University of Utah, Department of Mathematics
Spring 2009, Algebra Preliminary Exam

Four correct solutions count as a pass; eight half-correct solutions may not!

1. Consider the symmetric group S_p for p an odd prime. Let C be the centralizer of the p-cycle $\sigma = (1\ 2\ \ldots\ p)$ and let N be the normalizer of $\langle\sigma\rangle$ in S_p. What is the order of each of the groups C and N? Are they abelian?

2. Let U be the subgroup of $GL_2(\mathbb{Z}/p\mathbb{Z})$ consisting of upper triangular matrices. Is U solvable? Prove or disprove.

3. A group of order 81 acts on a set X with 30 elements. Show that some element of X is fixed by at least 27 elements. How many elements of X are fixed by precisely 3 elements, assuming there is at least one such element?

4. Let $\mathbb{S}$ be a finite subset of $\mathbb{C}$. Let X be the set of $n \times n$ complex matrices all of whose eigenvalues lie in $\mathbb{S}$. Consider the conjugation action of $GL_n(\mathbb{C})$ acting on X.

   Prove that this action has finitely many orbits. If $n = 3$, find a formula for the number of orbits in terms of the cardinality of $\mathbb{S}$.

5. Let R be a commutative ring. If for each prime ideal $\mathfrak{p}$ of R, the local ring $R_\mathfrak{p}$ contains no nonzero nilpotent elements, prove that R contains no nonzero nilpotent elements.

6. Let $\mathfrak{a}$ be the ideal of the polynomial ring $\mathbb{R}[x]$ that is generated by $x^3 - 5x^2 + 6x - 2$ and $x^4 - 4x^3 + 3x^2 - 4x + 2$. Is $\mathfrak{a}$ a principal ideal? Is it prime? Is it maximal?

7. Let $\mathbb{F}$ be a field with 81 elements. What is the number of roots of $x^{25} - 1 = 0$ in $\mathbb{F}$?

8. Suppose $f(x) \in \mathbb{Q}[x]$ is an irreducible polynomial with complex roots $\alpha_1, \ldots, \alpha_n$. Prove that $\alpha_i - \alpha_j \notin \mathbb{Q}$ for all $i \neq j$.

9. Let K be a subfield of a field L. If $\sigma\colon L \longrightarrow L$ is a homomorphism that fixes all elements of K, is σ necessarily an automorphism of L? Justify your answer.

10. Let $L = \mathbb{Q}(e^{2\pi i/8})$. Determine all subgroups of the Galois group $\mathrm{Gal}(L/\mathbb{Q})$, and the corresponding intermediate fields.