SYMMETRIC PRODUCTS AND THE HILBERT SCHEME OF POINTS

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Abstract. These notes aim at giving a first glance to the problems related to the study of configuration spaces in algebraic geometry, presenting symmetric products and Hilbert schemes of points. They include topics from a series of lectures given by Izzet Coskun at the school Stability conditions on triangulated categories and geometric applications, held in Nordfjordeid, Norway, June 2016.

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1. Introduction

Given a smooth projective variety \( X \) over the field of the complex numbers, we aim to understand the configuration spaces of \( X \), i.e. the space of unordered \( n \)-tuples of points of \( X \):

\[
C_n(X) = \{ \{ p_1, ..., p_n \} \text{ s. t. } p_i \in X \text{ distinct} \}.
\]

A general remark about these spaces is that they are not compact. Indeed, endow \( X \) with the analytic topology, and consider the product \( X^n \), and the quotient map \( q : X^n \to X^n / S_n \) by the action of the symmetric group which permutes the copies of \( X \). The quotient

\[
X^{(n)} = X^n / S_n
\]

is called the \( n \)-th symmetric product of \( X \). Clearly, we have an inclusion \( C_n(X) \subset X^{(n)} \). If we let \( \Delta \subset X^n \) be the closed locus of ordered \( n \)-tuples in which two or more points collide, we see that the inverse image of \( C_n(X) \) is \( X - \Delta \), which is not compact. However, this argument also suggests a very natural compactification of \( C_n(X) \), that is the symmetric product itself. After introducing some concepts which will be useful to deal with this topic, we dedicate a section to the properties of
the symmetric powers. Despite being the most obvious compactifications, it turns out that the Hilbert scheme of points is better behaved, at least in dimension 2. In particular, a finite length subscheme of a surface is always smoothable, i.e. it is a deformation of a smooth subscheme. When we increase the dimension to 3, we see that the smoothable locus does not cover the whole Hilbert scheme, and many subschemes are not smoothable.

2. Preliminaries

In this section we introduce some concepts and tools that are going to be useful in the notes. Rather than an exhaustive introduction, this will be a summary of useful definitions, facts and examples. Hartshorne’s book [5] offers a thorough tration of these topics.

Schemes. The main object of our study is the spectrum of a ring. The idea behind this concept is to study an algebraic variety by understanding polynomial (often called regular) functions defined on it. In this way, an affine variety is identified with the ring \( A \) of its regular functions. The spectrum of the ring \( A \), \( \text{Spec} A \), is the set of all its prime ideals. Among these, we can identify the maximal ideals with points of \( Z \). A scheme is the result of a glueing of spectra of rings. We are particularly interested in schemes of length zero, we will only see affine examples. Let us work in the field of complex numbers \( \mathbb{C} \). A scheme of length zero is then \( \text{Spec} A \), where \( A \) is a finite dimensional vector space over \( \mathbb{C} \).

Example 2.1. Consider the affine line \( \mathbb{A}^1 \), and \( n \) distinct points \( a_i \) on it. Let \( f = \prod (x - a_i) \) be the ideal defining those points on the line. The scheme

\[
X = \text{Spec} \mathbb{C}[x]/(f)
\]

is a subscheme of length \( n \) of \( \mathbb{A}^1 \). Indeed, the module \( \mathbb{C}[x]/(f) \) is a finitely generated complex vector space, with basis \((1, x, x^2, \ldots, x^{n-1})\). The topological space corresponding to \( X \) is the set of maximal ideals of \( \mathbb{C}[x]/(f) \). This corresponds exactly to the points \( a_i \) in \( \mathbb{A}^1 \). Regular functions on \( X \) are described by an \( n \)-tuple of values, one for every point, hence they are given by a skyscraper sheaf consisting of one copy of the field \( \mathbb{C} \) for every point in the support.

