Spring 2015
Algebra Qualifying Exam Solutions

Hannah Hoganson, Allechar Serrano
August 5, 2016

In the problems below, $K$ denotes a field; $\mathbb{F}_p$ denotes the field with $p$ elements.

1. Let $\sigma$ be a 5-cycle in $S_5$. How many elements of $S_5$ commute with $\sigma$? How many elements of the subgroup $A_5$ commute with $\sigma$? How many conjugacy classes of 5 cycles are there in $A_5$?

Solution 1: The set of elements which commute with $\sigma$ in $S_5$ is the centralizer $C_{S_5}(\sigma)$ and from the orbit-stabilizer theorem, with the action of conjugation, we know

$$[S_5 : C_{S_5}(\sigma)] = |Cl_{S_5}(\sigma)|$$

Conjugacy classes in $S_5$ are elements of the same cycle type, so $|Cl_{S_5}(\sigma)|$ is the number of 5-cycles in $S_5$ which is $4 \cdot 3 \cdot 2 \cdot 1$, so $|C_{S_5}(\sigma)| = 5$.

Let $\sigma = (a_1, a_2, a_3, a_4, a_5)$, then we can write $\sigma = (a_1, a_5)(a_1, a_4)(a_1, a_3)(a_1, a_2)$ so $\sigma$ is even and is in $A_5$. For any integer $k$, $\sigma^k = ((a_1, a_5)(a_1, a_4)(a_1, a_3)(a_1, a_2))^k$ and is a product of $4k$ 2-cycles so is also even. Any power of $\sigma$ will commute with $\sigma$, and $\sigma$ has order five, so the elements in $S_5$ which commute with $\sigma$ are exactly

$$\{\sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5 = 1\} \subset A_5$$

Because $A_5 < S_5$ elements in $A_5$ commute with $\sigma$ in $A_5$ if and only if they commute as elements of $S_5$. Thus the number of elements in $A_5$ that commute with $\sigma$ is also five.

Now from the orbit stabilizer theorem with the conjugation action inside $A_5$ we know

$$|Cl_{A_5}(\sigma)| = [A_5 : C_{A_5}(\sigma)] = \frac{5 \cdot 4 \cdot 3}{5} = 12$$

So the size of the conjugacy class of $\sigma$ is twelve and there are 24 total five cycles, so there must be two distinct conjugacy classes of 5 cycles in $A_5$.

Solution 2: The number of $m$–cycles in $S_n$ is $\frac{n(n-1)\ldots(n-m+1)}{m}$. The number of 5–cycles in $S_5$ is 24. Hence, by Orbit-Stabilizer theorem there are 5 elements in $S_5$ which commute with $\sigma$. It is easy to see that these 5 elements are the powers of $\sigma$, hence they are in $A_5$. There are 12 number of elements in $A_5$ that commute with $\sigma$. Since the centralizer of $\sigma$ is contained in $A_5$, then the conjugacy class splits in $A_5$ (also, $24 \nmid |A_5|$).

2. Prove that up to isomorphism there are at most 4 different group of order 30.

Solution: $30 = 2 \cdot 3 \cdot 5$ so there are subgroups of order 2, 3, 5. From the Sylow theorems we know $n_5 \in \{1, 6\}$ and $n_3 \in \{1, 10\}$. If $n_5 = 6$ then $n_3 \neq 10$ because then

$$|G| \geq 6(5 - 1) + 10(3 - 1) + 1 > 30$$
5. Determine, up to conjugacy, the elements of order 4 in \( GL_3(\mathbb{Q}) \)

**Solution:** Elements of order 4 must satisfy the polynomial \( x^4 - 1 \), but not the polynomials \( x^k - 1 \) for \( k = 1, 2, 3 \). The minimal polynomial must divide \( x^4 - 1 = (x^2 + 1)(x - 1)(x + 1) \), then

<table>
<thead>
<tr>
<th>Minimal polynomial</th>
<th>Characteristic polynomial</th>
<th>Rational canonical form</th>
</tr>
</thead>
<tbody>
<tr>
<td>((x^2 + 1)(x - 1))</td>
<td>( x^3 - x^2 + x - 1 )</td>
<td>[\begin{pmatrix} 0 &amp; 0 &amp; 1 \ 1 &amp; 0 &amp; -1 \ 0 &amp; 1 &amp; 1 \end{pmatrix}]</td>
</tr>
<tr>
<td>((x^2 + 1)(x + 1))</td>
<td>( x^3 + x^2 + x + 1 )</td>
<td>[\begin{pmatrix} 0 &amp; 0 &amp; -1 \ 1 &amp; 0 &amp; -1 \ 0 &amp; 1 &amp; -1 \end{pmatrix}]</td>
</tr>
</tbody>
</table>

6. Let \( M \) be a \( 3 \times 3 \) matrix over \( \mathbb{C} \) with trace\((M^k) = 0 \) for \( k = 1, 2, 3 \). Prove that \( M \) is nilpotent.

