

Variational Methods for k -Hessian Equations

David Hartenstine
University of Utah

University of Utah
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1 The k -Hessian Operators

We will work in a bounded, connected and open subset of \mathbb{R}^n , denoted Ω . For simplicity, we assume that $\partial\Omega$, the boundary of Ω , is smooth and strictly convex, although many of the following results hold under less restrictive hypotheses.

We use the standard notation $C^m(\Omega)$ (where m is a nonnegative integer) to indicate the space of real-valued functions defined on Ω all of whose derivatives of order less than or equal to m are continuous on Ω . $C^m(\bar{\Omega})$ is the space of functions in $C^m(\Omega)$ all of whose derivatives have continuous extensions to $\bar{\Omega}$. Let $0 < \alpha \leq 1$; a function u defined on Ω is said to be **uniformly Hölder continuous with exponent α on Ω** if

$$[u]_{\alpha, \Omega} = \sup_{x, y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

is finite. The space $C^{m, \alpha}(\bar{\Omega})$ is the subset of $C^m(\bar{\Omega})$ for which all derivatives of order m or less are Hölder continuous. We will use the following norms on these spaces:

$$\|u\|_{C^m(\bar{\Omega})} = \sup_{\Omega} |u| + \sum_{j=1}^n \sup_{|\beta|=j} \sup_{\Omega} |D^\beta u|$$

$$\|u\|_{C^{m, \alpha}(\bar{\Omega})} = \|u\|_{C^m(\bar{\Omega})} + \sup_{|\beta|=m} [D^\beta u]_{\alpha, \Omega}$$

These spaces equipped with these norms are Banach spaces. The theorem of Arzela-Ascoli implies that a sequence bounded in $C^{m, \alpha}(\bar{\Omega})$ has a subsequence that converges in C^m .

For any function $u \in C^2(\Omega)$ and $x \in \Omega$, we can construct the symmetric $n \times n$ matrix $D^2u(x)$, the **Hessian** of u at x , whose i, j entry is $\frac{\partial^2 u}{\partial x_i \partial x_j}(x)$, i.e.

$$D^2u(x) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}(x) \right)_{i, j}$$

Because $D^2u(x)$ is symmetric, all of its eigenvalues are real.

A polynomial of n variables is **symmetric** if

$$P(x_1, \dots, x_n) = P(\sigma(x_1, \dots, x_n))$$

where σ is any permutation of n letters.

Examples:

$$P_1(x_1, \dots, x_n) = x_1 + x_2 + \dots + x_n$$

$$Q(x_1, \dots, x_n) = x_1^2 + x_2^2 + \dots + x_n^2$$

$$P_2(x_1, \dots, x_n) = x_1x_2 + x_1x_3 + \dots + x_1x_n + x_2x_3 + \dots + x_2x_n + \dots + x_{n-1}x_n$$

$$P_n(x_1, \dots, x_n) = x_1x_2 \dots x_n$$

In general, the k -th elementary symmetric polynomial of n variables is

$$P_k(x_1, \dots, x_n) = \sum_{i_1 < \dots < i_k} x_{i_1} \dots x_{i_k}$$

Note that $P_k(x)$ (where $x = (x_1, \dots, x_n)$) is homogeneous of degree k meaning that $P_k(tx) = t^k P_k(x)$ for any $t \geq 0$.

We can now define the k -Hessian operators. For $1 \leq k \leq n$, define

$$S_k(D^2u)(x) = P_k(\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x))$$

where the $\lambda_i(x)$ are the eigenvalues of $D^2u(x)$. Equivalently, $S_k(D^2u)$ can be defined as the sum of the $k \times k$ principal minors of the Hessian matrix.

There are two well-known cases:

$S_1(D^2u) = P_1(\lambda_1, \lambda_2, \dots, \lambda_n) = \lambda_1 + \dots + \lambda_n =$ sum of eigenvalues. The sum of the eigenvalues of a matrix is the same as the trace, so

$$S_1(D^2u) = \text{trace}(D^2u) = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = \Delta u.$$

In other words, the operator $S_1(D^2u)$ is the same as the Laplacian.

$S_n(D^2u) = P_n(\lambda_1, \dots, \lambda_n) = \lambda_1 \dots \lambda_n =$ product of eigenvalues. The product of the eigenvalues of a matrix is the same as the determinant, so

$$S_n(D^2u) = \det(D^2u).$$

So the case $k = n$ coincides with the Monge-Ampère operator.

2 k -convexity and Ellipticity

In studying k -Hessian equations of the form $S_k(D^2u) = f(x, u, Du)$, it is desirable (as we will see below) to work not in the space of C^2 functions, but in a smaller set of functions, the cone of k -convex functions.

2.1 k -convexity

A function $u \in C^2(\Omega)$ is called **(uniformly) k -convex** if $S_j(D^2u) \geq (>) 0$ for $j = 1, \dots, k$. Equivalently, a C^2 function is k -convex if all of the eigenvalues of its Hessian matrix lie in the convex cone Γ_k in \mathbb{R}^n defined by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : P_j(\lambda) \geq 0, j = 1, \dots, k\}.$$

The set of k -convex functions on Ω will be denoted $\Phi^k(\Omega)$, and $\Phi_0^k(\Omega)$ stands for the set of k -convex functions that vanish continuously on $\partial\Omega$. As an immediate consequence of the definition, we see that $\Phi^k(\Omega) \subset \Phi^l(\Omega)$ for $l \leq k$. Note also that 1-convex functions are subharmonic; further, n -convex functions are convex in the usual sense.

If $u \in \Phi_0^k(\Omega)$, then u is negative in Ω , unless $u \equiv 0$. This is because if u is k -convex then it is subharmonic, so by the strong maximum principle, u cannot attain its maximum in the interior of Ω . Since $u|_{\partial\Omega} = 0$, we must have $u < 0$ in Ω .