Example 2.2. The last example dealt with distinct points on a variety, but the interesting phenomena appear when we make any of two points collide. In this case, the ring of regular functions contains nilpotent elements. Schemes of this kind are called non-reduced. For example, consider the scheme

\[
\text{Spec} \mathbb{C}[x]/(x^2).
\]

This time, we have a scheme of length two (with basis \((1, x)\)) which is supported only on one point (the origin, corresponding to the unique maximal ideal \((x)\)). Functions defined on this scheme are given by a skyscraper sheaf supported at the origin, given by a copy of the ring \( A = \mathbb{C}[x]/(x^2) \). This is isomorphic to two copies of the field, but with a peculiar \( \mathbb{C}[x] \) module structure. We can think of this subscheme as specifying a point and a vector tangent to \( \mathbb{A}^1 \) at that point.

\[1\text{an affine variety is the zero set of some polynomials in the affine space, or an open set thereof.} \]
Example 2.3. In the case of a curve, there is only one possible tangent direction to it. Therefore it is intuitively clear that a scheme like the above is what we get when we allow points of a finite length scheme to collide. However, the situation isn’t as clear when we raise the dimension. Depending on the reciprocal position of the points we allow to collide, we get two different structures in the limit: consider $\mathbb{A}^2$ this time, and the subscheme of length 2 supported at the origin and at the point $(0, a)$ with $a \neq 0$. The ring corresponding to this scheme is

$$A = \mathbb{C}[x, y] / (x, y(y - a)).$$

If we allow $(0, a)$ to collide with the origin, we get the subscheme $\text{Spec} \mathbb{C}[x, y] / (x, y^2)$. It is supported at the origin, but it remembers it comes from colliding two points moving vertically. In this sense, it is different from $\text{Spec} \mathbb{C}[x, y] / (x^2, y)$ (this is supported at the origin but it carries an horizontal tangent direction). In fact, it turns out that the space parametrizing length 2 subschemes of the affine plane, also denoted $\mathbb{A}^{2[2]}$, is obtained by blowing up the symmetric product along the diagonal $\Delta$:

$$\mathbb{A}^{2[2]} = \text{Bl}_{\Delta}(\mathbb{A}^{2(2)}).$$

We will see that this is in fact a resolution of singularities of $\mathbb{A}^{2(2)}$.

Sheaves and cohomology. We have seen that there is a tight relation between rings and schemes, through the “geometric version” of a ring which is its spectrum. We can carry over to geometry the notion of a module over a ring, the corresponding notion will be a sheaf of modules. We will skip the details concerning these definitions, and focus on the sheaf theoretic correspondent to the following exact sequence: given a ring $A$ and an ideal $I \subseteq A$, we have

$$0 \rightarrow I \rightarrow A \rightarrow A / I \rightarrow 0.$$ (1)

Geometrically, this sequence relates regular functions on a given variety $X$ to the ones on a subvariety $Z$, the relation being regulated by the equations cutting out the subvariety. This will hopefully be clearer in the next example.

Example 2.4. Consider a variety $Z$ in the affine space $\mathbb{A}^n$. Assume that $Z$ is defined by the vanishing of polynomials $f_1, \ldots, f_n$. Regular functions on affine $n$-space are polynomials in $n$ variables $A = \mathbb{C}[x_1, \ldots, x_n]$. Indeed, for $f \in A$, we have

$$f : \mathbb{A}^n \rightarrow \mathbb{C}$$

$$(a_1, \ldots, a_n) \mapsto f(a_1, \ldots, a_n).$$

Denote by $I = (f_1, \ldots, f_n)$. The function $f|_Z$ on $Z$ is the same as any function $(f + I)|_Z$, since all elements in $I$ vanish on $Z$.

Therefore we can give the following interpretation to the sequence (1): regular functions on $Z$ are determined by polynomials, up to the defining equations of $Z$. To carry over the same concept from affine schemes to schemes, we need to give a “sheafified” version of (1). We will write

$$0 \rightarrow I_Z \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0$$ (2)
for a subvariety \(Z\) of \(X\). One way to go back to vector spaces from the ideal sheaf sequence is to apply the global section functor \(H^0\) to the sheaves in question. This functor however is not exact, i.e. it does not preserve the exactness of the sequence. Instead, we get a long exact sequence

\[
0 \rightarrow H^0(X, I_Z) \rightarrow H^0(X, \mathcal{O}_X) \rightarrow H^0(X, \mathcal{O}_Z) \rightarrow \\
H^1(X, I_Z) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_Z) \rightarrow \\
H^2(X, I_Z) \rightarrow H^2(X, \mathcal{O}_X) \rightarrow H^2(X, \mathcal{O}_Z) \rightarrow \ldots
\]

where each \(H^i(X, \mathcal{O}_X)\) is a finitely generated \(\mathbb{C}\)-vector space, the \(H^i\)'s are called the \(i\)-th cohomology groups of the sheaves.

