INDIRECT METHODS IN CALCULUS OF VARIATIONS

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1. Introduction

The problem of calculus of variations is finding functional forms for which given integrals assume maximum ,or in geometric language ,this calculus deals with the problem of finding paths of intergration for which integrals admit maximum or minimum values.
In one dimentional problem, we have the type

$$I(y) = \int_a^b f(x, y(x), y'(x))dx \tag{1}$$

called variational integral , over all real sufficiently smooth functions y over $[a, b]$

2. Indirect methods

To maximize (or) minimize I, one considers the necessary condition, namely the Euler-Lagrange equation :

$$\partial_y f - \frac{d}{dx}\partial_{y'} f = 0 \tag{2}$$

i.e if y is an extremal of I then y is a solution of (2). Therefore, instead of finding the mimimum of I one solves the associated Euler-Lagrange equation and use some tests to determine if the solution is actually a mimimum function of I.

2.1. Special cases of the E-L equation.

When f is independent of one of the three variables ,(2) has special forms :
1) $f(x, y, y') = f(x, y')$ then $f_{y'} = constant$
2) $f(x, y, y') = f(y, y')$ then $f - y'f_{y'} = constant$
3) $f(x, y, y') = f(x, y)$ then $f_y = 0$
The solution curves of (2) are called **extremals**.

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2.2. **Sufficient conditions for a weak extremum.**

2.2.1. *Second variation form.* Let y be an extremum and h be any curve in $C_0[a,b]$, then

$$\Delta I = I(y+h) - I(y)$$

using Taylor expansion we obtain

$$\Delta I = \int_a^b (f_y h + f_{y'} h')dx + \frac{1}{2}\int_a^b (f_{yy}h^2 + 2f_{yy'}hh' + f_{y'y'}h'^2)dx + o(\|h\|^2)$$

intergrating by parts gives

$$\int_a^b (f_y h + f_{y'} h')dx = \int_a^b (f_y h - \frac{d}{dx} f_{y'} h)dx + f_{y'} h|_a^b = 0$$

So

$$\Delta I = \frac{1}{2} I_2(h) + o(\|h\|^2)$$

where

$$I_2 = \int_a^b (f_{yy}h^2 + 2f_{yy'}hh' + f_{y'y'}h'^2)dx$$

Again integrating by parts we have

$$\int_a^b 2f_{yy'}hh'dx = -\int_a^b \frac{d}{dx} f_{yy'} h^2 dx$$

Thus

$$I_2 = \int_a^b (Ph'^2 + Qh^2)dx,$$

$$P(x) = f_{y'y'}, Q = f_{yy} - \frac{d}{dx} f_{yy'}$$

Now consider the Euler-Lagrange equation of I_2

$$-\frac{d}{dx}(Pk') + Qk = 0 \tag{3}$$

The ode above always has nonzero solutions whenever $P \neq 0$. We observe that y is a weak (local) minimum curve if $I_2 > 0$ for all h in $C_0[a,b]$ .We then proceed to find conditions which are both necessary and sufficient for I_2 to be > 0 , i.e , *positive definite* . We introduce an important concept :

Definition. The point $\tilde{a}$ ($\neq a$) is said to be conjugate to the point a if the equation (3) has a solution which vanishes for $x = a$ and $x = \tilde{a}$ but is not identically zero.

The following theorem is due to Legendre's idea.

Theorem 1 (Legrendre's test). *If the quadratic functional*

$$I_2 = \int_a^b (Ph'^2 + Qh^2)dx$$

where

$$P(x) > 0, (a \leq x \leq b)$$

is positive definite for all h *in* $C_0[a,b]$ *if and only if the interval* $[a,b]$ *contains no points conjugate to* a.

Therefore we have the necessary conditions for a weak extremum .

Theorem 2 (Jacobi's test). *Suppose that for some admissible curve* $y = y(x)$ *the functional (1) satisfies the following conditions :*
1. The curve $y = y(x)$ *is an extremum , i.e satisfies (2);*
2. Along the curve $y = y(x)$ *,*

$$P(x) = f_{y'y'}(x, y(x), y'(x) > 0$$

3. The interval $[a,b]$ *contains no points cinjugate to the point* a.
Then the functional (1) has a weak mimimum for $y = y(x)$.

Remark. There is a method in finding conjugate points .
If $y = y(x, c_1, c_2)$ is the general solution of (2) then the equation for any point conjugate to a is

$$\frac{\partial y}{\partial c_1} / \frac{\partial y}{\partial c_2} = (\frac{\partial y}{\partial c_1} / \frac{\partial y}{\partial c_2})_{x=a}$$

Finally , we introduce the sufficient conditions for a strong extremum.

Theorem 3. *The following conditions are sufficient to ensure a strong minimum for the functional (1)at an extremal curve* y *:*
1. y *satisfies (2);*
2. there is no points conjugate to either a *or* b*:*
3.

$$\frac{\partial^2 f(x, y, p)}{\partial p^2} > 0$$

for all finite values of p.

3. The brachristochrone problem

A partical moves without friction under the influence of gravity from the origin O to a point B below O . Find the least time curve, i.e the curve that takes least time to go from O to B .
Take O as the origin , take axis of y vertically downward . The velovity of the partical at a depth y is $\sqrt{2gy}$, g is the acceleration due to the gravity . For each curve y : O→B , the time is given by

$$I(y) = \int \frac{ds}{v} = \frac{1}{\sqrt{2g}} \int \sqrt{\frac{1+y'^2}{y}} dx$$

Since $f(y, y')$ is independent of x, the Euler-Lagrange equation if I is

$$f - y' f_{y'} = 2c$$

or

$$y(1+y'^2) = 2c$$

Let $y' = \tan\Psi$, then

$$y = c(1 + cos\Psi)$$

Since $dx = dy \cot\Psi = -2c \sin 2\Psi \cos\Psi d\Psi = -c(2 + 2cos2\Psi)d\Psi$,

$$x = d - c(2\Psi + \sin 2\Psi)$$

where d, c are constant determined by O and B .For example , at O , $x = y = 0$ so $\Psi = \pi/2$ and $d = \Psi c$.
The solution of the Euler-Lagrange, therefore, is a cycloid with the generating circle of radius c.
Next we will show that this cycloid is actually the least time curve.
Rewrite Ψ as a function of $\frac{d-x}{c}$, we have

$$\cos\Psi = \Phi(\frac{d-x}{c})$$

, where Φ is some function . Thus

$$y = c + c\Phi(\frac{d-x}{c})$$

Let

$$\eta_1 = \frac{\partial y}{\partial d} = \Phi'(\frac{d-x}{c}) = -\tan\Phi$$

$$\eta_2 = \frac{\partial y}{\partial c} = 1 + \Phi(\frac{d-x}{c}) - c\frac{d-x}{c}\Phi'(\frac{d-x}{c}) = 2(1 + \Psi\tan\Psi)$$

Thus

$$\frac{\eta_2}{\eta_1} = -2(\Psi + \cos\Psi)$$

$$\frac{d}{dx}(\frac{\eta_2}{\eta_1}) = -\frac{1}{2c\sin^2\Psi} < 0$$

This implies η_2/η_1 is strictly decreasing and thus there is no points conjugate to a or b. Finally , we have $f(x,y,p) = \sqrt{1+p^2}/\sqrt{y}$ and hence

$$f_{pp} = (1+p^2)^{-3/2}y^{-1/2} > 0$$

Therefore by the strong minimum theorem I achieves its minimum for the cycloid .