Because the k -Hessian operators are homogeneous, if $u \in \Phi^k(\Omega)$ (or $\Phi_0^k(\Omega)$), then so is tu for any $t \geq 0$. Furthermore, if u_1 and u_2 are in $\Phi^k(\Omega)$, then for $1 \leq j \leq k$, and any $0 \leq t \leq 1$,

$$S_j^{\frac{1}{j}}(D^2(tu_1 + (1-t)u_2)) = S_j^{\frac{1}{j}}(tD^2u_1 + (1-t)D^2u_2)$$

In [2] it is proved that $S_j^{\frac{1}{j}}$ is a concave function on Φ^k , so the quantity on the previous line is

$$\geq tS_j^{\frac{1}{j}}(D^2u_1) + (1-t)S_j^{\frac{1}{j}}(D^2u_2) \geq 0$$

so $tu_1 + (1-t)u_2$ is k -convex. This proves that Φ^k and Φ_0^k are convex cones in $C^2(\Omega)$.

2.2 Ellipticity of Second Order Operators

A second order linear partial differential operator (with real coefficients) has the form

$$Lu(x) = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{k=1}^n b_k(x) \frac{\partial u}{\partial x_k}(x) + c(x)u(x)$$

where a_{ij} , b_k , and c are real-valued functions defined on Ω , and $a_{ij} = a_{ji}$.

Such an operator is called **elliptic** if the symmetric matrix $(a_{ij}(x))$ is positive-definite at every $x \in \Omega$. This means that the matrix $(a_{ij}(x))$ has only positive eigenvalues at every x .

The Laplace operator ($S_1(D^2u)$) is linear with $a_{ij} = \delta_{ij}$ and $b_k \equiv c \equiv 0$. It is also elliptic because the matrix $(a_{ij}(x))$ is the identity matrix, which is positive-definite. The other k -Hessian operators are nonlinear.

We now define ellipticity for nonlinear second order operators. A second order partial differential equation can be written as $F[u] = F(x, u, Du, D^2u) = 0$, where $F = F(x, z, p, r)$ is a function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_s^{n \times n}$, where the last set in the product is the set of symmetric $n \times n$ real matrices. For example, the k -Hessian equation $S_k(D^2u) = f(x, u, Du)$ can be written as $F_k[u] = S_k(D^2u) - f(x, u, Du) = 0$. The operator F is **elliptic with respect to the function** u if for every $x \in \Omega$, the matrix

$$(F_{ij}) = \left(\frac{\partial F}{\partial r_{ij}} \right)_{i,j}$$

is positive-definite at $(x, u(x), Du(x), D^2u(x))$. If F is linear $(F_{ij}) = (a_{ij})$, so this new definition covers the special case of linear operators.

In [2] it is shown that if u is k -convex, then F_k is elliptic with respect to u . This is the reason for working with k -convex functions. The advantage of ellipticity is that there is a well-developed theory for elliptic equations. Many of the techniques used for linear elliptic equations have generalizations to nonlinear elliptic equations. The basic idea is that for elliptic operators, one can prove maximum and comparison principles which lead to *a priori* estimates (inequalities that solutions of must satisfy that depend on the operator and the right-hand side of the equation, but **not** on the solution itself). These estimates can be used to establish the existence and regularity of solutions to many problems. We will see the application of elliptic theory in some of the arguments below.

3 Optimal Transport and the Monge-Ampère Equation

An equation of Monge-Ampère type arises in the problem of optimal transport. In this section, we introduce the discrete version of the problem first, then present the continuous analog and show the connection with the Monge-Ampère equation. Two references for this problem not mentioned below are [1] and [10].

3.1 The Discrete Case: The Shipper's Dilemma

A shipper has n identical packages located at n sources x_1, \dots, x_n that need to be sent to n destinations y_1, \dots, y_n . The x_i and the y_i need not be distinct. The shipper knows $c(x_i, y_j)$, the cost of transporting a package from x_i to y_j for each i and j . To maximize profit, the total cost of shipping the packages should be minimized. Let S be the set of bijections of $\{x_i\}$ to $\{y_j\}$. Each $\sigma \in S$ represents one possible plan for getting the packages to their destinations. The cost associated with σ is

$$C(\sigma) = \sum_{i=1}^n c(x_i, \sigma(x_i)).$$

The problem then is to find $\sigma^* \in S$ such that $C(\sigma^*) \leq C(\sigma)$ for all $\sigma \in S$. Since S is a finite set, this problem clearly has a solution.

3.2 The Continuous Case: The Monge-Kantorovich Problem

Let X and Y be compact subsets of \mathbb{R}^n and let f and g be non-negative, integrable functions with $\text{supp } f = X$ and $\text{supp } g = Y$, and satisfying the **mass-balance condition**:

$$\int_X f(x) dx = \int_Y g(y) dy$$

The problem is to determine what is the most efficient way of moving f to g with reference to a given cost function. The function f describes the original distribution of some quantity and g describes the desired distribution. Monge considered the problem of moving a pile of soil (located on the set X ,

with height $f(x)$ at $x \in X$) and filling an excavation or a hole (the set Y , with depth at y equal to $g(y)$). The mass balance condition means that the rearrangement preserves the total amount of the quantity. We are given the cost function $c : X \times Y \rightarrow [0, \infty)$, where $c(x, y)$ is the cost of moving a unit mass from x to y . We want to minimize the total cost of the rearrangement over the set of **measure-preserving** maps:

$$S = \left\{ s : X \rightarrow Y : s \text{ is } 1-1 \text{ and } \int_X h(s(x))f(x) dx = \int_Y h(y)g(y) dy \text{ for } h \in C(Y) \right\}.$$

The total cost associated with $s \in S$ is

$$C(s) = \int_X c(x, s(x))f(x) dx.$$

The problem then is to find $s^* \in S$, such that $C(s^*) \leq C(s)$, for all $s \in S$. A consequence of a map s being in S is that

$$\int_{s^{-1}(B)} f(x) dx = \int_B g(y) dy$$

for all Borel sets $B \subset Y$. This can be seen by approximating χ_B by continuous functions and using the dominated convergence theorem. Then by the formula for change of variables, we obtain for any $s \in S$:

$$\int_{s^{-1}(B)} f(x) dx = \int_B g(y) dy = \int_{s^{-1}(B)} g(s(x)) |\det Ds(x)| dx$$

so $f(x) = g(s(x)) |\det Ds(x)|$ a.e.

