Definitions and Examples

A sequence is a list of numbers. That is, there is a first number, a second number, and for every natural number \( n \) an \( n \)th number. Sequences are written in the form

\[
a_1, a_2, a_3, a_4, \ldots ,
\]

where the \( \ldots \) means that the pattern goes on forever. Sometimes the pattern is clear:

\[
1, 2, 3, 4, \ldots
\]

is telling us that, for any \( n \), the \( n \)th term is \( n \). Sometimes the pattern is not so clear:

\[
4, 14, 42, 59, 86, 125, \ldots
\]

These are the numbered stops of the Lexington Avenue express in New York. In fact, this is not a sequence: there is no \( n \)th stop for any \( n \). So, besides being undecipherable, this was a trick. We’ll refrain from tricks, and make the sense of the sequence clear. For example,

\[
1, 3, 6, 10, 15, \ldots
\]

may not be so clear, so instead we’ll write

\[
1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, 1 + 2 + 3 + 4 + 5, \ldots
\]

Of course, the most clear way of identifying a sequence is by an explicit formula for the \( n \)th term. Example:

\[
6, 12, 20, 30, 42, \ldots
\]

is not so terribly clear, but this is:

\[
6, 12, 20, 30, 42, \ldots, n^2 + 3n + 2, \ldots .
\]

Another way of describing a sequence is to explain how, once we have reached a number in a sequence, to calculate the next one. This is called the recursive definition of the sequence (and a
powerful tool in the use of computers in mathematics). So, sequence (1) could have been described this way:

\[ a_1 = 6. \] Once we have \( a_n \), we define \( a_{n+1} = a_n + 2(n + 2) \).

So, here \( a_2 = a_1 + 2(1 + 2) = 6 + 2(3) = 12 \), and \( a_3 = a_2 + 2(2 + 2) = 12 + 8 = 20 \), and so forth.

\( n! \) (called \( n \) factorial) is defined in two ways:

1. For any \( n \), \( n! \) is the product of the first \( n \) natural numbers.
2. For any \( n \), \( n! = n(n-1)! \).

Basic Classes of Sequences

An arithmetic sequence \( a_1, a_2, a_3, \ldots \) is a sequence where there is some number \( d \) such that \( a_{n+1} = a_n + d \) for every natural number \( n \). In other words, a sequence \( \{a_n\} \) is arithmetic, if the differences \( a_n - a_{n-1} \) of successive terms are all the same.

Examples

a) \( \{-7, -1, 5, 11, 17, \ldots\} \) is (the beginning of) an arithmetic sequence with \( d = 6 \).

b) \( \{-0, 1, 3, 6, 10, \ldots\} \) is not an arithmetic sequence because the difference between successive terms changes.

c) \( \{5, 10, 20, 40, 80, \ldots\} \) is a geometric sequence with \( r = 2 \).

d) \( \{1, -1, 1, -1, 1, \ldots\} \) is a geometric sequence with \( r = -1 \).

e) \( \{100, 50, 25, 12.5, 6.25, \ldots\} \) is a geometric sequence with \( r = 1/2 \).

f) \( \{1, 1/2, 1/4, 1/8, 1/16, \ldots\} \) is a geometric sequence with \( r = 1/2 \).

Note that the definition of arithmetic and geometric sequences is recursive in the sense that the rule tells how to get from one term to its successor. So, the initial term always has to be made explicit, as examples e) and f) show: they have the same ratio, but start in different places. Sometimes we have to specify several initial terms to define the sequence unambiguously.

The Fibonacci sequence: \( a_k = a_{k-1} + a_{k-2} \). To get started, we have to specify the first two terms:

\[ 1, 1, 2, 3, 5, 8, 13, \ldots \quad 7, -1, 6, 5, 11, 16, \ldots \]

These are both Fibonacci sequences, (the first is the one studied by Fibonacci); and most generally:

\[ a, b, a + b, a + 2b, 2a + 3b, 3a + 5b, \ldots \]

Formula for the general term of a sequence

When a sequence is given recursively, in principle, we cannot find the 100th term until we know the 99th, and so forth. However, it may be possible to find a formula that will tell us that; indeed, what the \( n \)th term is for any \( n \). Sometimes this is easy, but often it is quite hard.
Problems 1-8

1. What is the formula for the $n$th term of the sequence $1, 2, 3, 4, 5, \ldots$?

2. What is the formula for the $n$th term of the sequence $1, 4, 9, 16, 25, \ldots$?

3. What is the formula for the $n$th term of the arithmetic sequence $2, 6, 10, 14, 18, \ldots$?

4. What is the 232nd term of the sequence in problem 3?

5. What is the formula for the $n$th term of the arithmetic sequence $a, a + d, a + 2d, a + 3d, \ldots$?

6. What is the formula for the $n$th term of the sequence $1, -1, 1, -1, 1, -1, \ldots$?

7. What is the formula for the $n$th term of the sequence $3, 6, 12, 24, 48, \ldots$?

8. What is the formula for the $n$th term of the sequence $a, ar, ar^2, ar^3, \ldots$?

Formula for sums of arithmetic and geometric sequences

Given a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ we can create a new sequence by summing the terms of the sequence:

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, \ldots$$

This sequence is sometimes called the series whose general term is $a_n$, meaning precisely this: the $n$ term is the sum of the first $n$ terms of the $a_n$ sequence. Letting $S_n$ represent the $n$th term of the series, we can write

$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i,$$

or, recursively: $S_1 = a_1$, $S_n = S_{n-1} + a_n$. Can we find a formula for the $n$th term of the series? In general, the answer is usually no, but for arithmetic and geometric series, it can be done, by special tricks more than 2000 years old.

Sums of Arithmetic Series

Consider the sequence of natural integers: $1, 2, 3, 4, 5, \ldots$. The series $S_n$ formed of this sequence has as its $n$th term the sum of the first $n$ integers. These are called triangular numbers; why?

Can you find a formula for $S_n$? Here is a trick that works: first, consider an $(n + 1) \times (n + 1)$ square: meaning a square made up of $n + 1$ boxes on the horizontal and $n + 1$ boxes on the vertical. There are $(n + 1)^2$ boxes. The diagonal has $n + 1$ boxes, and when the diagonal is taken out, there are two arrays of $S_n$ boxes each. Thus:

$$(n + 1)^2 - (n + 1) = 2S_n.$$

Problem 9. Solve for $S_n$. 

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