Twisted Sheaves on Flag Varieties: $\mathfrak{sl}(2, \mathbb{C})$ Calculations

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December 14, 2016

These are notes from a series of talks given in the representation theory student seminar at the University of Utah in November 2016. These talks explored the twists that arise in the geometry of the Borel Weil theorem and Beilinson and Bernstein's equivalence of categories. Throughout the talks, we assume $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. We discuss the Serre's twisted sheaves of sections of line bundles, sheaves of twisted differential operators, and sheaves of modules over twisted differential operators. We describe standard Harish-Chandra sheaves for Harish-Chandra pairs $(\mathfrak{g}, N)$ and $(\mathfrak{g}, K)$, where $N \subset SL(2, \mathbb{C})$ is a unipotent subgroup and $K \subset SL(2, \mathbb{C})$ is the complexification of the maximal compact subgroup $SO(2) \subset SL(2, \mathbb{R})$. We calculate the action of the Lie algebra $\mathfrak{g}$ on global sections of these sheaves and realize them as representations of $\mathfrak{g}$. We finish by describing standard η-twisted Harish-Chandra sheaves for η a character of Lie$N$, which are a geometric model for a class of $\mathfrak{g}$-modules called Whittaker modules.

1 A Family of $D_{\mathbb{C}^*}$-modules

We begin this document by describing a family of $D_{\mathbb{C}^*}$-modules and their direct images as $D_{\mathbb{C}}$-modules. We examine the conditions under which these modules are reducibile and the parameterizaton of isomorphism classes of these modules. These modules will appear in later calculations, and we will refer back to these reducibility conditions.

For $t \in \mathbb{C}$, let N_t be the $D_{\mathbb{C}^*}$-module spanned by elements e_k, $k \in \mathbb{Z}$ with $D_{\mathbb{C}^*}$-action given by

$$z \cdot e_k = e_{k+1} \qquad \partial \cdot e_k = (k - (t+1))e_{k-1}.$$

The associativity of $D_{\mathbb{C}^*}$-actions implies that $z^{-1} \in D_{\mathbb{C}^*}$ must act by $z^{-1} \cdot e_k = e_{k-1}$. Let $E = \partial z$ be the Euler operator. Then

$$E \cdot e_k = \partial \cdot z^{k+1} = (k - t)e_k,$$

so the spanning set elements are eigenvectors of E. We claim that N_t are irreducible for all values of $t \in \mathbb{C}$. The proof of this is as follows.

Let $W \subset N_t$ be a submodule. Then since $E \in D_{\mathbb{C}^*}$, W is E-stable. We can express any $w \in W$ as a linear combination of $\{e_k\}$,

$$w = \sum_{k \in \mathbb{Z}} \alpha_k e_k,$$

where all but finitely many α_k are zero. Let $\ell(w) = \text{Card}\{n \in \mathbb{Z} | \alpha_n \neq 0\}$. If $w \neq 0$, then $\ell(w) > 0$. Let $w \in W$ be a non-zero element such that $\ell(w)$ is minimal. If $\ell(w) = 1$, then $w = e_n$ for some $n \in \mathbb{Z}$, and $W = N_t$, since repeated action by z and z^{-1} maps e_n to any e_j for $j \in \mathbb{Z}$. Assume $\ell(w) > 1$. Then if $w = \sum_{k \in \mathbb{Z}} \alpha_k e_k$,

$$E \cdot w = \sum_{k \in \mathbb{Z}} (k - t)\alpha_k e_k \in W,$$

and $\ell(E \cdot w) \leq \ell(w)$. If $t = i \in \mathbb{Z}$ and $\alpha_{i-t} \neq 0$, then $\ell(E \cdot w) < \ell(w)$, which contradicts the minimality of $\ell(w)$. If $t = i \in \mathbb{Z}$ and $\alpha_{i-t} = 0$, or if $t \notin \mathbb{Z}$, then $\alpha_n \neq 0$ for some $n \in \mathbb{Z}$ such that $n - t \neq 0$. Then

$$\frac{1}{n-t} E \cdot w = \alpha_n e_n + \sum_{k \in \mathbb{Z}, k \neq n} (k-t)\alpha_k e_k,$$

so $\ell(\frac{1}{n-t} E \cdot w - w) < \ell(w)$, which contradicts the minimality of $\ell(w)$. Therefore, any submodule must contain a spanning set element, which generates the entire module, so no proper submodules exist. This completes the proof that N_t is irreducible.

Let $\{e_k\}$ be the spanning set of N_t and $\{f_k\}$ the spanning set of N_{t+p} for a fixed integer $p \in \mathbb{Z}$. The map

$$\begin{aligned} \phi_p : N_t &\longrightarrow N_{t+p} \\ e_k &\longmapsto f_{k+p} \end{aligned}$$

is an isomorphism of $D_{\mathbb{C}^*}$-modules. Indeed, it is clearly an isomorphism of vector spaces, and it the following calculation shows that it is also a $D_{\mathbb{C}^*}$-module morphism:

$$\partial \cdot \phi_p(e_k) = \partial \cdot f_{k+p} = (k+p-(t+p+1)) f_{k+p-1} = (k-(t+1)) f_{k+p-1} = \phi_p((k-(t+1)) e_{k-1}) = \phi_p(\partial \cdot e_k)$$

Therefore $N_k \simeq N_{k+p}$ for any $p \in \mathbb{Z}$, and isomorphism classes of N_t are parameterized by $\mathbb{C}/\mathbb{Z}$.

Let $i : \mathbb{C}^* \longrightarrow \mathbb{C}$ be inclusion of varieties. Then the sheaf theoretic direct image $i_\bullet(N_t)$ is a $D_{\mathbb{C}}$-module. It is isomorphic to N_t as a vector space with the restricted action of $D_{\mathbb{C}}$ as a subring of $D_{\mathbb{C}^*}$. Since $z^{-1} \notin D_{\mathbb{C}}$, the reducibility of $i_\bullet(N_t)$ depends on the integrality of t. If $t \notin \mathbb{Z}$, then $i_\bullet(N_t)$ is irreducible since neither ∂ nor powers of z annihilate any e_k. But if $t \in \mathbb{Z}$, then $\partial \cdot e_{t+1} = 0$, so $i_\bullet(N_t)$ has a submodule which is spanned by $\{e_k\}$ for $k \in \mathbb{Z}_{\geq t+1}$.

2 Serre's Twisted Sheaves

We start our $\mathfrak{sl}(2, \mathbb{C})$ calculations by exploring the twists that arise in the Borel Weil theorem. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, and $X = \mathbb{P}^1$ its flag variety. Let $0 = [1:0] \in \mathbb{P}^1$ and $\infty = [0:1] \in \mathbb{P}^1$. Then X has an open cover consisting of the sets $U_0 = \mathbb{P}^1 - \{\infty\}$ and $U_1 = \mathbb{P}^1 - \{0\}$. Let $V = U_0 \cap U_1 \simeq \mathbb{C}^*$. We identify U_0 with $\mathbb{C}$ using the usual coordinate $z : U_0 \longrightarrow \mathbb{C}$ so that $z([1:x_1]) = x_1$. We identify U_1 with $\mathbb{C}$ using the coordinate $\zeta : U_1 \longrightarrow \mathbb{C}$ where $\zeta([x_0:1]) = x_0$. Under these identifications, we see that on the intersection V, our coordinates are related by $z = \frac{1}{\zeta}$. Let $\mathcal{O}_X$ be the structure sheaf on X, and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module such that $\mathcal{L}|_{U_0} \simeq \mathcal{O}_{U_0}$ and $\mathcal{L}|_{U_1} \simeq \mathcal{O}_{U_1}$.

Restricting $\mathcal{L}|_{U_0}$ to V gives an isomorphism $\mathcal{L}|_V \simeq \mathcal{O}_V$, as does restricting $\mathcal{L}|_{U_1}$ to V. The identity map on $\mathcal{L}|_V$ induces an $\mathcal{O}_V$-module isomorphism

$$\varphi : \mathcal{O}_V \longrightarrow \mathcal{O}_V.$$

Since $V \simeq \mathbb{C}^*$ is affine, a theorm of Serre implies that there is an $\Gamma(V, \mathcal{O}_V)$-module isomorphism which we also denote by φ:

$$\varphi : R(\mathbb{C}^*) \longrightarrow R(\mathbb{C}^*).$$

This morphism is entirely determined by $\varphi(1) = p \in R(\mathbb{C}^*)$ since 1 generates $R(\mathbb{C}^*)$ as an $R(\mathbb{C}^*)$-module. For $q \in R(\mathbb{C}^*)$, $\varphi(q) = \varphi(q \cdot 1) = q\varphi(1) = qp$. Since φ is an isomorphism, it has an inverse φ^{-1} given by $\varphi^{-1}(q) = qr$ for $q \in R(\mathbb{C}^*)$, and $r(z)p(z) = 1$ for all $z \in \mathbb{C}^*$. This implies that p and r have no zeros or poles in $\mathbb{C}^*$, so they must be proportional to an integral power of z; i.e. $p(z) = cz^n$ for $c \in \mathbb{C}$ and $n \in \mathbb{Z}$.