3.3 The Monge-Kantorovich Problem and the Monge-Ampère Equation

We fix the cost function $c(x, y) = |x - y|^2$. Monge considered the cost function $c(x, y) = |x - y|$. These are the two most important cost functions in applications.

We now demonstrate the connection with the Monge-Ampère equation, following [6]. Suppose s^* is a minimizer of C over S . Fix a positive integer m and choose $\{x_k\}_{k=1}^m \subset X$, such that there exist disjoint balls

$$B_k = B(x_k, r_k) \subset X \text{ such that } \int_{B_k} f(x) dx = \epsilon, k = 1, \dots, m.$$

Let $y_k = s^*(x_k)$ and $\tilde{B}_k = s^*(B_k)$. Note that the \tilde{B}_k are disjoint. Because s^* is measure-preserving,

$$\int_{\tilde{B}_k} g(y) dy = \int_{B_k} f(x) dx = \epsilon.$$

Now define a new map $\bar{s} \in S$ by permuting the B_k :

$$\begin{cases} \bar{s}(x_k) = y_{k+1} & y_{m+1} = y_1 \\ \bar{s}(B_k) = \tilde{B}_{k+1} & \tilde{B}_{m+1} = B_1 \\ \bar{s} \equiv s^* \text{ on } X \setminus \bigcup_{k=1}^m B_k \end{cases}$$

By assumption, $C(s^*) \leq C(\bar{s})$. So we have:

$$\int_X |x - s^*(x)|^2 f(x) dx \leq \int_X |x - \bar{s}(x)|^2 f(x) dx.$$

Now, since s^* and \bar{s} agree on $X \setminus \bigcup_{k=1}^m B_k$, and the B_k are disjoint, the last inequality implies that

$$\sum_{k=1}^m \int_{B_k} (|x - s^*(x)|^2 - |x - \bar{s}(x)|^2) f(x) dx \leq 0$$

By expanding and simplifying, we obtain

$$\sum_{k=1}^m \int_{B_k} (|s^*(x)|^2 + 2\langle x, \bar{s}(x) - s^*(x) \rangle - |\bar{s}(x)|^2) f(x) dx \leq 0$$

Because s^* and \bar{s} are in S , we also have that

$$\sum_{k=1}^m \int_{B_k} |s^*(x)|^2 f(x) dx = \sum_{k=1}^m \int_{B_k} |\bar{s}(x)|^2 f(x) dx.$$

This holds because $\int_{B_k} |s^*(x)|^2 f(x) dx = \int_{\tilde{B}_k} |y|^2 g(y) dy$ and $\int_{B_k} |\bar{s}(x)|^2 f(x) dx = \int_{\tilde{B}_{k+1}} |y|^2 g(y) dy$, so summing over k we obtain equality. Therefore:

$$\sum_{k=1}^m \int_{B_k} \langle x, \bar{s}(x) - s^*(x) \rangle f(x) dx \leq 0.$$

Divide by ϵ (which is the $f(x) dx$ measure of B_k) so that the integrals become averages, then let $\epsilon \rightarrow 0$, to obtain by differentiation,

$$\sum_{k=1}^m \langle x_k, y_{k+1} - y_k \rangle \leq 0.$$

This means that the graph $\{(x, s^*(x)) : x \in X\}$ is **cyclically monotone**. A result of Rockafellar then implies that $s^* = D\phi^*$ a.e. for a convex function ϕ^* . Since ϕ^* is convex, it has second derivatives a.e. From the end of Section 3.2, we have for any $s \in S$

$$f(x) = g(s(x)) |\det Ds(x)|.$$

So ϕ^* satisfies

$$f(x) = g(D\phi^*(x)) \det(D^2\phi^*(x)).$$

Therefore, an optimal map s^* is the gradient of a convex function that satisfies (in a weak sense) the following Monge-Ampère problem:

$$\begin{cases} \det D^2\phi(x)g(D\phi(x)) = f(x) \\ D\phi : X \rightarrow Y \end{cases}$$

Notice that since f and g are non-negative, this is an elliptic problem. There are some technical issues involved here regarding the sense in which ϕ satisfies the Monge-Ampère equation. However, if we make some additional assumptions, the equation will be satisfied in the classical sense. More precisely, the following theorem holds:

Theorem 1 (see [3]) *If Y is convex and $0 < \gamma < f$, $g < \beta < \infty$, and f and g are C^α for some $\alpha \in (0, 1]$, then ϕ is locally $C^{2,\alpha}$.*

4 Eigenvalues

Knowledge of the eigenvalues of a differential operator has many applications. Examples include Fourier series, bifurcation theory and variational methods. In this section, we discuss the (Dirichlet) eigenvalue problem for the k -Hessian equations.

4.1 Eigenvalues of $-\Delta$

A real number λ is called an **eigenvalue** of $-\Delta$ (for the Dirichlet problem on Ω) if there exists a nontrivial solution (called an **eigenfunction** corresponding to λ) to the problem:

$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

If λ is an eigenvalue and w is a corresponding eigenfunction, then any nonzero constant multiple of w is also an eigenfunction corresponding to λ because of the linearity of the Laplacian, i.e.

$$-\Delta(\alpha w) = \alpha(-\Delta w) = \alpha(\lambda w) = \lambda(\alpha w).$$

We now list some facts about the eigenvalues for this operator. These can be found in [5] for example. All eigenvalues are positive and there are countably many of them. The smallest eigenvalue, denoted λ_1 , is simple, meaning that if w is an eigenfunction for λ_1 , then any other solution of

$$\begin{cases} -\Delta u = \lambda_1 u \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

is a constant multiple of w . Each eigenfunction corresponding to λ_1 does not change sign, and is in $C^\infty(\bar{\Omega})$ if $\partial\Omega$ is smooth. Finally, λ_1 is given by the following variational principle, known as Rayleigh's Formula:

$$\lambda_1 = \min_{u \neq 0} \frac{-\int_{\Omega} u \Delta u}{\int_{\Omega} u^2}$$

where the min is taken over a suitable function space. The min is attained by an eigenfunction corresponding to λ_1 .

