Spring 2016
Algebra Qualfying Exam Solutions

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1. Prove that any group of order 345 is cyclic.

   Solution: Let G be a group of order 345. Let P_p be the Sylow $p-$subgroups. By Sylow theorems, we have that P_{23} is normal in G. We know that $P_{23} \cong \mathbb{Z}_{23}$ is cyclic. P_5 is also normal in G and $P_5 \cap P_3 = \{1\}$, so $P_3 P_5 = H$ is a subgroup of G of order 15. This subgroup of order 15 is also cyclic. So we have that $G = P_{23} \rtimes_\varphi H$ where $\varphi : H \to \text{Aut}(P_{23})$. The only map is the trivial one so $G = P_{23} \times H \cong \mathbb{Z}_{23} \times \mathbb{Z}_{15}$ which is cyclic.

2. Prove that there are no simple groups of order 90.

   Solution: Let G be a group of order 90. Let n_p be the number of Sylow $p-$subgroups. Suppose G is not simple so $n_5 = 6$ and $n_3 = 10$. We have 24 nontrivial elements in the Sylow $5-$subgroups and 80 nontrivial elements in the Sylow $3-$subgroups, this accounts for more than 90 elements, a contradiction.

3. Write down the rational and Jordan forms of the following matrix (viewed over $\mathbb{C}$):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{pmatrix}$$

   Solution: The characteristic polynomial of the matrix is $(x-1)^2(x+1)^2$ and the minimal polynomial of the matrix is $(x-1)(x+1)^2$. The rational canonical form and the Jordan form are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

4. Prove that $n \times n$ matrices A and B have the same characteristic polynomial if and only if the trace of A^k equals the trace of B^k for each $k \leqslant 1$.

   Solution: If A and B have eigenvalues $\lambda_1, \ldots, \lambda_n$ and the eigenvalues of A^k and B^k are $\lambda_1^k, \ldots, \lambda_n^k$. So the trace is $\lambda_1^k + \cdots + \lambda_n^k$ for A^k and B^k.
   Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be the eigenvalues of A and B, respectively. Then the elementary polynomials are the same and $a_i = b_i$ after reordering, so A and B have the same eigenvalues. Therefore the same characteristic polynomial.

5. Let R be a Noetherian ring. Prove that any surjective ring homomorphism $\varphi : R \to R$ is an isomorphism.

   Solution: We have $\ker\varphi \subset \ker\varphi^2 \subset \ldots \ker\varphi^n \subset \ldots$ must stabilize, say at $\ker\varphi^n$. We want to show that $\ker\varphi^n \cap \operatorname{Image}\varphi^n = \{0\}$. Let $x \in \ker\varphi^n \cap \operatorname{Image}\varphi^n$ so $\varphi^n(x) = 0$ and $x = \varphi(y)$ for some $y \in R$. So $\varphi^{2n}(y) = 0$ and $y \in \ker\varphi^{2n} = \ker\varphi^n$, then $\varphi^n(y) = 0$ and $x = 0$. Therefore $\ker\varphi^n \cap \operatorname{Image}\varphi^n = \{0\}$. φ is surjective so $\operatorname{Image}\varphi^n = R$. However $\ker\varphi^n \subset R$, so $\{0\} = \ker\varphi \cap R = \ker\varphi$ and φ is an isomorphism.

6. Calculate the Galois group $p(x) = x^3 + 3x + 2$, viewed as a polynomial over $\mathbb{Q}$.

   Solution 1: The discriminant of the polynomial is $D = -4(3^3) - 27(2^2) = -216$ hence D is not a square in $\mathbb{Q}$. Therefore, the Galois group is S_3.

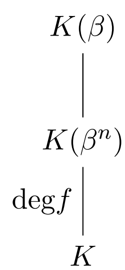
   Solution 2: Hannah's solution without using discriminants will be added soon.

7. Let K be a field and $f(x) \in K[x]$ an irreducible polynomial. Let $n \geqslant 2$ be an integer and set $g(x) = f(x^n)$. Prove that if $h(x)$ is an irreducible factor of $g(x)$, then the degree of f divides the degree of h.

   Solution: Let β be a root of $h(x)$. Then β is a root of $g(x)$ since $h(x)$ is a factor of $g(x)$. Thus

   $$g(\beta) = f(\beta^n) = 0,$$

   and β^n is a root of $f(x)$. Since $h(x)$ and $f(x)$ are irreducible, the degrees of the extensions $K(\beta)/K$ and $K(\beta^n)/K$ must be $\deg h$ and $\deg f$, respectively. Therefore, we have the following tower of fields:



   where $[K(\beta) : K] = \deg h$. So $\deg f | \deg h$.

8. Let p be a prime number. Prove that if $x^{p^n} - x + 1$ is irreducible over $\mathbb{F}_p$ then $n = 1$ or $n = 2 = p$.
   Hint: If α is a root, $\alpha^p - \alpha$ is in $\mathbb{F}_p$.

   Solution: Hannah's solution will be added soon.

9. Let $R = \mathbb{Z}[x]$, set $I = (x^3)$ and let U be the multiplicatively closed set $\{n \in \mathbb{Z} : n \text{ is odd}\}$. Find all the prime ideals in the ring $(U^{-1}R)/U^{-1}I$

   Solution: The total rings of fractions was not covered in the 2015-2016 algebra sequence, so we assume this material will not show up on the Fall 2016 qualifying exam.

10. Prove that the $\mathbb{Z}-$module $\mathbb{Q}$ is not projective.

    Solution: A projective module must be a summand of a free module and a $\mathbb{Z}-$ free module is not divisible, and so cannot contain $\mathbb{Q}$.