We can see from the relationship of the coordinate systems of U_0 and U_1 on the intersection V that global sections of $\mathcal{L}|_{U_0}$ are polynomials in z in $R(\mathbb{C}^*)$, and the global sections of $\mathcal{L}|_{U_1}$ are polynomials in $\zeta = z^{-1}$ in $R(\mathbb{C}^*)$. Global sections of $\mathcal{L}$ must restrict to functions in $R(\mathbb{C}^*)$ which are global sections of both $\mathcal{L}|_{U_0}$, and $\mathcal{L}|_{U_1}$, related by φ. We have two cases.

If $n > 0$, no elements of $\mathbb{C}[z]$ are mapped to elements of $\mathbb{C}[z^{-1}]$ under φ, so $\mathcal{L}$ has no global sections. If $n \leq 0$, then the subset of $\mathbb{C}[z]$ spanned by $\{z^{|n|}, z^{|n|-1}, \ldots, 1\}$ will be sent to a subset of $\mathbb{C}[z^{-1}]$, so $\Gamma(X, \mathcal{L})$ is $(n+1)$-dimensional.

Such $\mathcal{L} = \mathcal{O}(n)$ are completely determined by $n \in \mathbb{Z}$, and are called *Serre's Twisted Sheaves.* These appeared in our study of the Borel-Weil theorem as sections of G-equivariant line bundles constructed from irreducible representations of a maximal torus $T \subset G$. The Borel-Weil theorem states that for $n \in \mathbb{Z}_{\leq 0}$, $\Gamma(X, \mathcal{O}(n)) \simeq F_n$, where F_n is the finite dimensional irreducible representation of $\mathfrak{g}$ of highest weight $|n|$, and for $n \in \mathbb{Z}_{>0}$, $\Gamma(X, \mathcal{O}(n)) = 0$.

In this first example, we see that in the case of $\mathfrak{sl}(2, \mathbb{C})$, invertible $\mathcal{O}_X$-modules with respect to the cover $\{U_0, U_1\}$ are completely determined by their integral twist n, and the value of n dictates whether or not such sheaves will have any global sections.

3 Sheaves of Twisted Differential Operators

In this section, we explore differential operators on the twisted sheaves of regular functions from the previous section. Such sheaves of rings are examples of sheaves of twisted differential operators. Sheaves of twisted differential operators can be defined more generally and do not have to arise from sheaves of sections of line bundles. However, constructing them in this way gives a natural motivation for the general definition.

Let $\mathcal{O}(n)$ be the invertible $\mathcal{O}_X$-module determined by $\varphi(q) = z^n q$ for $q \in R(\mathbb{C}^*)$ as described in the §2. Let $\mathcal{D}_X$ be the ring of local differential operators on X. Let $\mathcal{D}_n$ be the sheaf of rings on X given by $\mathcal{D}_n(U_0) = \mathcal{D}iff(\mathcal{O}(n)|_{U_0}) \simeq \mathcal{D}_{U_0}$ and $\mathcal{D}_n(U_1) = \mathcal{D}iff(\mathcal{O}(n)|_{U_1}) \simeq D_{U_1}$. The sheaf $\mathcal{D}_n$ is locally isomorphic to $\mathcal{D}_X$, so it is an example of a sheaf of twisted differential operators. See [2] for a precise definition of these objects. Let $D_{\mathbb{C}^*} = \Gamma(\mathbb{C}^*, \mathcal{D}_{\mathbb{C}^*})$ be the ring of differential operators on $\mathbb{C}^*$. We can define a ring isomorphism $\psi : D_{\mathbb{C}^*} \longrightarrow D_{\mathbb{C}^*}$ in the following way. For $T \in D_{\mathbb{C}^*}$, let $\psi(T)$ be the differential operator which makes the diagram

$$\begin{tikzcd} R(\mathbb{C}^*) \arrow[r, "\varphi"] \arrow[d, "T"] & R(\mathbb{C}^*) \arrow[d, "\psi(T)"] \\ R(\mathbb{C}^*) \arrow[r, "\varphi"] & R(C^*) \end{tikzcd}$$

commute.

Since $D_{\mathbb{C}^*}$ is generated by z and differentiation with respect to z, which we denote by ∂, we can precisely describe ψ by calculating the image of z and ∂. First we notice that $\psi(z) = z$. Indeed, for $f \in R(\mathbb{C}^*)$,

$$\varphi(z \cdot f) = z^n z f = z^{n+1} f = z z^n f = z \cdot \varphi(f).$$

It remains to calculate the image of ∂. Since ψ is an isomorphism of rings of differential operators, it must preserve order. This implies that $\psi(\partial)$ must be a first order differential operator of the form $a\partial + b$ for $a, b \in R(\mathbb{C}^*)$. Since $\varphi(\partial(f)) = \varphi(f') = z^n f'$, the following calculation determines the value of a and b:

$$\psi(\partial)(\varphi(f)) = (a\partial + b)(z^n f) = a\partial(z^n f) + b z^n f = anz^{n-1} f + a z^n f' + f z^n f = (\frac{an}{z} + b) z^n f + a z^n f' = z^n f'$$

To make the final equality hold, we determine that $a = 1$ and $b = -\frac{n}{z}$. We conclude that our sheaf of twisted differential operators $\mathcal{D}_n$ is determined up to ismorphism by the ring isomorphism

$$\begin{aligned} \psi : D_{\mathbb{C}^*} &\longrightarrow D_{\mathbb{C}^*} \\ z &\mapsto z \\ \partial &\mapsto \partial - \frac{n}{z} \end{aligned}$$

We can observe that the differential operator $\partial - \frac{n}{z} = z^n \partial z^{-n}$. Indeed, for $f \in R(\mathbb{C}^*)$,

$$(z^n \partial z^{-n})(f) = z^z \partial(z^{-n} f) = z^n(-nz^{-n-1})f + z^n z^{-n} f' = f' - \frac{n}{z} = (\partial - \frac{n}{z})(f).$$

We conclude this section by noting that the formula $\psi(\partial) = \partial - \frac{n}{z}$ defines a ring endomorphism of $D_{\mathbb{C}^*}$ for any $n \in \mathbb{C}$. This is in contrast to the $R(\mathbb{C}^*)$-module endomorphism of $R(\mathbb{C}^*)$ given by $\varphi(q) = z^n q$, which is only well-defined for $n \in \mathbb{Z}$. This observation explains the general fact that the sheaves of sections of line bundles on the flag variety constructed in the Borel-Weil theorem are parameterized by integral lattices, whereas sheaves of homogeneous twisted differential operators on the flag variety are parameterized by complex vector spaces.

4 Sheaves of Modules over Twisted Differential Operators

In this section, we describe $\mathcal{D}_n$-modules which are homogeneous with respect to the action of $\text{Int}(\mathfrak{g}) = PSL(2, \mathbb{C})$. Such modules which also obey certain compatibility conditions with the action of a subgroup of $PSL(2, \mathbb{C})$ are called *Harish-Chandra sheaves.* For a precise construction of Harish-Chandra sheaves, see [2]. We examine standard objects in two categories of Harish-Chandra sheaves - those corresponding to the unipotent subgroup N of $PSL(2, \mathbb{C})$ and those corresponding to the complexification K of the maximal compact subgroup of $PSL(2, \mathbb{R})$. We explicitly calculate the action of $\mathfrak{g}$ on the global sections of these standard objects which allows us to recognize them as familiar representations of $\mathfrak{g}$. In the $(\mathfrak{g}, N)$ case, we see that for negative integral n, global sections of Harish-Chandra sheaves are dual Verma modules, and in the $(\mathfrak{g}, K)$ case, we see that for integral n, global setions of Harish-Chandra sheaves are principle series representations of $SL(2, \mathbb{R})$.

Before we analyze the standard Harish-Chandra sheaves, we need to realize elements of our Lie algebra as vector fields on X. To do so, we will need to give a more explicit description of $\mathcal{O}(n)$ as it appears in the context of the Borel-Weil theorem. Let $\{H, E, F\}$ be the standard basis for $\mathfrak{sl}(2, \mathbb{C})$. These elements are given by the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and satisfy the commutation relations $[H, E] = 2E$, $[H, F] = -2F$, and $[E, F] = H$. We have $G = \text{Int}(\mathfrak{g}) = PSL(2, \mathbb{C})$. This group acts transitively on the flag variety X. Let N be the unipotent subgroup of the Borel subgroup in $PSL(2, \mathbb{C})$ which stabilizes $0 \in X$, and $\bar{N}$ the unipotent subgroup of the stabilizer of $\infty \in X$. Let T be the maximal torus normalizing both N and $\bar{N}$. Then $\text{Lie}(N) = \text{span}\{E\}$, $\text{Lie}(\bar{N}) = \text{span}\{F\}$, and $\text{Lie}(T) = \text{span}\{H\}$. The subgroups N, $\bar{N}$, and T correspond to the subgroups

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{C} \right\}, \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \middle| c \in \mathbb{C} \right\}, \text{ and } \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{C} \right\}$$

of $SL(2, \mathbb{C})$.

