other hand, the left-hand side of (38) is negative. Note that from existence and uniqueness of the initial value problem we must have $u'(1)\neq 0$. □

Finally, we prove the "critical dimension" Theorem 3.2:

Proof. We again apply (36) with $R=1$ and f as in (37), now with

$$a=a_1+a_2r^m \qquad b=-r+r^{m+1},$$

where a_1,a_2,m are constants to be determined. Since $b(1)=0$ and $u(1)=0$, all the boundary terms vanish. We choose a_1 and a_2 so that the integrals containing $|u'|^{\beta+2}$ vanish, i.e.,

$$a_1=-\frac{\alpha-\beta-1}{\beta+2}\qquad a_2=\frac{\alpha-(m+1)(\beta+1)}{\beta+2}.$$

With the free parameter m left we have:

$$I_5=\int_0^1 r^\alpha a'uu'|u'|^\beta=I_1+I_2+I_3+I_4,\tag{39}$$

where

$$I_1 = \lambda\int_0^1\left[a_1+\frac{\delta+1}{\beta+2}\right]r^\delta|u|^{\beta+2}\tag{40}$$

$$I_2 = \lambda\int_0^1\left[a_2-\frac{\delta+m+1}{\beta+2}\right]r^{\delta+m}|u|^{\beta+2}\tag{41}$$

$$I_3 = \int_0^1\left[a_1+\frac{\gamma+1}{q}\right]r^\gamma|u|^q\tag{42}$$

$$I_4 = \int_0^1\left[a_2-\frac{\gamma+m+1}{q}\right]r^{\gamma+m}|u|^q\tag{43}$$

From (33) it follows that $I_1 > 0$. Since $q = q^*$, $I_3 = 0$.

Let us examine I_5. From (28) we observe

$$-r^\alpha u'(r)|u'(r)|^\beta = \int_0^r \lambda r^\delta u|u|^\beta + r^\gamma u|u|^{q-2} dr > 0,$$

for positive solutions of (28). We conclude $u'(r) < 0$ for all $r \in (0,1]$. If $a_2 < 0$ (the choice of m will imply this!), then

$$I_5 = \int_0^1 r^\alpha a' uu'|u'|^\beta = m|a_2| \int_0^1 r^{\alpha+m-1} u|u'|^{\beta+1} = C \int_0^1 r^{\alpha+m-1} \left|(u^{\frac{\beta+2}{\beta+1}})'\right|^{\beta+1},$$

where $C = C(m, |a_2|, \beta) > 0$. It follows from an embedding theorem (see §5) that

$$\int_0^1 r^{\alpha+m-1} \left|(u^{\frac{\beta+2}{\beta+1}})'\right|^{\beta+1} \geq c \int_0^1 r^\delta \left(u^{\frac{\beta+2}{\beta+1}}\right)^{\beta+1} = c \int_0^1 r^\delta u^{\beta+2}, \qquad (44)$$

provided $m \leq \delta - \alpha + \beta + 2$. We then choose

$$m = \delta - \alpha + \beta + 2,$$

which is positive in view of the hypothesis of the theorem. From (44) it follows that

$$I_5 \geq \tilde{c} I_1,$$

for some $\tilde{c} > 0$. If our choice of m renders $a_2 < 0$, then $I_2 < 0$ and $I_4 < 0$ and a sign analysis of (39) implies there must exist a $\lambda^* > 0$ such that there is no solution for $\lambda \leq \lambda^*$. Thus to complete the proof we need to show $a_2 < 0$, i.e.,

$$\alpha - (\delta - \alpha + \beta + 3)(\beta + 1) < 0.$$

But this is equivalent to our hypothesis (34) and the proof is complete. ☐

4 Appendix

4.1 Variational form for S_k

Recall $S_k(D^2u)$ is defined in terms of the elementary symmetric polynomials acting on the eigenvalues of D^2u. In the two extreme cases $k=1$ and $k=n$, the fact that $\sum \lambda_i$ and $\Pi\lambda_i$ are, respectively, the trace and determinant of the matrix allows us to see immediately the partial differential operator defined by S_k, i.e., $S_1(D^2u)=\Delta u$ and $S_n(D^2u)=\det D^2u$. In general, for a symmetric matrix r, $S_k(r)$ is the sum of all principal $k\times k$ minors of r, i.e.,

$$S_k(r)=\frac{1}{k!}\sum \delta\begin{pmatrix} i_1,\dots,i_k \\ j_1,\dots,j_k\end{pmatrix} r_{i_1j_1}\cdots r_{i_kj_k}, \tag{45}$$

where $\delta\binom{i_1,\cdots,i_k}{j_1,\cdots,j_k}$ is 1 (resp. -1) if $(i_1,\dots,i_k)$ are distinct and $(j_1,\dots,j_k)$ is an even (resp. odd) permutation of $(i_1,\dots,i_k)$, otherwise it is 0. From this we can determine $\frac{\partial S_k}{\partial r_{ij}}=S^{ij}(r)$:

$$S^{ij}(r)=\frac{1}{(k-1)!}\sum \delta\begin{pmatrix} i_1,\dots,i_{k-1},i \\ j_1,\dots,j_{k-1},j\end{pmatrix} r_{i_1j_1}\cdots r_{i_{k-1}j_{k-1}}. \tag{46}$$

