Fall 2010
Algebra Qualfying Exam Solutions

Allechar Serrano

June 29, 2016

1. Determine the number of $5-$Sylow subgroups of $SL_2(\mathbb{F}_5)$

   Solution: We have that $|SL_2(\mathbb{F}_p)| = (p^2-1)(p^2-p)/p$, so $|SL_2(\mathbb{F}_5)| = 120$. Let n_5 be the number of Sylow $5-$subgroups of $SL_2(\mathbb{F}_5)$, we have that $n_5 \in \{1, 6, 24\}$. Consider $M \in GL_2(\mathbb{F}_5)$ such that $M^5 = I$, since the field has characteristic 5, then $M^5 - I = (M - I)^5 = 0$. So $x^2 - 2x + 1 = (x - 1)^2$ is the characteristic polynomial of M, then $\det M = 1$ and $\operatorname{trace} M = 2$. There are 24 elements, excluding the identity, of order 5 satisfying these conditions. Therefore, the number of Sylow $5-$subgroups is $n_5 = 6$.

2. Let G be the subgroup of $GL_2(\mathbb{R})$ consisting of matrices of the form $\begin{pmatrix} \text{a} & \text{c} \\ 0 & \text{b} \end{pmatrix}$. Is G solvable?

   Solution: Consider $\begin{pmatrix} 1 & 0 \\ 0 & \text{d} \end{pmatrix}$ with $d \neq 1$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with $d \neq 1$. These matrices do not commute. Consider the map $\varphi : \begin{pmatrix} \text{a} & \text{c} \\ 0 & \text{b} \end{pmatrix} \longmapsto (a, b)$, φ is surjective and $\ker \varphi$ is normal. Also, $\ker \varphi$ is isomorphic to the additive group of $\mathbb{R}$. We get the abelian tower $\{I\} \subset \ker \varphi \subset G$. Therefore G is solvable.

3. Consider the automorphisms σ, τ of $\mathbb{Q}(x)$ with $\sigma : x \longmapsto \frac{1}{x}$ and $\tau : x \longmapsto 1 - x$. What is the order of the group generated by these two elements? Determine the group.

   Solution: We have that $|\sigma| = |\tau| = 2$. Since τ and σ do not commute, $|\tau\sigma| = 3 = |\sigma\tau|$ and $\tau\sigma = (\sigma\tau)^2$. Then $|\langle \sigma, \tau \rangle| = 6$, so the group is isomorphic to S_3.

4. Set $R = \mathbb{Q}[x]$, and consider the submodule M of R^2 generated by the elements $(1-2x, -x^2)$ and $(1-x, x-x^2)$. Express R^2/M as a direct sum of cyclic modules.

   Solution: The corresponding Smith form is $\operatorname{diag}\big(x, (x-1)^2\big)$. Therefore $R^2/M \cong R/(x) \oplus R/\big((x-1)^2\big)$.

5. Recall that a Hermitian matrix is a complex matrix which equals to its conjugate transpose. Determine the conjugacy classes of 5×5 Hermitian matrices A satisfying $A^5 + A^3 + 3A = 6I$.

   Solution: A matrix is Hermitian if and only if all its eigenvalues are real. The minimal polynomial must divide $x^5 + 2x^3 + 3x - 6$ and the only real solution of this polynomial is $x = 1$. Therefore, there is at most one conjugacy class of 5×5 Hermitian matrices satisfying the polynomial which corresponds to the identity matrix.

6. Determine the number of conjugacy classes of 4×4 complex matrices satisfying $A^3 - 2A^2 + A = 0$.

   Solution: Since the minimal polynomial of A must divide $x(x-1)^2$, we get the following possibilities:

Minimal polynomial	Characteristic polynomial	Jordan form
x	x^4	$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$
$x-1$	$(x-1)^4$	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$
$(x-1)^2$	$(x-1)^4$	$\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$
$x(x-1)$	$x^3(x-1)$	$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}$
	$x^2(x-1)^2$	$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$
	$x(x-1)^3$	$\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$
$x(x-1)^2$	$x^2(x-1)^2$	$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$
	$x(x-1)^3$	$\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$

Therefore, there are 9 conjugacy classes.

7. Let α be the positive real root of x^6-7. What is the number of elements of the ring $\mathbb{Z}[\alpha]/(\alpha^2)$? Is every ideal in this ring principal?