Next we realize Serre's twisted sheaves as sheaves of sections of line bundles. Let $\sigma_n : T \to GL(\mathbb{C})$ be the irreducible representation of T mapping $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto a^n$. As we discussed in previous seminars, we can use σ_n to build a homogeneous G-equivariant line bundle over X in the following way: we extend our representation σ_n to N trivially to get a one-dimensional representation of $B = TN = \mathrm{stab}_G\{0\}$. Then let $E = \sim\backslash G \times V$ where $(g, v) \sim (bg, \sigma_n(b)v)$ for $g \in G, b \in B, v \in \mathbb{C}$. We have a G-equivariant line bundle $\pi : E \to X$ mapping the equivalence class $[g, v]$ to the coset Bg. The line bundle (E, X, π) is the homogeneous vector bungle attached to the representation $(\sigma_n, \mathbb{C})$. Let $\mathcal{O}(n)$ be the sheaf of sections of (E, X, π). In our study of Borel-Weil, we showed that $\mathcal{O}(n)$ is isomorphic to the sheaf of holomorphic functions on $\mathbb{C}^2 - \{0\}$ which are homogeneous of degree $-n$. Let $p : \mathbb{C}^2 - \{0\} \longrightarrow \mathbb{P}^1$ be the quotient by the relation $(x_0, x_1) \sim (\lambda x_0, \lambda x_1)$ for $(x_0, x_1) \in \mathbb{C}^2 - \{0\}$ and $\lambda \in \mathbb{C}^*$. Then we can consider $\mathcal{O}(n)$ as a sheaf on $X = \mathbb{P}^1$ by letting $\mathcal{O}(n)(U_i)$ be the vector space of homogeneous holomorphic functions on $p^{-1}(U_i)$ of degree $-n$. These $\mathcal{O}(n)$ are Serre's twisted sheaves that we discussed in §2, so they are locally isomorphic to $\mathcal{O}_X$. More explicitly, for U_0 we have an isomorphism between $\mathcal{O}(n)(U_0)$ and $\mathcal{O}_X(U_0)$ which associates $f \in \mathcal{O}(n)(U_0)$ with $g := f(1, -) \in \mathcal{O}_X(U_0)$. For U_1 our isomorphism associates $f \in \mathcal{O}(n)(U_1)$ with $h := f(-, 1) \in \mathcal{O}_X(U_1)$.

In order to calculate the action of $\mathfrak{g}$ on our standard Harish-Chandra sheaves, we need to explicitly realize H, E, F as differential operators on regular functions on each of our charts U_0 and U_1. We begin with U_0. As in §2, we identify U_0 with $\mathbb{C}$ in the usual coordinate $z([1 : x_1]) = x_1$. The group G acts on the flag variety X in the following way. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $[x_0 : x_1] \in X$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x_0 : x_1] = [ax_0 + bx_1 : cx_0 + dx_1].$$

If we restrict this action to $\bar{N}$, we see that U_0 is invariant. This gives us an action map:

$$a : \bar{N} \times U_0 \longrightarrow U_0$$
$$\left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, [1 : x_1] \right) \mapsto [1 : c + x_1]$$

By restricting the action map to $0 = [1 : 0] \in U_0$ and composing with z we have an isomorphism

$$\bar{N} \times \{0\} \xrightarrow{a} U_0 \xrightarrow{z} \mathbb{C}$$
$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \longmapsto [1 : c] \longmapsto c$$

This induces an isomorphism of the collection of vector fields on $\bar{N}$ with the collection of vector fields on $\mathbb{C}$. Let ∂ be differentiation with respect to z, considered as a vector field on $\mathbb{C}$. Elements of the Lie algebra of $\bar{N}$ can be associated to vector fields on $\bar{N}$ in the following way. For $X \in \mathrm{Lie}\,\bar{N}$, the corresponding vector field acts on a function $f \in C^\infty(\bar{N})$ by $X(f)(x) = \frac{d}{dt}\big|_{t=0}(1 + tX) \cdot f(x)$. The vector field on $\bar{N}$ associated to F acts on $f \in C^\infty$ by

$$\begin{aligned}
F(f)\left(\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}\right) &= \frac{d}{dt}\Big|_{t=0} f\left((1+tF)^{-1}\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}\right)\\
&= \frac{d}{dt}\Big|_{t=0} f\left(\begin{pmatrix}1 & 0\\ -t & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}\right)\\
&= \frac{d}{dt}\Big|_{t=0} f\left(\begin{pmatrix}1 & 0\\ -t+c & 1\end{pmatrix}\right)\\
&= -f'\left(\begin{pmatrix}1 & 0\\ c & 1\end{pmatrix}\right)
\end{aligned}$$

From this calculation we see that under the isomorphism between $\bar{N}$ and $\mathbb{C}$, $-\partial$ is associated with $F \in \mathfrak{g}$. To find the vector fields which correspond to H and E, we can use the bracket relations on $\mathfrak{g}$ and the isomorphism between $\mathcal{O}(n)(U_0)$ and $\mathcal{O}_X(U_0)$.

Since the torus T acts on U_0 by left multiplication, we have a natural action of T on $\mathcal{O}(n)(U_0)$ by the left regular representation. For $A = \begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix} \in T$ and $f \in \mathcal{O}(n)(U_0)$,

$$A \cdot f(x_0, x_1) = f(p^{-1}(A^{-1} \cdot [x_0, x_1])) = f(a^{-1}x_0, ax_1).$$

So under the isomorphism described earlier, T acts on $g(z) \in \mathcal{O}_X(U_0)$ by

$$A \cdot g(z) = A \cdot f(1, z) = f(a^{-1}, az) = a^n f(1, a^2 z) = a^n g(a^2 z).$$

Here the second to last equality comes from the fact that $f \in \mathcal{O}(n)(U_0)$ is homogeneous of degree $-n$. We can differentiate this action to get an action of Lie(T) =span$\{H\}$:

$$\begin{aligned}
\begin{pmatrix}a & 0\\ 0 & -a\end{pmatrix} \cdot g(z) &= \frac{d}{dt}\Big|_{t=0}\left(1 + t\begin{pmatrix}a & 0\\ 0 & -a\end{pmatrix}\right) \cdot g(z)\\
&= \frac{d}{dt}\Big|_{t=0}\begin{pmatrix}1+ta & 0\\ 0 & \frac{1}{1+ta}\end{pmatrix} \cdot g(z)\\
&= \frac{d}{dt}\Big|_{t=0}(1+ta)^n g((1+ta)^2 z)\\
&= ((1+ta)^n g'((1+ta)^2 z)(2az(1+ta)) + g((1+ta)^2 z)(na(1+ta)^{n-1}))\big|_{t=0}\\
&= 2azg'(z) + ang(z)
\end{aligned}$$

From this calculation we see that in the chart U_0, H acts by the vector field $2z\partial + n$. We can use the bracket relations on $\mathfrak{g}$ to find the vector field associated to E in this chart. Since E must be associated to a first order differential operator, $E = a\partial + b$ for $a, b \in \mathbb{C}[z]$. Since $[E, F] = H$, the following calculation,

$$[E, F] = [a\partial + b, -\partial] = (\partial a - a\partial)\partial + (\partial b - b\partial) = [\partial, a]\partial + [\partial, b] = a'\partial + b' = 2z\partial + n,$$

lets us conclude that $a = z^2 + a_0$ and $b = nz + b_0$, where $a_0, b_0 \in \mathbb{C}$. In the calculation above, we use the fact that as differential operators on $\mathbb{C}$, $[\partial, a] = a'$, since for $f \in C^\infty(\mathbb{C})$,

$$[\partial, a] = \partial af - a\partial f = \partial(a)f + a\partial(f) - a\partial(f) = a'f.$$

Finally we use the bracket relation $[H,E]=2E$ to conclude

$$\begin{aligned}[H,E] &= [2z\partial+n, z^2\partial+a_0\partial+nz+b_0]\\ &= 2[z\partial, z^2\partial]+2a_0[z\partial,\partial]+2n[z\partial,z]\\ &= 2z^2\partial-2a_0\partial+2nz\\ &= 2(z^2\partial+a_0\partial+nz+b_0),\end{aligned}$$

so $a_0=b_0=0$. Summing up, in U_0, we have

$$E=z^2\partial+nz, \qquad H=2x\partial+n, \qquad F=-\partial.$$

Then, our map $\psi : D_{\mathbb{C}^*} \longrightarrow D_{\mathbb{C}^*}$ lets us conclude that in the other chart U_1,

$$E=z^2\partial \qquad H=2z\partial-n \qquad F=-\partial+\frac{n}{z}.$$

At this point in time, we wish to generalize our twisted sheaves of differential operators to allow for the possibility of non-integral twists while maintaining the structure present in the previous construction. We also introduce a ρ-shift for symmetry purposes. Let $\mathcal{D}_X$ be the sheaf of local differential operators on X. For $t\in\mathbb{C}$, let $\mathcal{D}_t$ be the twisted sheaf of differential operators on X such that $\mathcal{D}_t|_{U_0}\simeq\mathcal{D}_{U_0}$ and $\mathcal{D}_t|_{U_1}\simeq\mathcal{D}_{U_1}$ with the relationship on the intersection given by

$$\begin{aligned}\psi : \mathcal{D}_{C^*} &\longrightarrow \mathcal{D}_{\mathbb{C}^*}\\ \partial &\longmapsto \partial-\frac{t+1}{z}\end{aligned}$$

Let the local isomorphisms between $\mathcal{D}_t$ and $\mathcal{D}_X$ be given in such a way that the Lie algebra basis elements are associated with the following vector fields in our two charts: In U_0,

$$E=z^2\partial+(t+1)z, \qquad H=2x\partial+(t+1), \qquad F=-\partial.$$

In U_1,

$$E=z^2\partial \qquad H=2z\partial-(t+1) \qquad F=-\partial+\frac{n}{t+1}.$$

If we specialize to integral values of t, these are the sheaves of twisted differential operators on $\mathcal{O}(t+1)$ described previously.