The global section functor isn’t the only one we can apply to the ideal sheaf sequence. For example, later on we will need to apply the functor \(\text{Hom}(-, \mathcal{O}_Z)\) which is not exact but only left exact. It returns the long exact sequence:

\[
0 \rightarrow \text{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \rightarrow \text{Hom}(\mathcal{O}_X, \mathcal{O}_Z) \rightarrow \text{Hom}(I_Z, \mathcal{O}_Z) \rightarrow \\
\text{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \rightarrow \text{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) \rightarrow \text{Ext}^1(I_Z, \mathcal{O}_Z) \rightarrow \\
\text{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \rightarrow \text{Ext}^2(\mathcal{O}_X, \mathcal{O}_Z) \rightarrow \text{Ext}^2(I_Z, \mathcal{O}_Z) \rightarrow \ldots
\]

### 3. (Non) Smoothness of Symmetric Products

In this section we study the smoothness of the symmetric products of \(X\), and have a look at some examples.

**Proposition 3.1.** If \(X\) is a smooth curve, the symmetric product \(X^{(n)}\) is also smooth.

**Proof.** Here we only give an idea of the argument. The claim is easy to prove for \(\mathbb{A}^1\): the ring of invariants \(k[x_1, \ldots, x_n]^S_n\) is a polynomial ring in \(n\) variables, generated by the elementary symmetric polynomials in the \(x_i\), hence the symmetric powers of \(\mathbb{A}^1\) are isomorphic to the affine space of the same dimension, hence smooth. It is left to show that this is enough, we are not going to show this in detail. The idea is the following: locally analytically every smooth point of a curve is isomorphic to the origin of the affine line. The same holds for the products of the curve and of the line. Then it is enough to check that taking completions commutes with taking \(S_n\) invariants. \(\square\)

**Exercise 3.2.** This result can be proved with more powerful technology. Recognize that symmetric powers of curves are Hilbert schemes (looking at the Hilbert-Chow morphism), and use deformation theory (cfr. Theorem 4.4) as in the proof of Theorem 4.2.

**Remark 3.3.** If \(X\) is a curve of genus \(g\), the symmetric product \(X^{(n)}\) comes equipped with a natural map

\[
X^{(n)} \xrightarrow{h} \text{Pic}^n(X) \\
\{p_1, \ldots, p_n\} \mapsto \mathcal{O}_X(\Sigma p_i).
\]

By Riemann-Roch we know that if \(n \geq 2g - 1\) (so that \(h^1(K - \Sigma p_i) = 0\)) then \(h\) is a \(\mathbb{P}^{n-\delta}\) fibration, the fiber being the complete linear system of the divisor \(\Sigma p_i\). For lower \(n\), Brill-Noether theory gives a description of this map.
Example 3.4. There is a nice description of the symmetric powers of $\mathbb{P}^1$, in fact we have $\mathbb{P}^1(n) \simeq \mathbb{P}^n$. Consider an unordered $n$-tuple of points in $\mathbb{P}^1$ and write it $\{[u_i : v_i] | i = 1, \ldots, n\}$. This is an element of $\mathbb{P}^1(n)$. This datum can be used to give a polynomial that vanishes exactly at the points $[u_i : v_i]$. Take for example

$$\prod (u_i y - x v_i)$$

and write it $a_0 x^n + a_1 x^{n-1} y + \ldots + a_n y^n$ for some coefficients $a_i$. Now, the roots of the polynomial are uniquely determined by the homogeneous tuple $[a_0 : \ldots : a_n]$ of the coefficients, this is a point in $\mathbb{P}^n$.

If the dimension of $X$ exceeds 1, the symmetric product $X^{(n)}$ is singular.

