Bridgeland stability conditions on orbifolds
Franco Rota

Introduction

In this work, we:
- define numerical stability conditions on smooth Deligne-Mumford stacks and study them in a specific example of a root stack over a curve;
- study the stability manifold, and show that it’s contractible;
- show an example of wall-crossing which matches the Hecke correspondence for moduli of parabolic bundles.

Setup

The definition of a Bridgeland stability $\sigma = (P, Z)$ condition on a triangulated category $D$ includes:
- a slicing of $D$: a collection $P = \{P(\phi)\}_{\phi \in R}$ of full additive subcategories satisfying:
  - semi-orthogonality;
  - Harder-Narasimhan filtrations;
- a central charge $Z: K(D) \rightarrow \mathbb{C}$ where
  - $Z$ factors through a fixed finite rank quotient $K(D) \rightarrow \Lambda$;
  - $Z$ is compatible with the slicing: for any non-zero $E \in P(\phi)$, $Z(\gamma(E)) \in \mathbb{R}_{<0} e^{i\phi};$
  - $(P, Z)$ satisfies the support property.

Let $\text{Stab}_Y(D)$ denote the topological space of stability conditions on $D$.

**Theorem 0.1.** ([Bru07, Thm. 1.2]). The map $\text{Stab}_Y(X) \rightarrow \text{Hom}(K(C), \mathbb{C})$ given by $(P, Z) \mapsto Z$ is a local homeomorphism. In particular, $\text{Stab}_0(X)$ is a complex manifold of dimension $rk(\Lambda)$.

We consider stability conditions of the bounded derived category of coherent sheaves on an orbifold $X$.

There is a Chern character defined for smooth Deligne-Mumford stacks [AR03] taking values in the Chen-Ruan orbifold cohomology [CR04]:

$$\text{Ch}: K(X) \rightarrow H^{CR}_k(X, \mathbb{Q}).$$

We define $\Lambda$ to be the image of $\text{Ch}$, and $\text{Stab}(X)$ to be the corresponding stability manifold.

Example: a root stack

We consider the second root stack $X := \mathbb{C}^{\sqrt{2}} \to C$ of a point $p$ on a smooth curve $C$ of positive genus.

i. The effective “root” line bundle $O(p/2)$ produces objects:

$$O \rightarrow O(p/2) \rightarrow t \quad \quad O\left(p\right) \rightarrow O(p) \rightarrow t';$$

ii. $t$ and $t'$ described above are exceptional, $\text{Hom}(t, t') = \text{Hom}(t', t) = 0$ and the only non-trivial extensions are

$$t \to \mathbb{C} \to t' \quad \quad t' \to \mathbb{C} \to t.$$

There is a semi-orthogonal decomposition for $D(X) = (\tau, \tau(D(C)))$ [BLS16], so there are extra summands

$$K(X) = K(C) \oplus [t] \quad \quad H^2_{CR}(X) = H^*(C) \oplus [t].$$

The stability manifold

To construct stability conditions, we focus on the slice $S \subset \text{Stab}(X)$ whose central charges restrict to Mumford slope:

$$\sigma = (P, Z) \in S \quad \iff \quad Z = -\text{deg} + i \text{rk}.$$

The only conditions with heart $\text{Coh}(X)$ have $Z(t), Z(t') \in (-1, 0)$. With analytic continuation, one finds two charts of $S$:

$$U = \{\sigma \in S \text{ s.t. } t \text{ is } \sigma\text{-stable} \} \quad \quad V = \{\sigma \in S \text{ s.t. } t' \text{ is } \sigma\text{-stable} \}$$

which are included in $\mathbb{C}$ by coordinates

$$\sigma \in U \mapsto \ln|Z_\sigma(t)| + i\pi \phi(g(t)) \quad \quad \sigma \in V \mapsto \ln|Z_\sigma(t')| + i\pi \phi(g(t'))$$

Roughly speaking, $U$ is a subset of the universal cover

$$C' = \{Z(t) : \sigma \in U\}.$$

The hearts appearing are built out of $\tau^n \text{Coh}(C)$ and $t$. For example:

- $B_1 = \tau^n \text{Coh}(C) \oplus [(2t)]$
- $C_2 = \tau^n \text{Coh}(C)(p/2) \oplus [(t-2)]$
- $A_2 = (t', t'[−1])$

In progress: Up to the action of $GL^+(2, \mathbb{R})$, there are all numerical stability conditions.

Wall-crossing and parabolic bundles

Consider the moduli space of parabolic bundles of rank 2 and degree 0 on a curve of genus at least 2. A parabolic bundle is a vector bundle $E$ with a given surjection $E_p \rightarrow \mathbb{C}_p$ at a point of $C$. Then the short exact sequence $F \rightarrow E \rightarrow \mathbb{C}_p$ determines maps:

$$M_{par}(2, 0) \quad \quad M(2, 0) \quad \quad M(2, -1)$$

This is called the Hecke correspondence.

By a result of Borne [Bor07], there’s an equivalence between parabolic bundles on $C$ and vector bundles over the second root stack $X = \mathbb{C}^{\sqrt{2}}$. The space $M_{par}(2, 0)$ corresponds to $M(C)$ with $\rho = (2,0) \oplus [t] \in H^2_{CR}(X, \mathbb{Q})$, for $\sigma$ in the inverse image of the shaded region in the picture. The walls are given by the triangles for $E$ in $M(C)$.

Perspectives

- quotient stacks of elliptic curves;
- kleinian singularities;
- mirror symmetry for orbifold del Pezzo surfaces.

Acknowledgements

This work wouldn’t have been possible without the guidance of my advisor, Aaron Bertram.

References