Before we construct standard Harish-Chandra sheaves, we will calculate the operator corresponding to the Casimir element in our two charts. The Casimir element for $\mathfrak{sl}(2,\mathbb{C})$ is $\Omega = H^2+2EF+2FE\in\mathfrak{z}(\mathfrak{g})$. The following calculation shows that the Casimir operator in the chart U_0 acts by the scalar $(t+1)^2-2(t+1)$. We use the fact mentioned previously that $[\partial,z]=1$.

$$\begin{aligned}\Omega &= (2z\partial+(t+1))(2z\partial+(t+1))+2(z^2\partial+(t+1)z)(-\partial)+2(-\partial)(z^2\partial+(t+1)z)\\ &= 4z\partial z\partial+4(t+1)z\partial+(t+1)^2-2z^2\partial^2-2(t+1)z\partial-2\partial z^2\partial-2(t+1)\partial z\\ &= 2(z\partial z+z\partial z-z^2\partial-\partial z^2)\partial+2(t+1)(z\partial-\partial z)+(t+1)^2\\ &= (t+1)^2-2(t+1)+2((z\partial-\partial z)z+z(\partial z-z\partial))\partial\\ &= (t+1)^2-2(t+1)+2z\partial-2z\partial\\ &= (t+1)^2-2(t+1)\end{aligned}$$

Similarly, in the other chart U_1 a different calculation yields the same result. Here we use the fact mentioned previously that $z\partial z^{-1} = \partial - z^{-1}$.

$$\begin{aligned}
\Omega &= (2z\partial - (t+1))(2z\partial - (t+1)) + 2(z^2\partial)(-\partial + (t+1)z^{-1}) + 2(-\partial + (t+1)z^{-1})(z^2\partial) \\
&= 4z\partial z\partial - -4(t+1)z\partial + (t+1)^2 - 2z^2\partial^2 + 2(t+1)z^2\partial z^{-1} - 2\partial z^2\partial + 2(t+1)z\partial \\
&= 2(2z\partial z - z^2\partial - \partial z^2)\partial - 2(t+1)z(\partial - z\partial z^{-1}) + (t+1)^2 \\
&= 2((z\partial - \partial z)z + z(\partial z - z\partial))\partial - 2(t+1)z(\partial - (\partial - z^{-1})) + (t+1)^2 \\
&= 2z\partial - 2z\partial - 2(t+1)zz^{-1} + (t+1)^2 \\
&= (t+1)^2 - 2(t+1)
\end{aligned}$$

Since the Casimir element is in the center of the universal enveloping algebra of $\mathfrak{g}$, this computation confirms the fact that it acts by a scalar.

Our next step is to construct standard Harish-Chandra sheaves on X and realize their global sections as representations of $\mathfrak{g}$ by calculating the actions of H, E, F in U_0 and U_1. Before we can construct standard Harish-Chandra sheaves in our specific setting, we will give a basic idea of their general construction. A precise construction of standard Harish-Chandra sheaves can be found in [2].

Definition 4.1. A pair $(\mathfrak{g}, K)$, where $\mathfrak{g}$ is a complex semisimple Lie algebra and $K \subset \mathrm{Int}(\mathfrak{g})$ is a subgroup, is called a *Harish-Chandra pair* if K acts on the flag variety X of $\mathfrak{g}$ with finitely many orbits.

A *Harish-Chandra sheaf* for the Harish-Chandra pair $(\mathfrak{g}, K)$ is a K-homogeneous $\mathcal{D}_\lambda$-module on X where the action of $\mathfrak{g}$ restricted to the Lie algbra of K agrees with the differential of the action of K. For a precise definition of this compatibility condition, see [2]. For intuition, the reader may consider Harish-Chandra sheaves to be the $\mathcal{D}$-module analogue to Harish-Chandra modules, where the compatibility conditions between the actions are more straightforward. Standard Harish-Chandra sheaves are parameterized by pairs of geometric data (Q, τ), where Q is a K-orbit on X and τ is an irreducible K-homogeneous connection on Q. They are constructed in the following way. Let

$$i : Q \longrightarrow X$$

be inclusion. Then the standard Harish-Chandra sheaf corresponding to (Q, τ) is $\mathcal{I}(Q, \tau) := i_+(\tau)$, where i_+ is the $\mathcal{D}$-module direct image functor. Some properties of $\mathcal{I}(Q, \tau)$ that fall from its construction are the following: First, $\mathcal{I}(Q, \tau)$ has a unique irreducible submodule, $\mathcal{L}(Q, \tau)$ and all irreducible Harish-Chandra modules arise in this way. Second, the support of $\mathcal{I}(Q, \tau)$ is $\bar{Q}$. Third, if Q is an open affine subvariety, $\mathcal{I}(Q, \tau)|_Q = \tau$.

4.1 The Harish-Chandra Pair $(\mathfrak{sl}(2, \mathbb{C}), N)$

Now, we specialize to the Harish-Chandra pair $(\mathfrak{sl}(2, \mathbb{C}), N)$, where N is the unipotent subgroup of $PSL(2, \mathbb{C})$ described previously. The N-orbits on X are $\{0\}$ and U_1. The only N-homogeneous irreducible connections on $\{0\}$ and U_1 are $\mathcal{O}_{\{0\}}$ and $\mathcal{O}_{U_1}$. This implies that there are two standard Harish-Chandra sheaves for a choice of $t \in \mathbb{C}$, one corresponding to each N-orbit. We denote the standard Harish-Chandra sheaves in the category of D_t-modules by $\mathcal{I}(\{0\}, t)$ and $\mathcal{I}(U_1, t)$. Here the t indicates the twist in D_t since there is only once choice of irreducible connection on each orbit.

We begin by analyzing $\mathcal{I}(\{0\}, t)$. This is a $\mathcal{D}_t$-module supported at $\{0\}$, so $\mathcal{I}(\{0\}, t)|_{U_1}$=0. Therefore it is enough to examine $\mathcal{I}(\{0\}, t)|_{U_0}$. We claim that $\mathcal{I}(\{0\}, t)$ is isomorphic to the $\mathcal{D}_{\mathbb{C}}$-module of truncated Laurent series centered at 0. We begin by discussing precisely what we mean by the $\mathcal{D}_{\mathbb{C}}$-module of truncated Laurent series at 0.

Let $\mathcal{M}$ be the sheaf of Laurent series centered at 0 on $\mathbb{C}$. $\mathcal{M}$ is clearly a $\mathcal{D}_{\mathbb{C}}$-module. The structure sheaf $\mathcal{O}_{\mathbb{C}}$ on $\mathbb{C}$ is a maximal submodule of $\mathcal{M}$. The quotient $\mathcal{D}_{\mathbb{C}}$-module $\mathcal{P} = \mathcal{M}/\mathcal{O}_{\mathbb{C}}$ is isomorphic to the module of Laurent series with strictly negative terms. This is what we refer to as the $\mathcal{D}_{\mathbb{C}}$-module of *truncated Laurent series.* It is irreducible, generated by z^{-1}, and supported at $\{0\}$.

Since irreducible Harish-Chandra sheaves are determined uniquely by their geometric data, there is only one irreducible $\mathcal{D}_n$-module supported at $\{0\}$ up to isomorphism. Therefore, $\mathcal{I}(\{0\}, t) \simeq \mathcal{P}$ is irreducible for any $t \in \mathbb{C}$. Now that we have a concrete realization of our module, we can calculate the action of E, F and H. Since $\mathcal{I}(\{0\}, t)$ is supported in U_0, we can do our calculations entirely in the coordinate system on U_0. Consider the action of E, F and H on the generator z^{-1}:

$$E \cdot z^{-1} = (z^2\partial + (t+1)z) \cdot z^{-1} = z^2(-z^{-2}) + (t+1) = -1 + (t+1) = 0$$

$$F \cdot z^{-1} = -\partial \cdot z^{-1} = z^{-2}$$

$$H \cdot z^{-1} = (2z\partial + (t+1)) \cdot z^{-1} = 2z(-z^{-2}) + (t+1)z^{-1} = -2z^{-1} + (t+1)z^{-1} = (t-1)z^{-1}$$

We see that E annihilates z^{-1}, $F^m \cdot z^{-1} = (-1)^{m+1}m!z^{-(m+1)}$ span $\mathcal{P}$ for $m \in \mathbb{Z}_+$, and H acts on z^{-1} by the scalar $t-1$. From our previous calculation, we know that the Casimir element Ω acts on z^{-1} as multiplication by $(t+1)^2 - 2(t+1)$. From these actions we conclude that

$$\Gamma(X, \mathcal{I}(\{0\}, n)) \simeq M(t),$$

where $M(t)$ is the Verma module of highest weight $t - 1 \in \mathbb{C}$.

