Spring 2015
Algebra Qualfying Exam Solutions

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In the problems below, K denotes a field; $\mathbb{F}_p$ denotes the field with p elements.

1. Let σ be a 5-cycle in S_5. How many elements of S_5 commute with σ? How many elements of the subgroup A_5 commute with σ? How many conjugacy classes of 5 cycles are there in A_5?

   Solution 1: The set of elements which commute with σ in S_5 is the centralizer $C_{S_5}(\sigma)$ and from the orbit-stabilizer theorem, with the action of conjugation, we know

   $$[S_5 : C_{S_5}(\sigma)] = |Cl_{S_5}(\sigma)|$$

   Conjugacy classes in S_5 are elements of the same cycle type, so $|Cl_{S_5}(\sigma)|$ is the number of 5-cycles in S_5 which is $4 \cdot 3 \cdot 2 \cdot 1$, so $|C_{s_5}(\sigma) = 5$.

   Let $\sigma = (a_1, a_2, a_3, a_4, a_5)$, then we can write $\sigma = (a_1, a_5)(a_1, a_4)(a_1, a_3)(a_1, a_2)$ so σ is even and is in A_5. For any integer k, $\sigma^k = ((a_1, a_5)(a_1, a_4)(a_1, a_3)(a_1, a_2))^k$ and is a product of $4k$ 2-cycles so is also even. Any power of σ will commute with σ, and σ has order five, so the elements in S_5 which commute with σ are exactly

   $$\{\sigma, \sigma^2, \sigma^3, \sigma^4, \sigma^5 = 1\} \subset A_5$$

   Because $A_5 < S_5$ elements in A_5 commute with σ in A_5 if and only if they commute as elements of S_5. Thus the number of elements in A_5 that commute with σ is also five.

   Now from the orbit stabilizer theorem with the conjugation action inside A_5 we know

   $$|Cl_{A_5}(\sigma)| = [A_5 : C_{A_5}(\sigma)] = \frac{5 \cdot 4 \cdot 3}{5} = 12$$

   So the size of the conjugacy class of σ is twelve and there are 24 total five cycles, so there must be two distinct conjugacy classes of 5 cycles in A_5.

   Solution 2: The number of $m-$cycles in S_n is $\frac{n(n-1)\cdots(n-m+1)}{m}$. The number of $5-$cycles in S_5 is 24. Hence, by Orbit-Stabilizer theorem there are 5 elements in S_5 which commute with σ. It is easy to see that these 5 elements are the powers of σ, hence they are in A_5. There are 12 number of elements in A_5 that commute with σ. Since the centralizer of σ is contained in A_5, then the conjugacy class splits in A_5 (also, $24 \nmid |A_5|$).

2. Prove that up to isomorphism there are at most 4 different group of order 30.

   Solution: $30 = 2 \cdot 3 \cdot 5$ so there are subgroups of order $2, 3, 5$. From the Sylow theorems we know $n_5 \in \{1, 6\}$ and $n_3 \in \{1, 10\}$. If $n_5 = 6$ then $n_3 \neq 10$ because then

   $$|G| \geq 6(5-1) + 10(3-1) + 1 > 30$$

So there is only one three group, call it P_3, and it is normal. Let P_5 be one of the 5-groups, then $N = P_3P_5$ is a subgroup of order 15 and because $P_3 \cap P_5 = \{1\}$, $N = P_3 \rtimes_\psi P_5$ where $\psi : P_5 \to \mathrm{Aut}(P_3)$. That is $\psi : \mathbb{Z}/5\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ so $\psi = id$ and $N \cong \mathbb{Z}/15\mathbb{Z}$.

If $n_5 = 1$ then we also have a subgroup $N \cong \mathbb{Z}/15\mathbb{Z}$ because P_5 will be normal and $N = P_5P_3 = P_5 \rtimes_\psi P_3$ where $\psi : P_3 \to \mathrm{Aut}(P_5)$. That is $\psi : \mathbb{Z}/3\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$ so $\psi = id$ and $N = \mathbb{Z}/15\mathbb{Z}$.

