Math 4510-1 Summer, 2003 Introduction to Topology

There will be no problems due for this section. The final exam will be held during class hour on Tuesdaay, Jul 29 and will consist of short-answer questions on the material of the second half of the course: Elementary Topology.

Supplementary Notes, part IV

Connectness

Definition. Let X be a topological space. X is **connected** if and only if X cannot be written as a disjoint union of two nonempty open sets.

Here are some equivalent formulations:

1. A topological space X is connected if and only if X cannot be written as a disjoint union of two nonempty closed sets.

2. Let X be a connected topological space. Then whenever we write $X = U \cup V$ with U and V open and nonempty, then $U \cap V \neq \emptyset$.

Definition. Let A be a subset of a topological space X. A is **disconnected** if we can find sets U and V, open in X, such that $A \subset U \cup V$ and $A \cap U \neq \emptyset$, $A \cap V \neq \emptyset$. A is connected if it is not disconnected.

Examples

1. Let X be the unit interval $[0, 1]$. Then the set of rational points of X is disconnected. So is the set of irrational points.

2. Any subset of the Cantor set with two points is disconnected.

3. **Proposition**. $[0, 1]$ is connected. In fact, any interval is connected.

Proposition. The closure of a connected set is connected.

Propositon. Let A and B be connected subsets of X. If $A \cap B \neq \emptyset$, then $A \cup B$ is connected.

Propositions 9C and 9D state the appropriate generalization to larger unions.

Definition. Let X be a topological space. Define the equivalence relation on X: xEy if there exists a connected set which contains both x and y. For $x \in X$, the **connected component of** x is the set $C_x = \{y \in X : yEx\}$. The sets C_x are the **connected components** of X.

Proposition. Every point in X belongs to precisely one connected component. That is, X is the disjoint union of its components. The connected components are closed subsets of X.

Proposition. A connected subset of $\mathbf{R}$ is an interval. An open set on the line is a disjoint union of open intervals.

Proposition. Let $f : X \to Y$ be a continuous map. If A is connected in X, then $f(A)$ is connected in Y.

Proposition. Let f be a continuous function on the interval $[a, b]$. Then for every γ between $f(a)$ and $f(b)$, there is a $c \in [a, b]$ such that $f(c) = \gamma$.

Examples in $\mathbf{R^n}$.

1. A subset A of R^n is **starlike** if there is a point a_0 in A such that for every other point $a \in A$, the line segment joining a_0 to a is in A. Starlike sets are connected.

9.27 is an example of a subset of R^2 which is surprisingly, connected. Look also at the graph of $\sin(1/x)$.

Path-Connectedness

Definition. Let X be a topological space. A **path** in X is the image of a continuous map $s : [0, 1] \to X$.

Note that a path is directed: it has an initial point and an endpoint. If s is a path, then s^{-1} denotes the path in the reverse direction. If s_1 and s_2 are paths such that the endpoint of s_1 is the initial point of s_2, we can combine them to form the product path $s_1 s_2$.

Definition. X is **path-connected** if , for any two points x, y in X, there is a path from x to y.

Note that 9.27 provides an example of a connected set in R^2 which is not path connected. (We could define path-components as we did components; then this example has 2 path-components.)

Proposition. An open set in R^n is connected if and only if it is path-connected.

Hausdorff spaces

Definition. A topological space X is **Hausdorff** if and only if, for $x \neq y$, x and y have disjoint neighborhoods. More precisely, there are open sets U, V with $U \cap V = \emptyset$, and $x \in U,\ y \in V$.

Examples

Any metric space is Hausdorff.

Let X be the curve (in polar coordinates in the plane) $r = \theta, 0 \leq \theta \leq 2\pi$, as a point set. For two rays R_1, R_2 (straight lines emanating from the origin), let (R_1, R_2) be the set of points in X lying in the cone between R_1 and R_2 (going counterclockwise). Then the collection of (R_1, R_2) is a base for a non-Hausdorff topology on X, because the points $\theta = 0$ and $\theta = 2\pi$ have the same neighborhoods.

Compactness

Definition. Let (X, Ω) be a topological space, A a subset of X. A is **compact** if and only if every open covering of A has a finite subcover. To be precise, an **open covering** of A is a collection $\mathcal{C} \subset \Omega$ of open sets such that, for $x \in A$, there is a $U \in \mathcal{C}$ for which $x \in U$. Then A is compact if this condition is satisfied: for any open covering $\mathcal{C}$ of A, there are $U_1, \ldots, U_n$ in $\mathcal{C}$ which form an open covering of A.

Examples

1. A finite set is compact. A space X with the trivial topology is compact. The arrow is compact.

2. $\mathbf{R^n}$ is not compact; in fact, any open ball $B(0, r)$ in R^n is not compact.

3. An unbounded set in $\mathbf{R^n}$ is not compact.

3. A closed subset of a compact set is compact.

Proposition. In a Hausdorff space a compact set is closed.

Proposition. A compact set in R^n is closed and bounded.

Proposition. Let $f : X \to Y$ be a continuous map. If X is compact, so is $f(X)$.

Proposition. Let X be a compact space, $f : X \to R$ a continuous function. Then there are points a, b in X such that

$$f(a) = \max\{f(x); x \in X\} \qquad f(b) = \min\{f(x); x \in X\} .$$

So, what are the compact subsets of $\mathbf{R^n}$?

Proposition. A closed bounded set in R^n is compact.

To prove this we will need an important theorem of Lebesgue;

Lebesgue's Theorem. Any covering of any set in R^n has a countable subcovering.

Now, the proof goes like this;

1. A closed bounded interval I in $\mathbf{R}$ is compact.

This is the heart of the proof (and is really just the Heine-Borel theorem - for those of you who remember that). Suppose we have a countable cover $U_1, U_2, ...$ of I which has no finite subcover. Then, for every n, there is an $x_n \in I$ not in $U_1 \cup \cdots \cup U_n$. Let a be the least upper bound of the set of x_n. Since I is closed, $a \in I$, so there is an N such that $a \in U_N$. But alas, all but finitely many of the x_n are also in U_N (since a is the least upper bound of the x_n), so for some $n > N$, $x_n \in U_1 \cup \cdots \cup U_n$, a contradiction.

2. A closed bounded rectangle in R^2 is compact.

This is a slightly compicated unravelling of what it means to be a covering in a product space, and is the basis of the induction proof to get to the general R^n.

3. A closed bounded set in R^2 is compact, because it is a closed subset of a closed bounded rectangle.

Proposition. Let X and Y be compact Hausdorff spaces. A continuous bijection $f : X \to Y$ is a homeomorphism.

Let $g : Y \to X$ be the inverse to f. We'll show that , for every closed set C in X, $g^{-1}(C)$ is closed in Y. Suppose that C is closed in X. Then C is compact, so $f(C)$ is compact in Y, so is closed in Y. But $f(C) = g^{-1}(C)$.