4.2 Eigenvalues for the k -Hessians

Consider the problem

$$\begin{cases} S_k(D^2u) = |\lambda u|^k \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

The question is for which positive numbers λ does this problem have a nontrivial k -convex solution. Such a number λ is called an **eigenvalue** (for the Dirichlet problem on Ω) for the operator S_k .

The exponent k is required to preserve scaling, so that if w is an eigenfunction corresponding to λ , so is αw for any $\alpha > 0$.

$$S_k(D^2w) = \alpha^k S_k(D^2w) = \alpha^k |\lambda w|^k = |\lambda(\alpha w)|^k.$$

Theorem 2 *There exists a unique positive constant λ_1 so that the above eigenvalue problem has a nontrivial k -convex solution. Moreover, this eigenvalue is simple, so that if u_1 and u_2 are k -convex eigenfunctions, then $u_2 = \alpha u_1$ for some positive constant α .*

This result was first proved by Wang in [11], but see [8] for an alternate proof. Wang also proved the following generalization of Rayleigh's Formula:

$$\lambda_1^k = \min_{u \neq 0} \frac{-\int_{\Omega} u S_k(D^2u)}{\int_{\Omega} |u|^{k+1}}$$

where the minimum is taken over all functions in Φ_0^k , and is attained by an eigenfunction.

4.3 An Example

The following result (see [4]) illustrates the importance of the eigenvalue in studying these equations.

Theorem 3 *Consider the problem*

$$\begin{cases} S_k(D^2u) = \Psi(x, u) \text{ in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $\Psi \in C(\overline{\Omega} \times \mathbb{R}^-) \cap C^{1,1}(\overline{\Omega} \times \mathbb{R}^-)$ and $\Psi(x, z) > 0$ for $z < 0$. Suppose also that

$$\lim_{z \rightarrow -\infty} \frac{\Psi(x, z)}{|z|^k} < \lambda_1 \text{ and}$$

$$\lim_{z \rightarrow 0^-} \frac{\Psi(x, z)}{|z|^k} > \lambda_1 \text{ uniformly in } \Omega.$$

Then there exists a nontrivial k -convex solution in $C^{3,\alpha}(\Omega) \cap C(\overline{\Omega})$ for some $0 < \alpha < 1$.

This theorem is proved using a variant of the mountain pass theorem.

4.4 Proof of the Rayleigh type formula for the k -Hessians

4.4.1 Preliminary Results

In this section we collect some results that are needed for the proof. We start with a comparison principle.

Theorem 4 ([7], **Theorem 17.1**) *Suppose $u, v \in \Phi^k(\Omega) \cap C(\bar{\Omega})$ satisfy*

$$\begin{cases} S_k(D^2v) \leq S_k(D^2u) \text{ in } \Omega \\ u|_{\partial\Omega} \leq v|_{\partial\Omega}. \end{cases}$$

Then $u \leq v$ in Ω .

The next result we will need is a uniqueness theorem.

Theorem 5 (Lemma 5.1 in [11]) *Suppose $f(x, u) \geq 0$ is nonincreasing for $u \leq 0$, and that $[f(x, u)]^{\frac{1}{k}} \in C^{1,1}(\bar{\Omega} \times \mathbb{R})$ and is strictly concave with respect to u . Then there exists at most one nonzero solution to*

$$\begin{cases} S_k(D^2u) = f(x, u) \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases} \quad (1)$$

The next result comes from Wang's proof of the existence of the eigenvalue for the k -Hessians ([11], p. 35).

$$\lambda_1 = \sup_{\lambda > 0} \{S_k(D^2u) = (1 - \lambda u)^k \text{ in } \Omega \text{ and } u|_{\partial\Omega} = 0 \text{ has a } C^2(\bar{\Omega}) \text{ solution}\} \quad (2)$$

Before presenting the next result, we need to introduce some terminology. The function $w \in \Phi^k$ is a **subsolution** (resp. **supersolution**) of (1) if

$$\begin{cases} S_k(D^2w) \geq (\leq) f(x, w) \text{ in } \Omega \\ w|_{\partial\Omega} \leq (\geq) 0. \end{cases}$$

Theorem 6 (Theorem 3.3 in [11]) *Suppose $f \in C^{1,1}(\bar{\Omega} \times \mathbb{R})$, $D_x(f^{\frac{1}{k}}) \in C^1(\bar{\Omega} \times \mathbb{R})$, $D_u^2(f^{\frac{1}{k}}) \geq -C > -\infty$ and $f(x, u) > 0$ for $u < 0$. Then if there exists a subsolution w and a supersolution v with $w|_{\partial\Omega} \leq 0$ and $v|_{\partial\Omega} = 0$, then problem (1) has a solution $u \in C^{3,\alpha}(\Omega) \cap C^{1,1}(\bar{\Omega})$ with $v \geq u \geq w$.*

The last result in this section concerns the regularity of solutions.

Theorem 7 (Theorem 4.3 in [4]) *Suppose u solves (1), where $f \in C^{1,1}(\overline{\Omega \times \mathbb{R}^-})$ and is bounded below by a positive constant. Then*

$$\|u\|_{C^{3,\alpha}(\bar{\Omega})} \leq C$$

for some $0 < \alpha < 1$ and a constant C depending on $n, k, \Omega, \sup_{\Omega} |u|$ and second order estimates on f .

4.4.2 The Proof

Let ϕ be an eigenfunction corresponding to λ_1 . Then

$$-\int_{\Omega} \phi S_k(D^2\phi) = -\lambda_1^k \int_{\Omega} \phi |\phi|^k = \lambda_1^k \int_{\Omega} |\phi|^{k+1}$$

so if it can be shown that $-\int_{\Omega} u S_k(D^2u) \geq \lambda_1^k \int_{\Omega} |u|^{k+1}$ for all $u \in \Phi_0^k$, we are done.

Let $\lambda \in (0, \lambda_1)$ and $p \in [0, k)$. By Theorem 5, there is at most one solution to

$$\begin{cases} S_k(D^2u) = (1 + |\lambda u|)^p & \text{in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases} \quad (3)$$

The theorem applies because $f(x, u) = (1 + |\lambda u|)^p > 0$ and is nonincreasing for $u \leq 0$ and we also have that $(1 + |\lambda u|)^{\frac{p}{k}}$ is strictly concave (because $0 < p < k$) and is $C^{1,1}$.