**Solution:** Let \( a, b, c \) be the eigenvalues of \( M \). Then the eigenvalues of \( M^k \) are \( a^k, b^k, c^k \). We want to show that 0 is the only eigenvalue of \( M \). We have that
Consider the polynomial ring \( \mathbb{Q}[x] \). Prove that \( p \) is irreducible by Eisenstein’s criteria with \( \sigma \) a generating set for the ideal of polynomials such that \( \sqrt{\sigma} \) is an element of \( \mathbb{Q} \) or \( \mathbb{Z} \). Hence one eigenvalue, say \( a \) is equal to 0. Now, since trace(\( M \)) = 0, we have that \( b = -c \). Also, \( \text{trace}(M^2) = a^2 + b^2 + c^2 = 2b^2 = 0 \) so \( b = c = 0 \). Therefore, \( M \) is nilpotent.

7. Consider the polynomial ring \( \mathbb{Q}[x, y] \) where \( x, y \) are indeterminates over \( \mathbb{Q} \). Determine a finite generating set for the ideal of polynomials \( f(x, y) \) with \( f(i, i) = 0 \)

**Solution:** We don’t know how to do this problem. If you figure it out, you should let us know!

8. Suppose \( K \subset L \subset M \) are fields with \( [M : L] = 2 = [L : K] \). Prove that \( M = K(\alpha) \), where \( \alpha \) is a root of an irreducible polynomial in \( K[x] \) of the form \( x^4 + bx^2 + c \).

**Solution:** We didn’t have time to get to this problem. If someone works on it and writes a solution, please send it to Anna and she’ll update this.

9. Prove that \( \mathbb{Q}(\sqrt{5} + \sqrt{5}) \) is Galois over \( \mathbb{Q} \), and compute the Galois group

**Solution:** \( \sqrt{5} + \sqrt{5} \) is a root of the polynomial \( (x^2 - 5)^2 - 5 = x^4 - 10x^2 + 20 \) which is irreducible by Eisenstein’s criteria with \( p = 5 \) and has roots \( \{\pm\sqrt{5} \pm \sqrt{5}\} \). Because \( (\sqrt{5} + \sqrt{5})^2 = 5 + \sqrt{5} \) we have \( \sqrt{5} \in \mathbb{Q}(\sqrt{5} + \sqrt{5}) \). Note that

\[
\sqrt{5 + \sqrt{5}} \cdot \sqrt{\sqrt{5} - \sqrt{5}} = \sqrt{20} = 2\sqrt{5}
\]

so \( \sqrt{5 - \sqrt{5}} \) is also in \( \mathbb{Q}(\sqrt{5} + \sqrt{5}) \). Thus, \( \mathbb{Q}(\sqrt{5} + \sqrt{5}) \) is a splitting field of an irreducible polynomial over \( \mathbb{Q} \) and as such is Galois.

\[ [\mathbb{Q}(\sqrt{5} + \sqrt{5}) : \mathbb{Q}] = 4 \] so the Galois group has size four and is isomorphic to either \( \mathbb{Z}/4\mathbb{Z} \) or \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). We know the Galois group acts transitively on the roots so there is an element \( \sigma \) such that \( \sigma(\sqrt{5} + \sqrt{5}) = \sqrt{5} - \sqrt{5} \). Then \( \sigma(5 + \sqrt{5}) = \sigma(\sqrt{5} + \sqrt{5})^2 = 5 - \sqrt{5} \) so \( \sigma(\sqrt{5}) = -\sqrt{5} \). Now

\[
\sigma^2\left(\sqrt{5} + \sqrt{5}\right) = \sigma\left(\sqrt{5 - \sqrt{5}}\right) = \sigma\left(\frac{2\sqrt{5}}{\sqrt{5} + \sqrt{5}}\right) = \frac{-2\sqrt{5}}{\sqrt{5} - \sqrt{5}} = -\sqrt{5} + \sqrt{5}
\]
So, $\sigma^2 \neq id$ and must have order four, so $\text{Gal}(\mathbb{Q}(\sqrt{5 + \sqrt{5}})/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$.

10. Let $K$ be a field of characteristic $p > 0$, and let $t$ be an indeterminate. Consider the automorphism $\sigma \in \text{Aut}_K K(t)$ with $\sigma(t) = t + 1$. Determine the subfield of $K(t)$ that is fixed by $\sigma$.

**Solution:** $\sigma$ has order $p$, then $K(t)/K(t)^\sigma$ has extension degree equal to $p$ since it is a Galois extension and its Galois group is isomorphic to $\langle \sigma \rangle$. We have that $K(t^p - t) \subseteq K(t)^\sigma$ and $[K(t) : K(t^p - t)] = p$ so $K(t)^\sigma = K(t^p - t)$. 