Dirichlet's Condition

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Abstract

A dominant technique in early variational approaches was Dirichlet's principle, which lost favor after some mathematicians (notably Weierstrass) pointed out its weaknesses. Here we will discuss Dirichlet's principle, its flaws, and its salvation via direct minimization methods. [3]

1 Introduction

The majority of the material in this paper was generated from course notes and handouts from the VIGRE minicourse on Variational Methods and Partial Differential Equations, held at the University of Utah from 28 May - 8 June 2002. [3].

The tie between extrema of integral expressions of the form

$$\int_a^b f(x, y(x), y'(x))\, dx$$

and solutions of partial differential equations (PDEs) of the form

$$\frac{d}{dx}\partial_{y'} f(x, y(x), y'(x)) - \partial_y f(x, y(x), y'(x)) = 0$$

was discovered independently by both Euler and Lagrange in the middle of the eighteenth century. At first mathematicians sought to exploit this relationship by solving the PDE in order to find maxima or minima of the integral expression. Near the middle of the 19th century, however, mathematicians began looking at this relationship in reverse. They sought to find solutions of the PDE by maximizing or minimizing the integral expression.

One of the first problems that was looked at this was was the so called *Dirichlet Problem*

$$\Delta y(x) = 0, \quad x \in \Omega, \quad y(x) = f(x), \quad x \in \partial\Omega.$$

The associated integral expression is

$$\int_\Omega |\nabla y(x)|^2 dx.$$

This integral expression is always nonnegative, and thus it was assumed that a function, y, which minimizes the expression must exist, and thus the PDE must have a solution. This became known as *Dirichlet's principle.*

In 1870, however this viewpoint was challenged when Weierstrass gave an example of an integral expression which is always nonnegative, yet fails to achieve its minimum. In the remainder of this paper, we will discuss two such counterexamples, and explore conditions that may be imposed on either the class of admissible functions or the integral expression itself to ensure that the minimum is achieved.

2 Counterexamples

Here we will look at two counterexamples to Dirichlet's principle, and discuss the behavior that causes them to fail.

Example 1. *Minimize*

$$\varphi(y) := \int_0^1 y^2 \, dx$$

over $C[0,1]$*, subject to*

$$y(0) = 0, \quad y(1) = 1.$$

Let $\mathcal{A} := \{y \in C[0,1] \mid y(0) = 0, \text{ and } y(1) = 1\}$, and consider the sequence $\{y_n\}$ defined by

$$y_n = x^n$$

Then for each $n \in \mathbb{N}$, y_n is continuous on $[0,1]$, $y_n(0) = 0$, and $y_n(1) = 1$. Therefore each $y_n \in \mathcal{A}$. Now, note that

$$\begin{aligned} \varphi(y_n) &= \int_0^1 y_n^2 \, dx \\ &= \int_0^1 x^{2n} \, dx \\ &= \frac{1}{2n+1} \end{aligned}$$

and we see that $\lim_{n\to\infty} \varphi(y_n) = 0$. Noting that $\varphi(y) \geq 0 \ \forall y$, this gives us that $\inf_{\mathcal{A}} \varphi(y) = 0$ and that $\{y_n\}$ is a minimizing sequence for φ.

So, we've identified the infimum of φ over $\mathcal{A}$, but does φ achieve this infimum? I claim it does not.

Suppose $y \in C[0,1]$ minimizes φ. Then

$$\varphi(y) = 0 \;\Rightarrow\; \int_0^1 y^2\,dx = 0 \;\Rightarrow\; y = 0 \text{ a.e.}$$

But y is continuous. Thus $y \equiv 0$, and so $y(1) \neq 1$.

The problem in this counterexample appears to be the domain. We have a nice, bounded minimizing sequence ($\|y_n\|_\infty \leq 1 \;\forall\; n \in \mathbb{N}$), yet $\{y_n\}$ does not converge to a point in our domain.

Now, our second counterexample:

Example 2. *Minimize*

$$\varphi(y) := \int_{-1}^1 [xy'(x)]^2\,dx$$

over $H^1[-1,1]$*, subject to*

$$y(-1) = a, \quad y(1) = b, \quad a \neq b$$

Without loss of generality, assume $b > a$. Let $\mathcal{A} := \{y \in H^1[-1,1] \mid y(-1) = a, \text{ and } y(1) = b\}$, and consider the sequence $\{y_n\}$ defined by

$$y_n(x) = \begin{cases} a & -1 \leq x \leq -\frac{1}{n} \\ \frac{a+b}{2} + \frac{(b-a)n}{2}x & -\frac{1}{n} \leq x \leq \frac{1}{n} \\ b & \frac{1}{n} \leq x \leq 1. \end{cases}$$

Note that $y_n \in \mathcal{A}\;\forall n \in \mathbb{N}$, and

$$y_n'(x) = \begin{cases} 0 & -1 \leq x \leq -\frac{1}{n} \\ \frac{(b-a)n}{2} & -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} \leq x \leq 1, \end{cases}$$

so

$$\begin{aligned} \varphi(y_n) &= \int_{-\frac{1}{n}}^{\frac{1}{n}} \left[x\frac{(b-a)n}{2}\right]^2 dx \\ &= \left(\frac{(b-a)n}{2}\right)^2 \int_{-\frac{1}{n}}^{\frac{1}{n}} x^2\,dx \\ &= \left(\frac{(b-a)n}{2}\right)^2 \left(\frac{x^3}{3}\right)\Bigg|_{-\frac{1}{n}}^{\frac{1}{n}} \\ &= \frac{(b-a)^2}{6n} \end{aligned}$$

So $\lim_{n\to\infty}\varphi(y_n) = 0$. As in the previous example, this tells us that $\inf_{\mathcal{A}}\varphi(y) = 0$ and that $\{y_n\}$ is a minimizing sequence for φ. So, does φ achieve its infimum? Again, I claim it does not. Note that if $y \in \mathcal{A}$, we know y is continuous and $y(-1) < y(1)$. Thus $y' > 0$ on some set of positive measure, and thus $\varphi(y) > 0$. So there is no $y \in \mathcal{A}$ such that $\varphi(y) = 0$.