Define the functional

$$I_{p,\lambda}(u) = \int_{\Omega} \left[-\frac{1}{(k+1)} u S_k(D^2u) - \frac{1}{\lambda(p+1)} (1 + \lambda|u|)^{p+1} \right] dx.$$

A critical point of $I_{p,\lambda}$ is the solution of (3); see the appendix of Jacobsen's lecture notes for the derivation. We claim that $\inf I_{p,\lambda}$ over Φ_0^k is attained by some function $u_{p,\lambda}$. This is the longest and most difficult part of the argument, so we postpone the proof of this claim to the end.

Because $\lambda < \lambda_1$, by (2), there exists $\phi_{\lambda} \in C^2(\bar{\Omega})$ solving

$$\begin{cases} S_k(D^2u) = (1 - \lambda u)^k = (1 + |\lambda u|)^k & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases}$$

Since $k > p$, $S_k(D^2\phi_\lambda) \geq (1 + |\lambda u|)^p$, so ϕ_λ is a subsolution of (3). The zero function is a supersolution, so by Theorem 6, there is a solution w_λ of (3) in $C^{3,\alpha}$. By the uniqueness theorem, we must have $w_\lambda = u_{p,\lambda}$.

Then by the regularity result, Theorem 7, the $u_{p,\lambda}$ are uniformly bounded in $C^{3,\alpha}(\bar{\Omega})$ for $p \in [0, k)$. To see this we apply the theorem to

$$\begin{cases} S_k(D^2u) = (1 - \lambda u)^p \equiv f_p(u) \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

The function $u_{p,\lambda}$ solves this problem, and $\sup_\Omega |u_{p,\lambda}|$ is uniformly bounded by $\sup_\Omega |\phi_\lambda|$. Further, we have uniform estimates for the f_p . Hence the $u_{p,\lambda}$ are uniformly bounded in $C^{3,\alpha}(\bar{\Omega})$.

Now choose a sequence $\{p_n\}$ that increases to k . By Arzela-Ascoli, the sequence $\{u_{p_n,\lambda}\}$ has a subsequence, also denoted $\{u_{p_n,\lambda}\}$ that converges in $C^3(\bar{\Omega})$ to a function $u_{k,\lambda}$. Now

$$I_{p_n,\lambda}(u_{p_n,\lambda}) = \int_\Omega \left[-\frac{1}{(k+1)} u_{p_n,\lambda} S_k(D^2u_{p_n,\lambda}) - \frac{1}{(p_n+1)} (1 + \lambda|u_{p_n,\lambda}|)^{p_n+1} \right] dx.$$

The right-hand side of this equation converges to

$$\int_\Omega \left[-\frac{1}{(k+1)} u_{k,\lambda} S_k(D^2u_{k,\lambda}) - \frac{1}{(k+1)} (1 + \lambda|u_{k,\lambda}|)^{k+1} \right] dx = I_{k,\lambda}(u_{k,\lambda}).$$

Since $u_{p_n,\lambda}$ minimizes $I_{p_n,\lambda}$ over Φ_0^k , $I_{k,\lambda}$ attains its minimum at $u_{k,\lambda}$. To see that $u_{k,\lambda}$ minimizes $I_{k,\lambda}$, suppose that there is a $u \in \Phi_0^k$ with $I_{k,\lambda}(u) < I_{k,\lambda}(u_{k,\lambda})$. Since $I_{p_n,\lambda}(u) \rightarrow I_{k,\lambda}(u)$, for large n we must have that $I_{p_n,\lambda}(u) < I_{k,\lambda}(u_{k,\lambda})$, but $I_{p_n,\lambda}(u_{p_n,\lambda}) \rightarrow I_{k,\lambda}(u_{k,\lambda})$, so again for large n , $I_{p_n,\lambda}(u) < I_{p_n,\lambda}(u_{p_n,\lambda})$, but this contradicts the fact that $I_{p_n,\lambda}$ is minimized by $u_{p_n,\lambda}$.

We now claim that this implies that

$$\min_{u \in \Phi_0^k(\Omega)} \frac{-\int_\Omega u S_k(D^2u) dx}{\int_\Omega |u|^{k+1} dx} \geq \lambda^k.$$

Suppose not; then there exists $\bar{u} \in \Phi_0^k(\Omega)$ such that

$$-\int_\Omega \bar{u} S_k(D^2\bar{u}) dx < \lambda^k \int_\Omega |\bar{u}|^{k+1} dx.$$

Without loss of generality, we may assume that $\lambda > 1$ (If the inequality is satisfied for $\lambda < 1$, it also holds for $\lambda > 1$). By homogeneity, the same inequality will hold for $t\bar{u}$ for any $t > 0$. Then:

$$\begin{aligned} I_{k,\lambda}(t\bar{u}) &= \int_{\Omega} \left[-\frac{1}{(k+1)} (t\bar{u}) S_k(D^2(t\bar{u})) - \frac{1}{(k+1)} (1 + \lambda|t\bar{u}|)^{k+1} \right] dx \\ &< \int_{\Omega} \left[-\frac{t^{k+1}}{(k+1)} \bar{u} S_k(D^2\bar{u}) - \frac{1}{(k+1)} \lambda^{k+1} t^{k+1} |\bar{u}|^{k+1} \right] dx \\ &< (1 - \lambda) \frac{\lambda^k}{(k+1)} t^{k+1} \int_{\Omega} |\bar{u}|^{k+1} < 0. \end{aligned}$$

By taking t large, we get that $I_{k,\lambda}(t\bar{u}) < I_{k,\lambda}(u_{k,\lambda})$, which contradicts the fact that $u_{k,\lambda}$ minimizes $I_{k,\lambda}$.

