1. Determine the number of 5−Sylow subgroups of \( SL_2(\mathbb{F}_5) \)

**Solution:** We have that \( |SL_2(\mathbb{F}_p)| = (p^2-1)(p^2-p)/p \), so \( |SL_2(\mathbb{F}_5)| = 120 \). Let \( n_5 \) be the number of Sylow 5−subgroups of \( SL_2(\mathbb{F}_5) \), we have that \( n_5 \in \{1, 6, 24\} \). Consider \( M \in GL_2(\mathbb{F}_5) \) such that \( M^5 = I \), since the field has characteristic 5, then \( M^5 - I = (M - I)^5 = 0 \). So \( x^2 - 2x + 1 = (x - 1)^2 \) is the characteristic polynomial of \( M \), then \( \det M = 1 \) and \( \text{trace} M = 2 \). There are 24 elements, excluding the identity, of order 5 satisfying these conditions. Therefore, the number of Sylow 5−subgroups is \( n_5 = 6 \).

2. Let \( G \) be the subgroup of \( GL_2(\mathbb{R}) \) consisting of matrices of the form

\[
\begin{pmatrix}
a & c \\
0 & b
\end{pmatrix}
\]

Is \( G \) solvable?

**Solution:** Consider \( \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \) with \( d \neq 1 \) and \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) with \( d \neq 1 \). These matrices do not commute. Consider the map \( \varphi : \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mapsto (a,b) \), \( \varphi \) is surjective and \( \ker \varphi \) is normal. Also, \( \ker \varphi \) is isomorphic to the additive group of \( \mathbb{R} \). We get the abelian tower \( \{I\} \subset \ker \varphi \subset G \). Therefore \( G \) is solvable.

3. Consider the automorphisms \( \sigma, \tau \) of \( \mathbb{Q}(x) \) with \( \sigma : x \mapsto \frac{1}{x} \) and \( \tau : x \mapsto 1 - x \). What is the order of the group generated by these two elements? Determine the group.

**Solution:** We have that \( |\sigma| = |\tau| = 2 \). Since \( \tau \) and \( \sigma \) do not commute, \( |\tau \sigma| = 3 = |\sigma \tau| \) and \( \tau \sigma = (\sigma \tau)^2 \). Then \( |\langle \sigma, \tau \rangle| = 6 \), so the group is isomorphic to \( S_3 \).
4. Set \( R = \mathbb{Q}[x] \), and consider the submodule \( M \) of \( R^2 \) generated by the elements \((1 - 2x, -x^2)\) and \((1 - x, x - x^2)\). Express \( R^2/M \) as a direct sum of cyclic modules.

**Solution:** The corresponding Smith form is \( \text{diag}(x, (x-1)^2) \). Therefore \( R^2/M \cong R/(x) \oplus R/((x-1)^2) \).

5. Recall that a Hermitian matrix is a complex matrix which equals to its conjugate transpose. Determine the conjugacy classes of \( 5 \times 5 \) Hermitian matrices \( A \) satisfying \( A^5 + A^3 + 3A = 6I \).

**Solution:** A matrix is Hermitian if and only if all its eigenvalues are real. The minimal polynomial must divide \( x^5 + 2x^3 + 3x - 6 \) and the only real solution of this polynomial is \( x = 1 \). Therefore, there is at most one conjugacy class of \( 5 \times 5 \) Hermitian matrices satisfying the polynomial which corresponds to the identity matrix.

6. Determine the number of conjugacy classes of \( 4 \times 4 \) complex matrices satisfying \( A^3 - 2A^2 + A = 0 \).

**Solution:** Since the minimal polynomial of \( A \) must divide \( x(x - 1)^2 \), we get the following possibilities:
<table>
<thead>
<tr>
<th>Minimal polynomial</th>
<th>Characteristic polynomial</th>
<th>Jordan form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$x^4$</td>
<td>$\begin{pmatrix} 0 &amp; 0 \ 1 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$x - 1$</td>
<td>$(x - 1)^4$</td>
<td>$\begin{pmatrix} 1 &amp; 1 \ 1 &amp; 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>$(x - 1)^2$</td>
<td>$(x - 1)^4$</td>
<td>$\begin{pmatrix} 1 &amp; 1 \ 1 &amp; 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>$x(x - 1)$</td>
<td>$x^3(x - 1)$</td>
<td>$\begin{pmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$x^2(x - 1)^2$</td>
<td>$x^3(x - 1)$</td>
<td>$\begin{pmatrix} 0 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$</td>
</tr>
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</tr>
</tbody>
</table>

Therefore, there are 9 conjugacy classes.