This time there is no problem with the domain. We are working on a nice Hilbert space. This time, the problem lies with the functional itself. Note that

$$\begin{aligned} \int_{-1}^{1} y_n'(x)^2\, dx &= \left(\frac{(b-a)n}{2}\right)^2 \int_{-\frac{1}{n}}^{\frac{1}{n}} dx \\ &= \frac{(b-a)^2 n}{2}. \end{aligned}$$

So, $\lim_{n\to\infty} \|y_n\|_{H^1} = \infty$. Our minimizing sequence blows up. This results from the fact that φ is not coercive.

3 Correcting the Problem

As the above counterexamples show, a functional can fail to achieve its minimum due either to a problem with the domain, or to a characteristic of the functional itself. The particular requirements that guarantee a functional will achieve its minimum were not nailed down until the early twentieth century. The following theorem and corollary outline these requirements.

Theorem 3. *Let E be a reflexive Banach space, $C \subset E$ weakly closed, and $\varphi : C \to \mathbb{R}$ weakly lower semicontinuous. Then φ has a minimum over C if and only if φ has a bounded minimizing sequence.*

Corollary 4. *Let E be a reflexive Banach space, $C \subset E$ closed, convex, and $\varphi : C \to \mathbb{R}$ weakly lower semicontinuous and coercive. Then φ has a minimum over C.*

We now state and prove the following related result. This theorem and its proof are taken from [1].

Theorem 5. *Let U be a bounded set in R^n. Let*

$$L : R^n \times R \times U \mapsto R$$

be continuously differentiable in each variable and define

$$I[u] = \int_U L(\nabla u, u, x)\, dx$$

for $u \in C^1(\bar{U})$. Assume that L is bounded below, and in addition the mapping $p \mapsto L(p, z, x)$ is convex for each $z \in R$, $x \in U$. Then $I[\cdot]$ is weakly lower semicontinuous on $H^1(U)$.

Proof. Choose a sequence $\{u_k\}$ with

$$u_k \rightharpoonup u \text{ weakly in } H^1(U), \tag{1}$$

and let

$$l = \liminf_{k\to\infty} I[u_k].$$

Passing to a subsequence, assume

$$l = \lim_{k\to\infty} I[u_k].$$

Due to (1), $\{u_k\}$ is bounded in H^1. By Rellich's compactness theorem (described in e.g. [2]), there is a subsequence such that

$$u_k \to u \text{ in } L^2(U). \tag{2}$$

Once again passing to a subsequence, we have

$$u_k \to u \text{ almost everywhere in } U. \tag{3}$$

Pick $\varepsilon > 0$. By Egoroff's Theorem (see e.g. [4]) there exists a measurable set E_ε such that

$$u_k \to u \text{ uniformly in } U \tag{4}$$

and $\text{meas}(U - E_\varepsilon) \leq \varepsilon$. Let

$$F_\varepsilon = \{x \in U \,|\, |u(x)| + |\nabla u(x)| \leq \frac{1}{\varepsilon}\}. \tag{5}$$

Clearly $\text{meas}(U - F_\varepsilon) \to 0$ as $\varepsilon \to 0$. Let $G_\varepsilon = E_\varepsilon \bigcap F_\varepsilon$, noticing that $\text{meas}(U - G_\varepsilon) \to 0$ as $\varepsilon \to 0$. Without loss of generality, assume $L \geq 0$. Letting $D_p L(p, z, x)$ denote the derivative of L with respect to its $p-$variable, and using the convexity of L in the same variable, we see

$$\begin{aligned} I[u_k] &= \int_U L\left(\nabla u_k, u_k, x\right) dx & (6)\\ &\geq \int_{G_\varepsilon} L\left(\nabla u_k, u_k, x\right) dx & (7)\\ &\geq \int_{G_\varepsilon} L\left(\nabla u, u_k, x\right) dx + \int_{G_\varepsilon} D_p L\left(\nabla u, u_k, x\right) \cdot \left(\nabla u_k - \nabla u\right) dx & (8)\end{aligned}$$

Since u and ∇u are bounded on G_ε,

$$\lim_{k\to\infty} \int_{G_\varepsilon} L\left(\nabla u, u_k, x\right) dx \to \int_{G_\varepsilon} L\left(\nabla u, u, x\right) dx$$

by (4). Since we also have $\nabla u_k \rightharpoonup \nabla u$ in L^2,

$$\lim_{k\to\infty} \int_{G_\varepsilon} D_p L\left(\nabla u, u_k, x\right) \cdot \left(\nabla u_k - \nabla u\right) dx = 0$$

by the uniform convergence of u. Therefore

$$l = \lim_{k\to\infty} I[u_k] \geq \int_{G_\varepsilon} L\left(\nabla u, u, x\right) dx.$$

Letting ε go to 0 and recalling the Monotone Convergence Theorem gives

$$\lim_{k\to\infty} I[u_k] \geq \int_U L\left(\nabla u, u, x\right) dx.$$

□

References

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[4] H. L. Royden. *Real analysis*. Macmillan Publishing Company, third edition.