Example 3.5. One can check that the symmetric power $\mathbb{C}^2(n)$ is singular by looking at its tangent space: if this exceeds the dimension of the variety at a point, then that point is singular. Let us consider, for example, the point $\{0, \ldots, 0\}$ in $\mathbb{C}^2(n)$. Recall that the tangent space at a point is dual to $m/m^2$, where $m$ is the maximal ideal defining the point. The coordinate ring of $\mathbb{C}^2(n)$ is the invariant part of the polynomial ring $\mathbb{C}[x_1, y_1, x_2, y_2, \ldots, x_n, y_n]$, and the ideal of $\{0, \ldots, 0\}$ in $\mathbb{C}^2(n)$ is generated by invariant polynomials which vanish at $\{0, \ldots, 0\}$. In particular, all symmetric polynomials in the $x_i$ and $y_i$ belong to $m$ (we get $2n$ generators), but also the polynomial $x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$ lies in $m$, and is linearly independent from the others in $m/m^2$. Therefore the symmetric power is singular at zero.

Example 3.6. More explicitly, let us compute $\mathbb{C}^2(2)$. We start with $(\mathbb{C}^2)^2 = \mathbb{C}^4$. The nontrivial element of $S_2$ acts on a point by $(a, b, c, d) \mapsto (c, d, a, b)$. Consider the points

$$(1, 0, 1, 0)$$
$$(0, 1, 0, 1)$$
$$(1, 0, -1, 0)$$
$$(0, 1, 0, -1).$$

This choice of a basis allows us to write the action of $S_2$ in a nice form: it acts as the identity $I$ on the first two points, and as $-I$ on the last two. Hence, we can write the ring of regular functions of $\mathbb{C}^2(2)$ as the invariant part of the ring $\mathbb{C}[x, y, w, z]$ under the action

$$x \mapsto x$$
$$y \mapsto y$$
$$w \mapsto -w$$
$$z \mapsto -z.$$
4. The Hilbert scheme of points

Definition 4.1. Let $X$ be a smooth projective variety. We denote by $X^{[n]}$ the scheme parametrizing subschemes of $X$ of length $n$. The scheme $X^{[n]}$ is called the Hilbert scheme of $n$ points of $X$.

There is a lot to be said about the Hilbert scheme, its construction revolves around a simple idea by Grothendieck: instead of studying subschemes of $X$, it is useful to consider the corresponding ideals, and to parametrize these ones with a scheme. A careful construction of these spaces is carried out in [3].

The following theorem solves the problem encountered with the symmetric products.

Theorem 4.2 (Fogarty, [1]). Let $X$ be a smooth, projective surface. Then the Hilbert scheme $X^{[n]}$ is an irreducible, smooth variety of dimension $2n$. Furthermore, there is a map

$$(6) \quad h : X^{[n]} \to X(n)$$

$$(7) \quad h : Z \mapsto \sum_{p \in X} \text{length}_p(Z) \cdot p$$

called the Hilbert-Chow morphism, which is a resolution of singularities of $X^{(n)}$.

Proof. Fix a point $Z$ in $X^{[n]}$, let $I_Z$ denote the sheaf of ideals defining the subscheme $Z$. The strategy is to study the tangent space of $X^{[n]}$ at $Z$. Using our deformation theory result (Theorem 4.4), we know we can identify the tangent space with $\text{Hom}(I_Z, \mathcal{O}_Z)$. Consider the sequence

$$(8) \quad 0 \to I_Z \to \mathcal{O}_X \xrightarrow{\varphi} \mathcal{O}_Z \to 0$$

and apply the functor $\text{Hom}(-, \mathcal{O}_Z)$ to it. We get a long exact sequence

$$(9) \quad 0 \to \text{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) \xrightarrow{\varphi} \text{Hom}(\mathcal{O}_X, \mathcal{O}_Z) \to \text{Hom}(I_Z, \mathcal{O}_Z) \xrightarrow{\beta} \text{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) \to 0 \to \text{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \to 0$$