So, any group of order 30 has a subgroup $N \cong \mathbb{Z}/15\mathbb{Z}$ which is normal because $[G : N] = 2$. Let P_2 be a 2-sylow subgroup, then $NP_2 \leq G$ and because $N \cap P_2 = \{1\}$ we have $NP_2 = G$. So, $G = N \rtimes_\varphi P_2$ where $\varphi : P_2 \to \mathrm{Aut}(N)$. Because $\mathrm{Aut}(\mathbb{Z}/15\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ we can interpret φ as a map from $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. φ is determined by where it maps the generator, 1, of $\mathbb{Z}/2\mathbb{Z}$. In $\mathbb{Z}/2\mathbb{Z}$ the element 1 has order 2 so $\varphi(1) \in \{(0,0),(0,2),(1,0),(1,2)\}$. Thus there are at most 4 nonisomorphic groups of order 30.

3. Let H be a proper subgroup of a finite group G. Prove that $\bigcup_{g\in G} gHg^{-1}$ does not equal G.

   Solution: Let $m = [G : H]$, so $|G| = m|H|$. There are at most m conjugate subgroups of H in G. Each of these subgroups contains the identity so

$$\left| \bigcup_{g\in G} gHg^{-1} \right| \leq m\left(|H| - 1\right) + 1 = |G| - m + 1$$

   and because H is a proper subgroup $m \neq 1$ so $\left|\bigcup_{g\in G} gHg^{-1}\right| < |G|$ and they can not be equal.

4. Set $G = GL_2(\mathbb{C})$. Construct a proper subgroup H of G such that $\bigcup_{g\in G} gHg^{-1}$ equals G.

   Solution: Let H be the subgroup of upper triangular matrices. Since every element in G is conjugate to an upper triangular matrix (for example, its Jordan form), then $G = \bigcup_{g\in G} gHg^{-1}$.

5. Determine, up to conjugacy, the elements of order 4 in $GL_3(\mathbb{Q})$

   Solution: Elements of order 4 must satisfy the polynomial $x^4 - 1$, but not the polynomials $x^k - 1$ for $k = 1, 2, 3$. The minimal polynomial must divide $x^4 - 1 = (x^2 + 1)(x - 1)(x + 1)$, then

Minimal polynomial	Characteristic polynomial	Rational canonical form
$(x^2+1)(x-1)$	$x^3 - x^2 + x - 1$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$
$(x^2+1)(x+1)$	$x^3 + x^2 + x + 1$	$\begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$

6. Let M be a 3×3 matrix over $\mathbb{C}$ with $\mathrm{trace}(M^k) = 0$ for $k = 1, 2, 3$. Prove that M is nilpotent.

   Solution: Let a, b, c be the eigenvalues of M. Then the eigenvalues of M^k are a^k, b^k, c^k. We want to show that 0 is the only eigenvalue of M. We have that

$$
\begin{aligned}
&(a+b+c)^3 = \text{trace}(M)^3 = 0\\
\Leftrightarrow &\underbrace{a^3+b^3+c^3}_{\text{trace}(M^3)=0} + 3ac^2 + 3a^2b + 3a^2c + 3bc^2 + 3b^2c + 3ab^2 + 6abc = 0\\
\Leftrightarrow &\, 3ac^2 + 3a^2b + 3a^2c + 3bc^2 + 3b^2c + 3ab^2 + 6abc = 0\\
\Leftrightarrow &\, 3a^2(b+c) + 3b^2(a+c) + 3c^2(a+b) + 6abc = 0\\
\Leftrightarrow &\underbrace{3a^2(-a) + 3b^2(-b) + 3(-a-b)^2(a+b)}_{\text{since trace}(M)=0,\text{ so } c=-a-b} + 6abc = 0\\
\Leftrightarrow &\, 9a^2b + 9ab^2 + 6abc = 0\\
\Leftrightarrow &\, ab\underbrace{(3a+3b+2c)}_{=a+b\text{ since trace}(M)=0} = 0\\
\Leftrightarrow &\, ab\underbrace{(a+b)}_{=-c} = 0\\
\Leftrightarrow &\, abc = 0
\end{aligned}
$$

Hence one eigenvalue, say a is equal to 0. Now, since $\text{trace}(M) = 0$, we have that $b = -c$. Also, $\text{trace}(M^2) = a^2 + b^2 + c^2 = 2b^2 = 0$ so $b = c = 0$. Therefore, M is nilpotent.