Next we study the Harish-Chandra sheaf on the open orbit, $\mathcal{I}(U_1, t)$. Since the support of this sheaf is all of X, we need to use both charts and the twist given by ψ to understand its structure. Recall that the coordinates on U_0 are z and ∂, which is differentiation with respect to z. The coordinates on U_1 are ζ and ∂_ζ, which is differentiation with respect to ζ. On the intersection of the charts, $V \simeq C^*$, we have the relationship $\zeta = \frac{1}{z}$ and $\partial_\zeta = -z^2\partial$. The last assertion falls from the following calculation: Let $f : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ be a regular function. Then by the chain rule,

$$\partial_\zeta f(z) = \partial_\zeta f(\frac{1}{\zeta}) = -\zeta^{-2} f'(\frac{1}{\zeta}) = -z^2 f'(z).$$

Let $i_0 : U_0 \longrightarrow X$ and $i_1 : U_1 \longrightarrow X$ be inclusions of varieties. Then by definition, $\mathcal{I}(U_1, t) = i_{1+}(\mathcal{O}_{U_1}) = i_{1\bullet}(\mathcal{O}_{U_1})$. Here, $i_{1\bullet}$ is the sheaf-theoretic direct image, which agrees with the $\mathcal{D}$-module direct image in this setting. By construction, $\mathcal{I}(U_1, t)|_{U_1} = \mathcal{O}_{U_1}$ is irreducible. Since irreducibility of $\mathcal{D}_t$-modules is a local property, to examine the reducibility of $\mathcal{I}(U_1, t)$, it remains to study its restriction to U_0. We do so by using the relationship between trivializations on the intersection V.

Since U_1 is affine, we can study $\mathcal{I}(U_1, t)|_{U_1} = \mathcal{O}_{U_1}$ by instead considering its global sections, $\Gamma(U_1, \mathcal{O}_{U_1}) = R(\mathbb{C}) = \mathbb{C}[\zeta]$. The $D_{\mathbb{C}}$-module $\mathbb{C}[\zeta]$ is generated by 1. When we restict $\mathbb{C}[\zeta]$ to $\mathbb{C}^*$, localization gives us the $D_{\mathbb{C}^*}$-module $R(\mathbb{C}^*) \otimes_{\mathbb{C}[\zeta]} \mathbb{C}[\zeta] = R(\mathbb{C}^*)$ on the intersection. Here $R(\mathbb{C}^*)$ is a $D_{\mathbb{C}^*}$-module with the standard actions $\partial \cdot z^k = kz^{k-1}$ and $z \cdot z^k = z^{k+1}$. The relationship between our trivializations

$$\begin{aligned} \psi : D_{\mathbb{C}^*} &\longrightarrow D_{\mathbb{C}^*} \\ \partial &\longmapsto \partial - \frac{t+1}{z} \end{aligned}$$

gives us another module structure on $R(\mathbb{C}^*)$, where

$$\partial \star z^k = \psi(\partial) z^k = (\partial - \frac{t+1}{z}) z^k = (k - (t+1)) z^{k-1}, \text{ and}$$

$$z \star z^k = \psi(z) z^k = z^{k+1}.$$

With this action, $R(\mathbb{C}^*)$ is isomorphic as a $D_{\mathbb{C}^*}$-module to N_t from §1. Therefore, global sections of our standard Harish-Chandra sheaf on U_0 are $\Gamma(U_1, \mathcal{I}(U_1,t)|_{U_0}) \simeq i_\bullet(N_t)$, where $i : \mathbb{C}^* \longrightarrow \mathbb{C}$. From our arguments in §1, we know that this $D_{\mathbb{C}}$-module is irreducible if and only if $t \notin \mathbb{Z}$. This leaves us with two cases.

Our first case is $t \notin \mathbb{Z}$. Here, $\mathcal{I}(U_1,t)|_{U_0}$ and $\mathcal{I}(U_1,t)|_{U_1}$ are both irreducible. Since irreducibility of $\mathcal{D}_t$-modules is a local property, we conclude that if $t \notin \mathbb{Z}$, $\mathcal{I}(U_1,t)$ is an irreducible $\mathcal{D}_t$-module.

Our second case is when $t \in \mathbb{Z}$. In this case, we claim that our twisted sheaf of functions $\mathcal{O}(t+1)$ is a sub-$\mathcal{D}_t$-module of $\mathcal{I}(U_1,t)$. Indeed, since sheaf theoretic direct image and restiction are adjoint functors, we have

$$Hom_{\mathcal{D}_t|_{U_1}}(\mathcal{O}(t+1)|_{U_1}, \mathcal{O}_{U_1}) = Hom_{\mathcal{D}_t}(\mathcal{O}(t+1), i_{1\bullet}(\mathcal{O}_{U_1})).$$

Since $\mathcal{O}(t+1)|_{U_1} \simeq \mathcal{O}_{U_1}$ and $i_{1\bullet}(\mathcal{O}_{U_1}) = \mathcal{I}(U_1,t)$, the identity morphism from $\mathcal{O}_{U_1}$ to itself corresponds to a non-trivial morphism $\phi : \mathcal{O}(t+1) \longrightarrow \mathcal{I}(U_1,t)$. $\mathcal{O}(t+1)$ is irreducible as a $\mathcal{D}_t$-module, so ϕ must be an injective morphism. Therefore, we have a short exact sequence of $\mathcal{D}_t$-modules

$$0 \longrightarrow \mathcal{O}(t+1) \longrightarrow \mathcal{I}(U_1,t) \longrightarrow \mathcal{R} \longrightarrow 0.$$

If we restrict this sequence to U_1, we get

$$0 \longrightarrow \mathcal{O}_{U_1} \longrightarrow \mathcal{O}_{U_1} \longrightarrow \mathcal{R}|_{U_1} \longrightarrow 0,$$

since $\mathcal{O}(t+1)|_{U_1} \simeq \mathcal{O}_{U_1}$ and $\mathcal{I}(U_1,t)|_{U_1} = \mathcal{O}_{U_1}$. This implies that $\mathcal{R}|_{U_1}$=0, so supp$\mathcal{R} = \{0\}$. If we restrict to U_0, we have

$$0 \longrightarrow \mathcal{O}_{U_0} \longrightarrow \mathcal{I}(U_1,t)|_{U_0} \longrightarrow \mathcal{R}|_{U_0} \longrightarrow 0.$$

Taking global sections of this, we have

$$0 \longrightarrow R(\mathbb{C}) \longrightarrow M \longrightarrow \mathcal{R}|_{U_0} \longrightarrow 0,$$

where $M = \Gamma(U_0, \mathcal{M})$ is the D_{U_0}-module of Laurent series mentioned earlier. This implies that $\mathcal{R}|_{U_0} \simeq \mathcal{P} \simeq \mathcal{I}(\{0\},t)$, the $\mathcal{D}$-module of truncated Laurent series. This short exact sequence gives a long exact sequence in sheaf cohomology.

$$\begin{aligned} 0 &\longrightarrow \Gamma(X, \mathcal{O}(t+1)) \longrightarrow \Gamma(X, \mathcal{I}(U_1,t)) \longrightarrow \Gamma(X, \mathcal{I}(\{0\},t)) \longrightarrow \\ &\longrightarrow H^1(\mathcal{O}(t+1)) \longrightarrow H^1(\mathcal{I}(U_1,t)) \longrightarrow H^1(\mathcal{I}(\{0\},t)) \longrightarrow 0 \end{aligned}$$

If $t+1 \in \mathbb{Z}_{<0}$, we know from the Borel-Weil theorem that $\Gamma(X, \mathcal{O}(t+1)) = F_{-(t+1)}$, the finite dimensional $\mathfrak{g}$-representation of highest weight $-(t+1)$, and $H^1(\mathcal{O}(t+1)) = 0$. From our previous analysis of the standard Harish-Chandra sheaf on the closed orbit, we know that $\Gamma(X, \mathcal{I}(\{0\},t)) = M(t)$, an irreducible Verma module. In this case, the final three terms of our long exact sequence vanish, and we have the short exact sequence

$$0 \longrightarrow F_{-(t+1)} \longrightarrow \Gamma(X, \mathcal{I}(U_1,t)) \longrightarrow M(t) \longrightarrow 0.$$

If $t+1 \in \mathbb{Z}_{\geq 0}$, we know from the Borel-Weil-Bott theorem that $\Gamma(X, \mathcal{O}(t+1)) = 0$ and $H^1(\mathcal{O}(t+1)) = F_{t+1}$, the finite dimensional $\mathfrak{g}$-module of highest weight $t+1$. From our previous analysis, we know that $\Gamma(X, \mathcal{I}(\{0\},t)) = M(t)$, a reducible Verma module. In this case, the final two terms of our long exact sequence vanish and we have the short exact sequence

$$0 \longrightarrow \Gamma(X, \mathcal{I}(U_1,t)) \longrightarrow M(t) \longrightarrow F_{t+1} \longrightarrow 0.$$