It remains to prove that the inf of $I_{p,\lambda}(u)$ over Φ_0^k is attained. Let $M > 1$ and let $f_M(t)$ be a smooth, positive function such that

$$f_M(t) = \begin{cases} (1 + |t|)^p, & |t| \leq M \\ |t|^{-2}, & |t| \geq 2M, \end{cases}$$

and $|t|^{-2} < f_M(t) < 2(1 + |t|)^p$ for $|t| \in (M, 2M)$. We now define the following functional on Φ_0^k :

$$J_M(u) = \int_{\Omega} \left[\frac{-1}{(k+1)} u S_k(D^2 u) - F_M(u) \right] dx,$$

where $F_M(u) = \int_0^{|u|} f_M(t) dt$. Then critical points of $J_M(u)$ are solutions of $S_k(D^2 u) = f_M(u)$ with zero boundary data.

Our first observation is that $J_M(u)$ is bounded below. Since $u \in \Phi_0^k$, $u < 0$ and $S_k(D^2 u) \geq 0$, the first term in the integral defining J_M is bounded below by 0. Because of the decay of $f_M(t)$ at infinity, F_M is bounded above, and therefore, because Ω is bounded, the second term in J_M is bounded below as well. Let $d_M = \inf\{J_M(u) : u \in \Phi_0^k\}$.

Our next task is to show that d_M is attained. Let $\epsilon > 0$, and let $\phi_\epsilon^* \in \Phi_0^k \cap C^4(\bar{\Omega})$ be such that $J_M(\phi_\epsilon^*) \leq d_M + \frac{\epsilon}{2}$. (We can do this by choosing a minimizing sequence and using the density of smoother functions in $C^2(\bar{\Omega})$).

Let ϕ_ϵ solve the following problem:

$$\begin{cases} S_k(D^2u) = 1 - \eta + \eta S_k(D^2(\phi_\epsilon^*)) & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \quad (4)$$

where $\eta \in C_0^\infty(\Omega)$, $0 \leq \eta \leq 1$, and $\eta \equiv 1$ in the set $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \delta\}$. This problem has a solution because the right-hand side of the equation is strictly positive and regular. See [9] for a statement of this result.

Then $S_k(D^2\phi_\epsilon) = S_k(D^2\phi_\epsilon^*)$ in Ω_δ . By the comparison principle, Theorem 4, we have that

$$\sup_{\Omega_\delta} |\phi_\epsilon(x) - \phi_\epsilon^*(x)| \leq \sup_{\partial\Omega_\delta} |\phi_\epsilon(x) - \phi_\epsilon^*(x)|. \quad (5)$$

We use the comparison principle in the following way. For any $x \in \partial\Omega_\delta$,

$$\phi_\epsilon(x) \leq \phi_\epsilon^*(x) + \sup_{\partial\Omega_\delta} |\phi_\epsilon(x) - \phi_\epsilon^*(x)|.$$

Now note that $S_k(D^2(\phi_\epsilon^*)) = S_k(D^2(\phi_\epsilon^* + C))$ for any constant C . So by the comparison principle, with $u = \phi_\epsilon$ and $v = \phi_\epsilon^* + \sup_{\partial\Omega_\delta} |\phi_\epsilon(x) - \phi_\epsilon^*(x)|$, we have for any $x \in \Omega_\delta$,

$$\phi_\epsilon(x) \leq \phi_\epsilon^*(x) + \sup_{\partial\Omega_\delta} |\phi_\epsilon(x) - \phi_\epsilon^*(x)|.$$

Now we can reverse the roles of ϕ_ϵ and ϕ_ϵ^* to get inequality (5).

Now observe that

$$\sup_{\partial\Omega_\delta} |\phi_\epsilon(x) - \phi_\epsilon^*(x)| \leq \delta \sup_{\Omega} (|D\phi_\epsilon(x)| + |D\phi_\epsilon^*(x)|).$$

To see this let $x \in \partial\Omega_\delta$ and take $\bar{x} \in \partial\Omega$ such that $|x - \bar{x}| = \delta$. Then since ϕ_ϵ and ϕ_ϵ^* are zero on $\partial\Omega$,

$$|\phi_\epsilon(x) - \phi_\epsilon^*(x)| \leq |\phi_\epsilon(x) - \phi_\epsilon(\bar{x})| + |\phi_\epsilon^*(x) - \phi_\epsilon^*(\bar{x})|.$$

Now we can use the mean value theorem to get the previous estimate.

Since we can find a bound for $|D\phi_\epsilon|$ that is independent of δ (see appendix below), by choosing δ small, we can guarantee that $J_M(\phi_\epsilon) \leq d_M + \epsilon$. Why is this estimation possible? First, the ϕ_ϵ are uniformly bounded below by ϕ , the solution to the problem:

$$\begin{cases} S_k(D^2u) = 1 + S_k(D^2\phi_\epsilon^*) \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

Then $\tilde{\phi}$ is a subsolution of (4) for any δ , and 0 is a supersolution. Now the problem (4) has a unique solution (by the comparison principle). Therefore the solution guaranteed by Theorem 6 must be identical with ϕ_ϵ , and we must have $0 \geq \phi_\epsilon \geq \tilde{\phi}$. Then $|J_M(\phi_\epsilon) - J_M(\phi_\epsilon^*)|$ can be made as small as we like by choosing δ small. This is easiest to see by writing out the integrals defining J_M and then splitting them into an integral over Ω_δ , where $S_k(D^2\phi_\epsilon)$ and $S_k(D^2\phi_\epsilon^*)$ are identical and $\sup_{\Omega_\delta} |\phi_\epsilon(x) - \phi_\epsilon^*(x)|$ is small, and an integral over $\Omega \setminus \Omega_\delta$, where the integrands are bounded and the measure of the domain is small. The claim follows when we recall that $J_M(\phi_\epsilon^*) \leq d_M + \frac{\epsilon}{2}$.

The next step in the argument is to introduce a time-dependent problem:

$$\begin{cases} u_t - \log(S_k(D^2u)) = -\log f_M(u) \text{ in } Q \equiv \Omega \times (0, \infty) \\ u(x, 0) = \phi_\epsilon \\ u = 0 \text{ on } \partial\Omega \times (0, \infty). \end{cases} \quad (6)$$

By Theorems 9 and 10 (in the appendix), there exists $w_\epsilon(x, t) \in C^{3+\alpha, 2+\alpha/2}(\bar{Q})$ and $\|w_\epsilon\|_{C^{3+\alpha, 2+\alpha/2}(\bar{Q})} \leq C$, that solves problem (6). Notice that $w_\epsilon(x, 0) = \phi_\epsilon(x)$.