7. Consider the polynomial ring $\mathbb{Q}[x, y]$ where x, y are indeterminates over $\mathbb{Q}$. Determine a finite generating set for the ideal of polynomials $f(x, y)$ with $f(i, i) = 0$

   Solution: We don't know how to do this problem. If you figure it out, you should let us know!

8. Suppose $K \subset L \subset M$ are fields with $[M : L] = 2 = [L : K]$. Prove that $M = K(\alpha)$, where α is a root of an irreducible polynomial in $K[x]$ of the form $x^4 + bx^2 + c$.

   Solution: We didn't have time to get to this problem. If someone works on it and writes a solution, please send it to Anna and she'll update this.

9. Prove that $\mathbb{Q}(\sqrt{5+\sqrt{5}})$ is Galois over $\mathbb{Q}$, and compute the Galois group

   Solution: $\sqrt{5+\sqrt{5}}$ is a root of the polynomial $(x^2-5)^2 - 5 = x^4 - 10x^2 + 20$ which is irreducible by Eisenstein's criteria with $p = 5$ and has roots $\{\pm\sqrt{5\pm\sqrt{5}}\}$. Because $(\sqrt{5+\sqrt{5}})^2 = 5+\sqrt{5}$ we have $\sqrt{5} \in \mathbb{Q}(\sqrt{5+\sqrt{5}})$. Note that

$$\sqrt{5+\sqrt{5}} \cdot \sqrt{5-\sqrt{5}} = \sqrt{20} = 2\sqrt{5}$$

   so $\sqrt{5-\sqrt{5}}$ is also in $\mathbb{Q}(\sqrt{5+\sqrt{5}})$. Thus, $\mathbb{Q}(\sqrt{5+\sqrt{5}})$ is a splitting field of an irreducible polynomial over $\mathbb{Q}$ and as such is Galois.

   $[\mathbb{Q}(\sqrt{5+\sqrt{5}}) : \mathbb{Q}] = 4$ so the Galois group has size four and is isomorphic to either $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We know the Galois group acts transitively on the roots so there is an element σ such that $\sigma(\sqrt{5+\sqrt{5}}) = \sqrt{5-\sqrt{5}}$. Then $\sigma(5+\sqrt{5}) = \sigma(\sqrt{5+\sqrt{5}})^2 = 5-\sqrt{5}$ so $\sigma(\sqrt{5}) = -\sqrt{5}$. Now

$$\sigma^2\left(\sqrt{5+\sqrt{5}}\right) = \sigma\left(\sqrt{5-\sqrt{5}}\right) = \sigma\left(\frac{2\sqrt{5}}{\sqrt{5+\sqrt{5}}}\right) = \frac{-2\sqrt{5}}{\sqrt{5-\sqrt{5}}} = -\sqrt{5+\sqrt{5}}$$

So, $\sigma^2 \neq id$ and must have order four, so $\text{Gal}(\mathbb{Q}(\sqrt{5+\sqrt{5}})/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$.

10. Let K be a field of characteristic $p > 0$, and let t be an indeterminate. Consider the automorphism $\sigma \in \text{Aut}_K K(t)$ with $\sigma(t) = t + 1$. Determine the subfield of $K(t)$ that is fixed by σ.

    Solution: σ has order p, then $K(t)/K(t)^\sigma$ has extension degree equal to p since it is a Galois extension and its Galois group is isomorphic to $\langle\sigma\rangle$. We have that $K(t^p - t) \subseteq K(t)^\sigma$ and $[K(t) : K(t^p - t)] = p$ so $K(t)^\sigma = K(t^p - t)$.