We end our computations for the Harish-Chandra pair $(\mathfrak{g}, N)$ by computing the action of H, E, and F on $\Gamma(X, \mathcal{I}(U_1, t))$. We do the computations on $\Gamma(U_1, \mathcal{I}(U_1, t)|_{U_1}) \simeq \mathbb{C}[\zeta]$. We start by remarking that by using the relationships $z = \frac{1}{\zeta}$ and $\partial = -\zeta^2\partial_\zeta$ between the coordinate systems, in U_1-coordinates,

$$E = -\partial_\zeta \qquad H = -2\zeta\partial_\zeta - (t+1) \qquad F = \zeta^2\partial_\zeta + (t+1)\zeta.$$

This allows us to compute the action of the Lie algebra basis on the constant function 1 and ζ^k for $k \in \mathbb{Z}_+$:

$$\begin{aligned} E \cdot 1 &= -\partial_\zeta \cdot 1 = 0 & E \cdot \zeta^k &= -\partial_\zeta \cdot \zeta^k = -k\zeta^{k-1} \\ H \cdot 1 &= (-2\zeta\partial_\zeta - (t+1)) \cdot 1 = -(t+1) & H \cdot \zeta^k &= (-2\zeta\partial_\zeta - (t+1)) \cdot \zeta^k = -(t+1+2k)\zeta^k \\ F \cdot 1 &= (-\zeta^2\partial_\zeta + (t+1)\zeta) \cdot 1 = (t+1)\zeta & F \cdot \zeta^k &= (-\zeta^2\partial_\zeta + (t+1)\zeta) \cdot \zeta^k = (k+t+1)\zeta^{k+1} \end{aligned}$$

We see from this computation that E annihilates constant functions, H acts on the generator 1 by the scalar $-(t+1)$ and F sends ζ^k to ζ^{k+1} and scales by a value depending on k and t. Again, the Casimir element Ω acts in this chart by the scalar $(t+1)^2 - 2(t+1)$. We see that this case splits into two more cases.

If $t+1 \geq 0$, 1 generates $\Gamma(X, \mathcal{I}(U_1, t))$, which is isomorphic to the Verma module of highest weight $-(t+1)$, $M(-t)$. If $t < 0$, $F \cdot \zeta^{-(t+1)} = 0$, so the irreducible finite dimensional $\mathfrak{g}$-module of highest weight $-(t+1)$ is a submodule of $\Gamma(X, \mathcal{I}(U_1, t))$, and the quotient of $\Gamma(X, \mathcal{I}(U_1, t))$ is the Verma module $M(t)$. In this case, we see that $\Gamma(X, \mathcal{I}(U_1, t))$ has the structure of a dual Verma module. We summarize our findings in the following theorem.

Theorem 4.2. *(i) $\mathcal{I}(\{0\}, t)$ is an irreducible $\mathcal{D}_t$-module for any $t \in \mathbb{C}$.*

(ii) $\Gamma(X, \mathcal{I}(\{0\}, t)) = M(t)$ for any $t \in \mathbb{C}$.

(iii) If $t \notin \mathbb{Z}$, $\mathcal{I}(U_1, t)$ is an irreducible $\mathcal{D}_t$-module.

(iv) If $t \in \mathbb{Z}$, then there is a short exact sequence of $\mathcal{D}_t$-modules

$$0 \longrightarrow \mathcal{O}(t+1) \longrightarrow \mathcal{I}(U_1, t) \longrightarrow \mathcal{I}(\{0\}, t) \longrightarrow 0.$$

(v) If $t \in \mathbb{Z}_{\geq 0}$, $\Gamma(X, \mathcal{I}(U_1, t)) = M(-t)$.

(vi) If $t \in \mathbb{Z}_{<0}$, there is a short exact sequence of $\mathfrak{g}$-modules

$$0 \longrightarrow F_{-(t+1)} \longrightarrow \Gamma(X, \mathcal{I}(U_1, t)) \longrightarrow M(t) \longrightarrow 0,$$

where $F_{-(t+1)}$ is the irreducible finite dimensional $\mathfrak{g}$-module of highest weight $-(t+1)$.

4.2 The Harish-Chandra Pair $(\mathfrak{sl}(2, \mathbb{C}), K)$

Now we turn our attention to our second Harish-Chandra pair. Let $K \subset PSL(2, \mathbb{C})$ be the complexification of the maximal compact subgroup $SO(2) \subset PSL(2, \mathbb{R})$. Then

$$K = T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{C} \right\} \simeq \mathbb{C}^*.$$

K acts on X with three orbits: $\{0\}, \{\infty\}$, and V. The standard Harish-Chandra sheaves supported on the closed orbits $\{0\}$ and $\{\infty\}$ have the same structure as the standard Harish-Chandra sheaf supported at $\{0\}$ in the previous example, so we will concentrate on the open orbit V. We use the coordinate z on U_0 to identify V with $\mathbb{C}^*$:

$$\begin{aligned} V &\longrightarrow \mathbb{C}^* \\ [1:x_1] &\longmapsto x_1 \end{aligned}$$

K acts on V by the restricted action of G on X:

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot [1:x_1] = [a : a^{-1}x_1] = [1 : a^{-2}x_1].$$

Therefore, under the associations $K \simeq \mathbb{C}^*$ and $V \simeq \mathbb{C}^*$, our action map corresponds to the action of $\mathbb{C}^*$ on itself via

$$\begin{aligned} \mathbb{C}^* \times \mathbb{C}^* &\longrightarrow \mathbb{C}^* \\ (a, z) &\longmapsto a^{-2}z \end{aligned}$$

We claim that there are two irreducible K-homogeneous connections on $\mathbb{C}^*$. The justification for this is as follows:

Let τ be a $\mathbb{C}^*$-homogeneous connection on $\mathbb{C}^*$. Then $M = \Gamma(\mathbb{C}^*, \tau)$ is a $\mathbb{C}^*$-equivariant $R(\mathbb{C}^*)$-module. More precisely, if $\pi : \mathbb{C}^* \longrightarrow \mathcal{L}(M)$ is the action of $\mathbb{C}^*$ on M giving it the structure of a $\mathbb{C}^*$-module and $\rho : \mathbb{C}^* \longrightarrow \mathcal{L}(R(\mathbb{C}^*))$ is the action of $\mathbb{C}^*$ on $R(\mathbb{C}^*)$ induced from the action of $\mathbb{C}^*$ on itself described above (for $f \in R(\mathbb{C}^*)$ and $a \in \mathbb{C}^*$, $(\rho(a)f)(z) = f(a^2z)$), then the $R(\mathbb{C}^*)$-module action map

$$R(\mathbb{C}^*) \times M \longrightarrow M$$

is $\mathbb{C}^*$-equivariant; i.e. for $f \in R(\mathbb{C}^*), v \in M$, and $a \in \mathbb{C}^*$,

$$\pi(a)(f \cdot v) = (\rho(a)f) \cdot \pi(a)v.$$

Since all irreducible representations of $\mathbb{C}^*$ are one dimensional vector spaces where $a \in \mathbb{C}^*$ acts by a^n for some $n \in \mathbb{Z}$, any semisimple $\mathbb{C}^*$-module has a spanning set $\{v_n\}$ parameterized by a subset of the integers such that $a \cdot v_n = a^n \cdot v_n$ for $a \in \mathbb{C}^*$. We can apply this observation to both $R(\mathbb{C}^*)$ and M to get spanning sets $\{f_n\}$ of $R(\mathbb{C}^*)$ and $\{v_\ell\}$ of M.

On $R(\mathbb{C}^*)$, this tells us that

$$f_n(a^2z) = \rho(a)f_n(z) = a^n f_n(z),$$

so $f_n(a^2) = a^n f_n(1)$. We can normalize $f_n(1) = 1$ to conclude that $f_n(a^2) = a^n$ so $f_n(z) = z^{n/2}$. This implies that n is even. In other words, the indexing set of our spanning set contains only even integers. So by re-indexing and letting $\{F_m\}$ be the new spanning set, we see that $R(\mathbb{C}^*)$ decomposes as a $\mathbb{C}^*$-module in the following way:

$$R(\mathbb{C}^*) = \bigoplus_{m \in \mathbb{Z}} \mathbb{C} \cdot F_m,$$

where $F_m(z) = z^m$ and $\rho(a)F_m(z) = a^{2m}F_m(z)$.