In particular, this implies

$$S_k(r)=\frac{1}{k}\sum_{i,j=1}^{n} r_{ij}S^{ij}(r), \tag{47}$$

i.e.,

$$S_k(D^2u)=\frac{1}{k}\sum_{i,j=1}^{n} u_{x_ix_j}S^{ij}(D^2u). \tag{48}$$

A computation (see [9, 10]) shows

$$\sum_{j=1}^{n}\frac{\partial}{\partial x_j}S^{ij}(D^2u)=0 \qquad \text{for each } i. \tag{49}$$

Together with (48) this implies S_k has a divergence form:

$$S_k(D^2u)=\frac{1}{k}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(u_{x_i}S^{ij}(D^2u)\right). \tag{50}$$

The Euler-Lagrange equation associated with the Lagrangian $L=\frac{-zS_k(r_{ij})}{k+1}+F(x,z)$ is

$$-\frac{1}{k+1}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i\partial x_j}(uS^{ij}(D^2u))-\frac{S_k(D^2u)}{k+1}+f(x,u)=0. \tag{51}$$

From (49) we may rewrite the first term as

$$-\frac{1}{k+1}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}(u_{x_i}S^{ij}(D^2u)) \tag{52}$$

from which it follows (51) is equivalent to

$$-\frac{1}{k+1}kS_k(D^2u)-\frac{1}{k+1}S_k(D^2u)+f(x,u)=0,$$

or

$$-S_k(D^2u)+f(x,u)=0,$$

which is precisely (1).

4.2 Radial form of S_k

If $u:\Omega\to\mathbb{R}$ is radially symmetric then a calculation show

$$\frac{\partial u}{\partial x_i}=u'(r)\frac{x_i}{r}\quad\text{and}\quad\frac{\partial^2 u}{\partial x^2}=u''(r)\frac{x_ix_j}{r^2}+u'(r)\left[\frac{r^2\delta_{ij}-x_ix_j}{r^3}\right], \tag{53}$$

for $i,j=1,\dots,n$. At the point $x=(r,0,\dots,0)$ the Hessian matrix D^2u is diagonal with $u_{11}=u''(r)$ and $u_{ii}=u'(r)/r$, for $i>1$. Since the operator S_k is invariant with respect to rotations, it follows that

$$\begin{aligned}S_k(D^2u) &= u''\binom{n-1}{k-1}\left(\frac{u'}{r}\right)^{k-1}+\binom{n-1}{k}\left(\frac{u'}{r}\right)^{k}\\ &= \frac{1}{k}\binom{n-1}{k-1}r^{1-n}\left(r^{n-k}(u')^k\right)',\end{aligned}$$

where $\binom{n}{k}$ is the binomial coefficient.

5 An embedding theorem

We quote an embedding theorem needed in §3:

Proposition 5.1 ([5]). *Let $u:(0,R]\to\mathbb{R}$ be absolutely continuous. If $u(R)=0$ and*

- *(i) for $1\le\beta+2\le q<\infty$ one has*
    - *(a) $\alpha>\beta+1, \gamma\ge\alpha\frac{q}{\beta+2}-q\frac{\beta+1}{\beta+2}-1$, or*
    - *(b) $\alpha\le\beta+1$, $\gamma>-1$,*
- *(i) for $1\le q<\beta+2<\infty$ one has*

(c) $\alpha > \beta + 1, \gamma > \alpha \frac{q}{\beta+2} - q\frac{\beta+1}{\beta+2} - 1$, *or*

(d) $\alpha \leq \beta + 1$, $\gamma > -1$,

then

$$\left(\int_0^R x^\gamma |u(x)|^q \, dx\right)^{1/q} \leq c \left(\int_0^R x^\alpha |u'(x)|^{\beta+2} \, dx\right)^{1/(\beta+2)}. \tag{54}$$

This proposition corresponds to a continuous embedding $X_R \subset L^q_\gamma(0, R)$, where $L^q_\gamma(0, R)$ is the Banach space of measurable functions $u : [0, R] \to \mathbb{R}$ with finite weighted norm:

$$\|u\|_{L^q_\gamma} = \left(\int_0^R x^\gamma |u(x)|^q \, dx\right)^{1/q},$$

and X_R is defined as follows. For $0 < R < \infty, \alpha > 0$, and $\beta > -1$ let $\widetilde{X}_R$ denote the set of real valued L^1_{loc} functions defined on $(0, R)$ with distributional derivatives in L^1_{loc} such that

$$\int_0^R x^\alpha |u(x)|^{\beta+2} \, dx < \infty \quad \text{and} \int_0^R x^\alpha |u'(x)|^{\beta+2} \, dx < \infty.$$

Then $\widetilde{X}_R$ is a Banach space with norm $\| \cdot \|_{\widetilde{X}_R}$ defined by

$$\|u|^{\beta+2}_{\widetilde{X}_R} = \int_0^R x^\alpha |u|^{\beta+2} \, dx + \int_0^R x^\alpha |u'|^{\beta+2} \, dx.$$

It follows that $u \in \widetilde{X}_R$ is absolutely continuous in $(0, R]$ and, thus, we can consider the subspace X_R of those $u \in \widetilde{X}_R$ such that $u(R) = 0$. By Proposition 5.1 above it follows that for $u \in X_R$:

$$\int_0^R x^\alpha |u|^{\beta+2} \, dx \leq C \int_0^R x^\alpha |u'|^{\beta+2} \, dx.$$

Thus $\| \cdot \|_{\widetilde{X}_R}$ and $\| \cdot \|_{X_R}$ are equivalent norms on X_R. For different values of α and β, the spaces X_R are "weighted Sobolev spaces" [5]. In this way we can understand the critical exponent results above in terms of loss of compactness of the embedding of the weighted Sobolev space X_R into the weighted L^q_γ space.

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