   Solution: x^6-7 is the minimal polynomial of $\alpha = \sqrt[6]{7}$, hence $|\mathbb{Q}(\alpha) : \mathbb{Q}| = 6$. We have that $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^6-7)$, so $\mathbb{Z}(\alpha) \cong \mathbb{Z}[x]/(x^6-7)$.

Therefore $\mathbb{Z}[\alpha]/(\alpha^2) \cong \mathbb{Z}[x]/(x^2, 7) \cong \mathbb{Z}_7[x]/(x^2)$. So $\mathbb{Z}[\alpha]/(\alpha^2) = \{a + b\alpha : a, b \in \mathbb{Z}_7\}$, which has 49 elements.
The elements $a \neq 0$, $a \in \mathbb{Z}_7$ are units and every ideal is principal since if it is proper and nonzero, then it contains an element of the form $b\alpha$ for some $b \neq 0$, $b \in \mathbb{Z}_7$: so it contains $b^{-1}b\alpha = \alpha$, and the ideal is (α).

8. Find the degree of the splitting field of $x^6 - 3$ over (i) $\mathbb{Q}(\sqrt{-3})$ and (ii) $\mathbb{F}_5$.

   Solution: (i)The splitting field of $x^6 - 3$ is $\mathbb{Q}(\sqrt[6]{3}, \sqrt{-3})$. We have that $\mathbb{Q}(\sqrt[6]{3}, \sqrt{-3})/\mathbb{Q}(\sqrt[6]{3})$ is an extension of degree 2 and $\mathbb{Q}(\sqrt[6]{3})/\mathbb{Q}$ is an extension of degree 6. Thus $|\mathbb{Q}(\sqrt[6]{3}, \sqrt{-3}) : \mathbb{Q}(\sqrt{-3})| = 6$ since $\mathbb{Q}(\sqrt[6]{3}, \sqrt{-3})/\mathbb{Q}$ is a degree 12 extension.
   (ii) Since $3^2 \cong -1 \bmod 5$ and $3^4 \cong 1 \bmod 5$, then 3 is a primitive root of order 4. Then a $6-$th root of 3 in any extension of $\mathbb{F}_5$ is a root of unity of order n, for n a factor of 24. Therefore, all the roots of $x^6 - 3$ are in a field containing the $24-$th roots of unity, which is $\mathbb{F}_{25}$.

9. Prove that $x^4 + 1$ is reducible over any field of positive characteristic.

   Solution: It suffices to show that $x^4 + 1$ is reducible over any field $\mathbb{F}_p$. If $p = 2$, then $x^4 + 1 = (x + 1)^4$ is reducible. Otherwise, we have $p \equiv 1 \bmod 8$. Then $x^4 + 1 | x^8 - 1 | x^{p^2-1} - 1 | x^{p^2} - x$. If α is a root of $x^4 + 1$, then it is a root of $x^{p^2} - x$. Recall that the solutions of $x^{p^2} - x$ form the field $\mathbb{F}_{p^2}$. Therefore $\mathbb{F}_p(\alpha)$ is a subfield of $\mathbb{F}_{p^2}$. Thus $|\mathbb{F}_p(\alpha) : \mathbb{F}_p| \leqslant 2 \neq 4$, so $x^4 + 1$ is reducible.

10. For a prime p, determine the Galois group of $x^p - 2$ over $\mathbb{Q}$. What is its order? Is it abelian?

    Solution: The roots of the irreducible polynomial $x^p - 2$ are $\sqrt[p]{2}, \zeta\sqrt[p]{2}, \ldots, \zeta^{p-1}\sqrt[p]{2}$ where ζ is a primitive $p-$th root of unity. Thus the splitting field of $x^p - 2$ is $\mathbb{Q}(\zeta, \sqrt[p]{2})$, this extension has degree $p(p-1)$ over $\mathbb{Q}$ since $|\mathbb{Q}(\sqrt[p]{2}) : \mathbb{Q}| = p$ and the minimal polynomial of ζ is the cyclotomic polynomial Φ_p. Therefore, the Galois group has order $p(p-1)$ and is given by the automorphisms $\sigma : \sqrt[p]{2} \longmapsto \zeta\sqrt[p]{2}, \zeta \longmapsto \zeta$ and $\tau : \sqrt[p]{2} \longmapsto \sqrt[p]{2}, \zeta \longmapsto \zeta^2$. The group is abelian if $p = 2$. If p is an odd prime, then the Galois group is not abelian.