Now, the $\mathbb{C}^*$-equivariance of M implies that

$$\pi(a)F_m v_\ell = (\rho(a)F_m)\pi(a)v_\ell = (a^{2m}F_m)a^\ell v_\ell = a^{2m+\ell}F_m v_\ell.$$

If we now assume that M is irreducible as a $\mathbb{C}^*$-equivariant $R(\mathbb{C}^*)$-module, we can conclude that if $v_\ell \in M$ such that $v_\ell \neq 0$, then $F_m v_\ell \neq 0$ and $F_m v_\ell$ spans a one dimensional sub-$\mathbb{C}^*$-module $M_{2m+\ell}$, where $a \in \mathbb{C}^*$ acts by $a^{2m+\ell}$. Therefore, M decomposes as

$$M = \bigoplus_{m \in \mathbb{Z}} M_{2m+\ell}.$$

This leaves us two cases. If ℓ is even, then $M \simeq R(\mathbb{C}^*)$ as a $\mathbb{C}^*$-equivariant $R(\mathbb{C}^*)$-module. If ℓ is odd, then

$$M = \bigoplus_{m \in \mathbb{Z}} M_{2m+1}.$$

Denote $v_1 = z^{1/2}$. Then in the second case, $M = R(\mathbb{C}^*) \cdot z^{1/2}$. This completes the proof of our claim. Since $\mathbb{C}^*$ is affine, we can conclude that there are only two irreducible K-homogeneous connections on $\mathbb{C}^*$ up to isomorphism: $\tau_0 = \mathcal{O}_{C^*}$ and τ_1 such that $\Gamma(\mathbb{C}^*, \tau_1) = R(\mathbb{C}^*) \cdot z^{1/2}$.

An alternate way to conclude that there are only two irreducible K-homogeneous connections on $\mathbb{C}^*$ is to examine the representations of the stabilizer in K of a point in $\mathbb{C}^*$. For a generic $z \in \mathbb{C}^*$, $\text{stab}_K\{z\} = \{\pm 1\} \simeq \mathbb{Z}/2$. There are two irreducible representations of $\mathbb{Z}/2$, the trivial representation and the sign representation, and these correspond to the two irreducible K-homogeneous connections on $\mathbb{C}^*$ described above. This parameterization of K-homogeneous connections on an orbit Q by irreducible representations of the stabilizer in K of a point in Q holds in general, but we will not discuss the details in this document.

We can differentiate the action of K on $R(\mathbb{C}^*)$ described earlier to get an action of the Lie algebra basis element H on $R(\mathbb{C}^*)$. Let $f \in R(\mathbb{C}^*)$. Then,

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \cdot f(z) &= \frac{d}{dt}\bigg|_{t=0} \left(1 + t\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \cdot f(z) \\
&= \frac{d}{dt}\bigg|_{t=0} \begin{pmatrix} 1+t & 0 \\ 0 & \frac{1}{1+t} \end{pmatrix} \cdot f(z) \\
&= \frac{d}{dt}\bigg|_{t=0} f((1+t)^2 z) \\
&= (2(1+t) z f'(1+t)^2 z))\big|_{t=0} \\
&= 2z f'(z).
\end{aligned}$$

From this we conclude that $H = 2z\partial$. We can see from our previous calculations that this is not the differential operator associated to H in our trivializations of $\mathcal{D}_t$ on U_0 and U_1. To remedy this, we can introduce another trivialization of $\mathcal{D}_t$ on $\mathbb{C}^*$ where H corresponds to the differential operator $2z\partial$ on $\mathbb{C}^*$. We obtain this trivialization by resticting our original z-trivialization on U_0 to $\mathbb{C}^*$ and twisting it by the $D_{\mathbb{C}^*}$-automorphism given by

$$\partial \longmapsto \partial - \frac{t+1}{2z}.$$

This gives a new trivialization of $\mathcal{D}_t|_{\mathbb{C}^*}$ where our Lie algebra basis elements correspond to the following vector fields:

$$E = z^2\partial + \frac{t+1}{2}z \qquad F = -\partial + \frac{t+1}{2} \qquad H = 2z\partial$$

We can calculate the operator associated to the Casimir element in this trivialization as well, and we see that again Ω acts by $(t+1)^2 - 2(t+1)$:

$$\begin{aligned}
\Omega &= (2z\partial)(2z\partial) + 2(z^2\partial + \frac{t+1}{2}z)(-\partial + \frac{t+1}{2}z^{-1}) + 2(-\partial + \frac{n}{2}z^{-1})(z^2\partial + \frac{t+1}{2}z) \\
&= 4z\partial z\partial - 2z^2\partial^2 + nz^2\partial z^{-1} - nz\partial + \frac{n^2}{2} - 2\partial z^2\partial - n\partial z + nz\partial + \frac{n^2}{2} \\
&= 2(z(\partial z - z\partial) + (z\partial - \partial z)z)\partial + (t+1)z(\partial - z^{-1}) + (t+1)^2 - (t+1)\partial z \\
&= (t+1)z\partial - (t+1)zz^{-1} + (t+1)^2 - (t+1)\partial z \\
&= (t+1)(z\partial - \partial z) - (t+1) + (t+1)^2 \\
&= (t+1)^2 - 2(t+1)
\end{aligned}$$

Our last step is to study the structure of the two standard Harish-Chandra sheaves corresponding to the two connections on $\mathbb{C}^*$. We denote these two $\mathcal{D}_t$-modules by $\mathcal{I}(\mathbb{C}^*, \tau_0, t)$ and $\mathcal{I}(\mathbb{C}^*, \tau_1, t)$. To analyze the reducibility of $\mathcal{I}(\mathbb{C}^*, \tau_i, t)$, we will examine $\mathcal{I}(\mathbb{C}^*, \tau_i, t)|_{U_j}$ for $i, j \in \{0, 1\}$, as we did in with the previous Harish-Chandra pair. First we discuss $\mathcal{I}(\mathbb{C}^*, \tau_1, t)$. The global sections of $\mathcal{I}(\mathbb{C}^*, \tau_1, t)$ on $\mathbb{C}^*$ are spanned by $e_k = z^{k+\frac{1}{2}}$ for $k \in \mathbb{Z}$, since by construction,

$$\Gamma(\mathbb{C}^*, \mathcal{I}(\mathbb{C}^*, \tau_1, t)|_{\mathbb{C}^*}) = \Gamma(\mathbb{C}^*, \tau_1) = R(\mathbb{C}^*) \cdot z^{1/2}.$$

This is a $D_{\mathbb{C}^*}$-module with action of ∂ and z given by

$$\partial \cdot e_k = (k + \frac{1}{2})e_{k-1} \qquad z \cdot e_k = e_{k+1}.$$

We can recognize this module as the $D_{\mathbb{C}^*}$-module $N_{-\frac{1}{2}}$ from §1. Now we have three trivializations of $\mathcal{D}_t$. They are related on $\mathbb{C}^*$ by ring isomorphisms

$$D_{\mathbb{C}^*} \xrightarrow{\psi_0} D_{\mathbb{C}^*} \xleftarrow{\psi_1} D_{\mathbb{C}^*},$$

where $\psi_0(\partial) = \partial - \frac{t+1}{2z}$ and $\psi_1(\partial) = \partial + \frac{t+1}{2z}$. Note that the composition $\psi_0 \circ \psi_1^{-1} = \psi$, our twist described earlier. These two maps give us two alternate module structures on $\Gamma(\mathbb{C}^*, \tau_1)$, where the actions are:

$$\partial \star e_k = \psi_0(\partial) \cdot e_k = (\partial - \frac{t+1}{2z})e_k = (k + \frac{1}{2})e_{k-1} - (\frac{t+1}{2})e_{k-1} = (k - \frac{t}{2})e_{k-1}, \text{ and}$$

$$\partial * e_k = \psi_1(\partial) \cdot e_k = (\partial + \frac{t+1}{2z})e_k = (k + \frac{1}{2})e_{k-1} + (\frac{t+1}{2})e_{k-1} = (k + \frac{t+2}{2})e_{k-1}.$$

The first action gives a module isomorphic to $N_{\frac{t}{2}}$ and the second action gives a module isomorphic to $N_{-\frac{t+2}{2}}$. Therefore, we conclude that

$$\begin{aligned}
&\Gamma(U_0, \mathcal{I}(\mathbb{C}^*, \tau_1, t)|_{U_0}) \simeq i_\bullet(N_{\frac{t}{2}}), \text{ and} \\
&\Gamma(U_1, \mathcal{I}(\mathbb{C}^*, \tau_1, t)|_{U_1}) \simeq i_\bullet(N_{-\frac{t+2}{2}}).
\end{aligned}$$

From our previous analysis, we know that each of these are irreducible as $D_{\mathbb{C}}$-modules if and only both $\frac{t}{2}$ and $-\frac{t+2}{2}$ are not integers, which happens exactly when t is not an even integer. We conclude that $\mathcal{I}(\mathbb{C}^*, \tau_1, t)$ is irreducible if and only if $t \notin 2\mathbb{Z}$.

Similarly, global sections of $\mathcal{I}(\mathbb{C}^*, \tau_0, t)$ on $\mathbb{C}^*$ are spanned by z^k for $k \in Z$, so $\Gamma(U_0, \mathcal{I}(\mathbb{C}^*, \tau_0, t)|_{U_0}) \simeq i_\bullet(N_{\frac{t-1}{2}})$, and $\Gamma(U_1, \mathcal{I}(\mathbb{C}^*, \tau_0, t)|_{U_1}) \simeq i_\bullet(N_{-\frac{t+1}{2}})$. Therefore, by the same argument we conclude that $\mathcal{I}(\mathbb{C}^*, \tau_0, t)$ is irreducble if and only if $t \notin 2\mathbb{Z}+1$.