where $\text{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) = \text{Ext}^2(\mathcal{O}_X, \mathcal{O}_Z) = 0$ using the fact that $Z$ is zero dimensional. The map $\alpha$ is an isomorphism (all maps from $\mathcal{O}_X$ to $\mathcal{O}_Z$ factor through the quotient $q$), hence $\beta$ is an isomorphism, and $\text{Hom}(\mathcal{O}_Z, \mathcal{O}_Z)$ has dimension $n$. So does $\text{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z)$ by Serre duality. We claim that $\chi(\mathcal{O}_Z, \mathcal{O}_Z) = 0$, hence $\text{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z)$ has dimension $2n$ and so does the tangent space. This shows that $X^{[n]}$ is smooth. If we show that it is connected, then these two together imply that it is irreducible. First, let us prove the claim by direct computation: take a locally free resolution of $\mathcal{O}_Z$, apply the functor $\text{Hom}(-, \mathcal{O}_Z)$ to get a complex. Check that the ranks of the homology of such complex with alternating signs sum to zero. The connectedness requires an inductive argument. Since $X^{[1]} \simeq X$, it is connected. Suppose that $X^{[n-1]}$ is connected, and consider the diagram

...
\[ I = \{(F, F') \text{ s.t. } F \in X^{[n-1]}, F' \in X^{[n]}, F \subset F'\} \]

The fibre of \( \gamma \) over \( F \) is a copy of \( X \) blown up at the support of \( F \) so it’s connected. This also shows that \( \gamma \) is a projective map. By [4, TAG 0377], a closed map with connected fibres induces a bijection on connected components, hence \( I \) is connected.

Since \( \delta \) is surjective, then \( X^{[n]} \) is connected. □

In the next example, we present another kind of problem that arises when the dimension of \( X \) increases to 3. Unlike the case of a surface, in which all non reduced subschemes \( Z \) deform to a smooth one, in dimension 3 there are plenty of non reduced, non smoothable subschemes of finite length. It is still an open question to describe the smoothable subschemes.

**Example 4.3** (Iarrobino, [2]). Consider \( \mathbb{C}^3 \), pick a point \( p \) and choose coordinates so that \( p \) is the origin. Now, let \( V \) be a 24 dimensional subspace of the space of homogeneous degree 7 polynomials. Let \( m_p \) be the ideal generated by all homogeneous degree 8 polynomials, and define \( I_{V,p} = (V + m_p) \). This ideal vanishes at the origin and only at the origin (it is enough to notice that \( x^8 = y^8 = z^8 = 0 \) already only have the origin as zero). The length of this scheme is

\[ \dim \mathbb{C}[x, y, z]/I_{V,p} = 96 \]

because all degree 8 or higher polynomials are in \( I_V \), all degree 0, ..., 6 polynomials survive and only a 12 dimensional space of septics survive. Counting these we have \( 1 + 3 + 6 + 10 + 15 + 21 + 28 + 12 = 96 \). The dimension of schemes constructed as \( I_{V,D} \) is \( \dim \text{Gr(24,36)} + 3 = 291 \) (choice of the point \( p \) and the 24 dimensional subspace \( V \) of degree 7 polynomials). The dimension of the locus of smoothable schemes is \( \dim \mathbb{C}^{3(96)} = 3 \times 96 = 288 \). Hence, the general scheme we constructed cannot be smoothable.

**A result in deformation theory.** There is a very general result concerning Hilbert schemes, we state it here in the way that is useful for us. The theorem is stated and proved in [3] or, in full generality, in [6].

**Theorem 4.4.** Let \( X \) be a smooth projective variety, and \( Z \) a length \( n \) subscheme defined by the ideal \( I_Z \). Then, the Zariski tangent space of \( X^{[n]} \) at \( Z \) is isomorphic to \( \text{Hom}_{O_Z}(O_X, I_Z) \).

**Remark 4.5.** We can have a bit of intuition for the result above if we consider the similar case in which the subscheme is a divisor \( D \) of a smooth projective \( X \). In that case, \( \text{Hom}_{O_X}(I_D, O_D) \simeq \text{Hom}_{O_X}(O_X(-D), O_D) \simeq H^0(X, O_D(D)) \). It makes sense to think of a deformation of the subscheme as a global section of its normal bundle.

5. **From here**

The Hilbert scheme of points on a surface provides an interesting testing ground to some recent techniques in birational geometry. In particular, the birational geometry of the Hilbert scheme of points on \( \mathbb{P}^2 \) is well known ([7]), there is a
precise correspondence between wall-crossing in the Mori cone and wall-crossing in
the Bridgeland stability manifold. The more general setting of moduli spaces of
sheaves on $\mathbb{P}^2$ is surveyed in [8], which collects results from a series of papers on
the topic.

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