We now compute $\frac{d}{dt}J_M(w_\epsilon(\cdot, t))$, and use the fact that w_ϵ solves (6).

$$\begin{aligned} \frac{d}{dt}J_M(w_\epsilon(\cdot, t)) &= - \int_{\Omega} [S_k(D_x^2w_\epsilon) - f_M(w_\epsilon)] \frac{\partial}{\partial t}w_\epsilon(x, t) dx \\ &= - \int_{\Omega} [S_k(D_x^2w_\epsilon) - f_M(w_\epsilon)] \log \left(\frac{S_k(D_x^2w_\epsilon)}{f_M(w_\epsilon)} \right) dx. \end{aligned}$$

This is less than or equal to zero because if $S_k(D_x^2w_\epsilon)(x) \geq f_M(w_\epsilon(x))$, then the log term is positive, and if $S_k(D_x^2w_\epsilon)(x) < f_M(w_\epsilon(x))$, the log term is negative. Hence the integrand is nonnegative.

This implies that $d_M \leq J_M(w_\epsilon(\cdot, t)) \leq d_M + \epsilon$ for all $t \geq 0$. By the definition of d_M , $d_M \leq J_M(w_\epsilon(\cdot, t))$, because for each fixed t , $w_\epsilon(x, t) \in \Phi_0^k$. We also have that $J_M(w_\epsilon(\cdot, t))$ is nonincreasing in t and $J_M(w_\epsilon(\cdot, 0)) \leq d_M + \epsilon$.

Therefore, there must be a sequence $t_j \rightarrow \infty$ such that $\frac{d}{dt}J_M(w_\epsilon(\cdot, t_j)) \rightarrow 0$ as $j \rightarrow \infty$. If this were not true, then there would exist an $\epsilon_0 > 0$ such that for every positive integer N , there exists $t_N > N$ such that $\frac{d}{dt}J_M(w_\epsilon(\cdot, t_N)) <$

$-\epsilon_0$, but this would imply that for t sufficiently large, $J_M(w_\epsilon(\cdot, t)) < d_M$, contradicting the definition of d_M .

Because of the bound on w_ϵ , we can extract a subsequence $w_\epsilon(\cdot, t_j)$ that converges to a function $\tilde{v}_\epsilon(x)$ in $C^3(\bar{\Omega})$. We claim that \tilde{v}_ϵ solves the problem:

$$\begin{cases} S_k(D^2u) = f_M(u) \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

and we have that $d_M \leq J_M(\tilde{v}_\epsilon) \leq d_M + \epsilon$. The latter statement is true because it is true for $J_M(w_\epsilon(\cdot, t))$, and we have convergence in $C^3(\bar{\Omega})$. A similar observation shows that $\tilde{v}_\epsilon|_{\partial\Omega} = 0$. To see that the differential equation is satisfied, note that:

$$0 = \lim_{t_j \rightarrow \infty} \int_{\Omega} [S_k(D_x^2 w_\epsilon(x, t_j)) - f_M(w_\epsilon)(x, t_j)] \log \left(\frac{S_k(D_x^2 w_\epsilon(x, t_j))}{f_M(w_\epsilon(x, t_j))} \right) dx$$

but, the integrand is nonnegative for all $t \geq 0$, so it must go to zero. By the convergence, we have $S_k(D_x^2 w_\epsilon(x, t_j)) \rightarrow S_k(D^2 \tilde{v}_\epsilon(x))$, and $f_M(w_\epsilon(x, t_j)) \rightarrow f_M(\tilde{v}_\epsilon(x))$. Hence, $S_k(D^2 \tilde{v}_\epsilon) = f_M(\tilde{v}_\epsilon)$.

Since f_M is a bounded function, the \tilde{v}_ϵ are uniformly bounded. To see this let $K > 0$ be an upper bound for f_M , and let u_K solve

$$\begin{cases} S_k(D^2u) = K \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

Then by Theorem 4, $0 \geq \tilde{v}_\epsilon(x) \geq u_K(x)$. We also have that $f_M(\tilde{v}_\epsilon(x))$ is bounded below by a positive constant. Therefore by Theorem 7, $\|\tilde{v}_\epsilon\|_{C^{3,\alpha}(\bar{\Omega})} \leq C$, where C is a constant that does not depend on ϵ . We can therefore select a subsequence from the \tilde{v}_ϵ that converges (as $\epsilon \rightarrow 0$) in $C^3(\bar{\Omega})$ to a function v_M , which satisfies:

$$\begin{cases} S_k(D^2v_M) = f_M(v_M) \text{ in } \Omega \\ v_M|_{\partial\Omega} = 0. \end{cases} \quad (7)$$

We also have that $d_M = J_M(v_M)$, so we have finally established that the inf of J_M over Φ_0^k is attained.

The next step is to show that the v_M are uniformly bounded for $M \geq 1$. This claim is proved by contradiction. If the claim is not true then we can find a subsequence for which $R_M = -\inf_{\Omega} v_M \rightarrow \infty$ as $M \rightarrow \infty$. Define $u_M = \frac{v_M}{R_M}$. By the homogeneity of the k -Hessian, u_M satisfies

$$\begin{cases} S_k(D^2 u_M) = R_M^{-k} f_M(R_M u_M) \text{ in } \Omega \\ u_M|_{\partial\Omega} = 0. \end{cases}$$

Now since

$$R_M^{-k} f_M(R_M u_M) \leq \frac{2(1 + R_M |u_M|)^p}{R_M^k} \leq 2(1 + R_M)^p R_M^{-k}$$

which goes to zero as $M \rightarrow \infty$ (because $p < k$), we get that $R_M^{-k} f_M(R_M u_M)$ converges to zero uniformly. This implies that $u_M \rightarrow 0$ uniformly, as well, contradicting the fact that $\sup_{\Omega} |u_M| = 1$. To show that u_M converges uniformly to zero, we use the comparison principle. Let $\rho > 0$. Then for all M large enough, $R_M^{-k} f_M(R_M u_M) < \rho$. Let \tilde{u} be the unique solution of $S_k(D^2 u) = 1$, $u|_{\partial\Omega} = 0$. Then $\rho^{1/k} \tilde{u}$ is the unique solution of

$$\begin{cases} S_k(D^2 u) = \rho \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

By the comparison principle, $0 \geq u_M \geq \rho^{1/k} \tilde{u}$. Letting $\rho \rightarrow 0$, we get the uniform convergence of the u_M to zero. But this is a contradiction, because $\sup_{\Omega} |u_M| = 1$ for all M .