7. Let $\alpha$ be the positive real root of $x^6 - 7$. What is the number of elements of the ring $\mathbb{Z}[\alpha]/(\alpha^2)$? Is every ideal in this ring principal?

**Solution:** $x^6 - 7$ is the minimal polynomial of $\alpha = \sqrt[6]{7}$, hence $|\mathbb{Q}(\alpha) : \mathbb{Q}| = 6$. We have that $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(x^6 - 7)$, so $\mathbb{Z}(\alpha) \cong \mathbb{Z}[x]/(x^6 - 7)$. 

Therefore $\mathbb{Z}[\alpha]/(\alpha^2) \cong \mathbb{Z}[x]/(x^2, 7) \cong \mathbb{Z}_7[x]/(x^2)$. So $\mathbb{Z}[\alpha]/(\alpha^2) = \{a + b\alpha : a, b \in \mathbb{Z}_7\}$, which has 49 elements.

The elements $a \neq 0, a \in \mathbb{Z}_7$ are units and every ideal is principal since if it is proper and nonzero, then it contains an element of the form $b\alpha$ for some $b \neq 0, b \in \mathbb{Z}_7$: so it contains $b^{-1}b\alpha = \alpha$, and the ideal is $(\alpha)$.

8. Find the degree of the splitting field of $x^6 - 3$ over (i) $\mathbb{Q}(\sqrt{-3})$ and (ii) $\mathbb{F}_5$.

**Solution:** (i) The splitting field of $x^6 - 3$ is $\mathbb{Q}(\sqrt[6]{3}, \sqrt{-3})$. We have that $\mathbb{Q}(\sqrt[6]{3}, \sqrt{-3})/\mathbb{Q}(\sqrt[6]{3})$ is an extension of degree 2 and $\mathbb{Q}(\sqrt[6]{3})/\mathbb{Q}$ is an extension of degree 6. Thus $|\mathbb{Q}(\sqrt[6]{3}, \sqrt{-3}) : \mathbb{Q}(\sqrt{-3})| = 6$ since $\mathbb{Q}(\sqrt[6]{3}, \sqrt{-3})/\mathbb{Q}$ is a degree 12 extension.

(ii) Since $3^2 \equiv -1 \mod 5$ and $3^4 \equiv 1 \mod 5$, then 3 is a primitive root of order 4. Then a 6-th root of 3 in any extension of $\mathbb{F}_5$ is a root of unity of order $n$, for $n$ a factor of 24. Therefore, all the roots of $x^6 - 3$ are in a field containing the 24-th roots of unity, which is $\mathbb{F}_{25}$.

9. Prove that $x^4 + 1$ is reducible over any field of positive characteristic.

**Solution:** It suffices to show that $x^4 + 1$ is reducible over any field $\mathbb{F}_p$. If $p = 2$, then $x^4 + 1 = (x + 1)^4$ is reducible. Otherwise, we have $p \equiv 1 \mod 8$. Then $x^4 + 1 | x^8 - 1 | x^{p^2-1} - 1 | x^p - x$. If $\alpha$ is a root of $x^4 + 1$, then it is a root of $x^p - x$. Recall that the solutions of $x^p - x$ form the field $\mathbb{F}_{p^2}$. Therefore $\mathbb{F}_p(\alpha)$ is a subfield of $\mathbb{F}_{p^2}$. Thus $|\mathbb{F}_p(\alpha) : \mathbb{F}_p| \leq 2 \neq 4$, so $x^4 + 1$ is reducible.

10. For a prime $p$, determine the Galois group of $x^p - 2$ over $\mathbb{Q}$. What is its order? Is it abelian?

**Solution:** The roots of the irreducible polynomial $x^p - 2$ are $\sqrt[2]{2}, \zeta \sqrt[2]{2}, \ldots, \zeta^{p-1} \sqrt[2]{2}$ where $\zeta$ is a primitive $p$-th root of unity. Thus the splitting field of $x^p - 2$ is $\mathbb{Q}(\zeta, \sqrt[2]{2})$, this extension has degree $p(p - 1)$ over $\mathbb{Q}$ since $|\mathbb{Q}(\sqrt[2]{2}) : \mathbb{Q}| = p$ and the minimal polynomial of $\zeta$ is the cyclotomic polynomial $\Phi_p$. Therefore, the Galois group has order $p(p - 1)$ and is given by the automorphisms $\sigma : \sqrt[2]{2} \leftrightarrow \zeta \sqrt[2]{2}, \zeta \leftrightarrow \zeta$ and $\tau : \sqrt[2]{2} \leftrightarrow \sqrt[2]{2}, \zeta \leftrightarrow \zeta^2$. The group is abelian if $p = 2$. If $p$ is an odd prime, then the Galois group is not abelian.