We have just shown that the v_M are uniformly bounded, say $|v_M| \leq C$. For any $M > C$, $f_M(v_M(x)) = (1 + |v_M(x)|)^p$. Since each v_M solves (7), this means that for M large, each v_M solves the problem

$$\begin{cases} S_k(D^2 u) = (1 + |u|)^p \text{ in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases}$$

But by Theorem 5, this problem has a unique solution. Therefore, for $M > C$, all of the v_M are identical to the solution of this problem, which we denote u_p . For all M , $J_M(v_M) = d_M$, so for M large enough, $d_M = J_M(u_p)$.

We next claim that u_p is a minimizer for $I_p \equiv I_{p,1}$. We compute

$$d_M = J_M(u_p) = \int_{\Omega} -\frac{1}{k+1} u_p S_k(D^2 u_p) - F_M(u_p).$$

When $M > \sup_{\Omega} |u_p|$, $f_M(t) = (1 + t)^p$ on the interval $(0, |u_p|)$, so

$$F_M(u_p) = \int_0^{|u_p(x)|} (1 + t)^p dt = \frac{1}{p+1} ((1 + |u_p(x)|)^{p+1} - 1),$$

so that for M large enough,

$$\begin{aligned}
d_M = J_M(u_p) &= \int_{\Omega} -\frac{1}{k+1} u_p S_k(D^2 u_p) - \frac{1}{p+1} ((1 + |u_p(x)|)^{p+1} - 1) \\
&= I_p(u_p) + \frac{1}{p+1} |\Omega|.
\end{aligned}$$

By modifying the functional we may ignore the constant term $\frac{1}{p+1}|\Omega|$. To complete the argument, let u be any function in Φ_0^k . Then for any M , $J_M(v_M) \leq J_M(u)$. For M large enough, $J_M(v_M) = I_p(u_p)$, so $I_p(u_p) \leq J_M(u)$. By the same argument used above, for $M > \sup_{\Omega} |u(x)|$, $J_M(u) = I_p(u)$. Therefore, $I_p(u_p) \leq I_p(u)$, proving the claim. The same argument can be used for the functional $I_{p,\lambda}$.

4.4.3 Appendix

In this section we list some additional auxiliary results used in the proof of the theorem.

We first show that $|D\phi_{\epsilon}|$ can be bounded independently of δ . We use Theorem 3.4 from [4]:

Theorem 8 *Suppose a k -convex u satisfies:*

$$\begin{cases} S_k(D^2 u) = \psi(x, u, Du) \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $\psi = \psi(x, z, p) \geq 0$ and is Lipschitz continuous. Then:

$$|Du| \leq C(n, k, h, \sup_{\Omega} |u|, \partial\Omega)$$

where h is a nonnegative function satisfying $\frac{h(t)}{t} \rightarrow 0$ as $t \rightarrow \infty$, and

$$|\psi_x| + |\psi_z| \cdot |p| + |\psi_p| \cdot |p|^2 \leq h(|p|^{2k+1})$$

as $|p| \rightarrow \infty$.

We can use this theorem to show that $|D\phi_{\epsilon}|$ is uniformly bounded if we can find such a function h that is independent of δ . Note in our case, the right-hand side of the pde depends only on x . We use $h(t) = t^{1/2}$. Now ϕ_{ϵ} solves

$$\begin{cases} S_k(D^2u) = 1 - \eta + \eta S_k(D^2\phi_\epsilon^*) \equiv f_\eta \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

We may assume that $|D\eta| \leq C\frac{1}{\delta}$, so $|Df_\eta| \leq C\frac{1}{\delta}$. Then for any δ , $|Df_\eta| \leq h(|p|^{2k+1}) = |p|^{k+1/2}$ for $|p|$ large. So the theorem gives us a uniform bound on $|D\phi_\epsilon|$.

We now turn to the results used for the time-dependent problem. See Appendix A of [11] for the proofs.

Suppose that $\Omega \subset \mathbb{R}^n$ is bounded, smooth and strictly convex. Let $Q = \Omega \times (0, \infty)$ and $Q_T = Q \cap \{0 < t < T\}$. Consider the problem

$$\begin{cases} Lu = u_t - \log(S_k(D_x^2u)) = g(x, t, u) \text{ in } Q \\ u|_{\partial Q} = \phi \end{cases} \quad (8)$$

where $\phi \in C^{4,3}(\bar{Q})$ and $g \in C^2(\bar{Q} \times \mathbb{R})$.

Theorem 9 *Suppose $\phi(x, 0) \in \Phi^k(\Omega)$, $L\phi = g(x, t, \phi)$ on $\partial\Omega \times (\{t = 0\})$, and there exist nonnegative constants C_1 and C_2 such that*

$$g(x, t, u) \geq -C_1 - C_2|u|.$$

Then, if $u \in C^4(Q) \cap C^3(\bar{Q})$ solves (8),

$$\|u\|_{C^{2,1}(\bar{Q}_T)} \leq C = C(\Omega, T, \|\phi\|_{C^{4,3}(\bar{Q}_T)}, C_1, C_2, g).$$

Furthermore, if $C_2 = 0$, $\|\phi\|_{C^{4,3}(\bar{Q})}$ is finite, and g is independent of t , then C is independent of T .

Theorem 10 *Under the same hypotheses of Theorem 9, there exists a solution $u \in C^{3+\alpha, 2+\alpha/2}(\bar{Q})$. Furthermore, if g is bounded below, then u satisfies $\|u\|_{C^{3+\alpha, 2+\alpha/2}(\bar{Q})} \leq C$.*

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