Finally, we realize the global sections of $\mathcal{I}(\mathbb{C}^*, \tau_i, t)$ as $\mathfrak{g}$-modules. The global sections of $\mathcal{I}(\mathbb{C}^*, \tau_i, t)$ are spanned by $e_k = z^{k+\frac{i}{2}}$. For $i \in \{0, 1\}$, the actions of H, E, and F are:

$$
\begin{aligned}
H \cdot e_k &= 2z\partial \cdot z^{k+\frac{i}{2}} = 2(k+\frac{i}{2})z^{k+\frac{i}{2}} = 2(k+\frac{i}{2})e_k \\
E \cdot e_k &= (z^2 + \frac{t+1}{2}z) \cdot z^{k+\frac{i}{2}} = (k+\frac{i}{2}+\frac{t+1}{2})z^{k+\frac{i}{2}+1} = (k+\frac{i}{2}+\frac{t+1}{2})e_{k+1} \\
F \cdot e_k &= (-\partial + \frac{t+1}{2z}) \cdot z^{k+\frac{i}{2}} = (-k-\frac{i}{2}+\frac{t+1}{2})z^{k+\frac{i}{2}-1} = (-k-\frac{i}{2}+\frac{t+1}{2})e_{k-1}
\end{aligned}
$$

We can see that $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_i, t))$ is reducible only when E or F annihilates some e_k; i.e. when $k+\frac{i}{2}+\frac{t+1}{2} = 0$ or $-k-\frac{i}{2}+\frac{t+1}{2} = 0$ for some $k \in \mathbb{Z}$. This happens precisely when $t+1 \in 2\mathbb{Z}+i$. The structure of this reducible module depends on the sign of $t+1$. If $t+1 \in \mathbb{Z}_{>0}$, then $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_i, t))$ has two submodules - one spanned by $\{e_k\}$ for $k \geq \frac{t+1-i}{2}$, and the other spanned by $\{e_k\}$ for $k \leq -\frac{t+1+i}{2}$. If $t+1 \in \mathbb{Z}_{<0}$, then $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_i, t))$ has a finite dimensional submodule spanned by $\{e_k\}$ for $\frac{t+1-i}{2} \leq k \leq -\frac{t+1+i}{2}$. We can see from this structure that $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_i, t))$ as a principle series representation of $SL(2, \mathbb{R})$. In the case where $i = 0$ and $t+1 = 0$, the representation $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_i, t))$ is a limit of two discrete series representations. ***DOUBLE CHECK THIS. LANG?**

We summarize our conclusions about the reducibilty of the Harish-Chandra sheaves for $(\mathfrak{sl}(2, \mathbb{C}), K)$ in the following theorem.

Theorem 4.3. *(i)* $\mathcal{I}(\mathbb{C}^*, \tau_1, t)$ *is irreducible as a* $\mathcal{D}_t$*-module if and only if* $t \notin 2\mathbb{Z}$.

(ii) $\mathcal{I}(\mathbb{C}^*, \tau_0, t)$ *is irreducible as a* $\mathcal{D}_t$*-module if and only if* $t \notin 2\mathbb{Z}+1$.

(iii) $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_1, t))$ *is irreducible as a* $\mathfrak{g}$*-module if and only if* $t \in 2\mathbb{Z}$.

(iv) $\Gamma(X, \mathcal{I}(\mathbb{C}^*, \tau_0, t))$ *is irreducible as a* $\mathfrak{g}$*-module if and only if* $t \in 2\mathbb{Z}+1$.

5 η-twisted Harish-Chandra Sheaves

In this final section, we discuss the $\mathcal{D}_t$-modules which correspond to a class of representations of $\mathfrak{g}$ called Whittaker modules. These are η-twisted Harish-Chandra sheaves, where $\eta : \mathrm{Lie}N \longrightarrow \mathbb{C}$ is a character of the the Lie algebra of N. For non-degenerate η, there is only one standard object in this category, and this standard object is irreducible. This is the geometric analogue to the main algebraic result of Kostant in [1]. In this section, we prove this fact directly for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. We begin with the general definition of η-twisted Harish-Chandra sheaves.

Definition 5.1. Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair, and let $\mathfrak{k} = \mathrm{Lie}K$ be the Lie algebra of K. Let $\eta \in \mathfrak{k}^*$ be a character of $\mathfrak{k}$, and $\lambda \in \mathfrak{h}^*$, where $\mathfrak{h}$ is the abstract Cartan subalgebra of $\mathfrak{g}$. We say $\mathcal{V}$ is a $(\mathcal{D}_\lambda, K, \eta)$-module (also called an *η-twisted Harish-Chandra sheaf*) if the following conditions are satisfied.

(i) $\mathcal{V}$ is a coherent $\mathcal{D}_\lambda$-module,

(ii) $\mathcal{V}$ is a K-equivariant $\mathcal{O}_X$-module, and

(iii) $\pi(\xi) = \mu(\xi) + \eta(\xi)$ for all $\xi \in \mathfrak{k}$, and the morphism

$$\mathcal{D}_\lambda \otimes_{\mathcal{O}_X} \mathcal{V} \longrightarrow \mathcal{V}$$

is K-equivariant. Here π is the action of $\mathfrak{k}$ on $\mathcal{V}$ induced by the $\mathcal{D}_\lambda$-action, and μ is the differential of the K-action.

Standard objects in the category $\mathcal{M}(\mathcal{D}_\lambda, K, \eta)$ of η-twisted Harish-Chandra sheaves are constructed analogously to those in the untwisted category. That is, standard objects in $\mathcal{M}(\mathcal{D}_\lambda, K, \eta)$ are parameterized by pairs (Q, τ), where $Q \subset X$ is a K-orbit and τ is an irreducible η-twisted K-homogeneous connection on Q. If $i : Q \longrightarrow X$ is inclusion, then the standard η-twisted Harish-Chandra sheaf corresponding to the geometric data (Q, τ) is $\mathcal{I}(Q, \tau, \eta) = i_+(\tau)$. If $\eta = 0$, this construction gives our previous definition of Harish-Chandra sheaves. As in the untwisted case, constructing standard objects in this category amounts to constructing irreducible K-homogeneous connections on K-orbits. But in contrast to the untwisted case, in this setting we seek connections where the two natural actions of $\mathfrak{k}$ differ by η. We will see in the following calculations that not all K-orbits admit such connections.

We dedicate the rest of this section to to calculating the η-twisted Harish-Chandra sheaves for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and N the unipotent subgroup of $PSL(2, \mathbb{C})$ described in §4. Let $\mathfrak{n} = \mathrm{Lie}(N)$ and fix a nonzero character $\eta \in \mathfrak{n}^*$. Let $t \in \mathbb{C}$. The two N-orbits are $\{0\}$ and U_1. First we analyze the closed orbit.

As described in §4.1, the only irreducible connection on the closed orbit $\{0\}$ is $\mathcal{P}$, the $\mathcal{D}_{\mathbb{C}}$-module of truncated Laurent series. Recall that $\mathcal{P}$ is generated by z^{-1}. Since this orbit is a single point, N acts trivially on $\mathcal{P}$, so the differential μ of the N-action is zero; i.e. for $\xi \in \mathfrak{n}$, $\mu(\xi) = 0$. Since $\mathfrak{n} = \mathrm{span}\{E\}$, this implies $\mu(E) = 0$. We calculated previously that as a differential operator on U_0, $E = z^2\partial + nz$. As above, we denote this action of $\mathfrak{n}$ on $\mathcal{P}$ induced by the $\mathcal{D}_t$-action by π. We calculated previously that $\pi(E)z^{-1} = 0$. Therefore, the equality

$$\pi(E)z^{-1} = \mu(E)z^{-1} + \eta(E)z^{-1}$$

can only hold if $\eta = 0$. We conclude that for non-zero η, there is no irreducible η-twisted N-homogeneous connection on the closed orbit $\{0\}$, and in turn, there is no standard η-twisted Harish-Chandra sheaf corresponding to the orbit $\{0\}$.

Next we turn our attention to the open orbit U_1. Let $\mathcal{R}$ be the $\mathcal{D}_{U_1}$-module with $R := \Gamma(U_1, \mathcal{R}) = \mathbb{C}[\zeta]$ as a vector space, with $D_{\mathbb{C}}$-action π given by

$$\begin{aligned}
\pi(\partial_\zeta)\zeta^k &= k\zeta^{k-1} - \eta(E)\zeta^k \text{ for } k > 0,\\
\pi(\partial_\zeta)1 &= -\eta(E), \text{ and}\\
\pi(\zeta)\zeta^k &= \zeta^{k+1} \text{ for } k \in \mathbb{Z}_{\geq 0}.
\end{aligned}$$

Here ζ and ∂_ζ are the coordinates on U_1 described in §4. Since U_1 is affine, specifying the global sections of $\mathcal{R}$ completely determines its structure, so we will work in the setting of rings. We showed previously that in these coordinates on U_1, $E = -\partial_\zeta$, so $\pi(E) = -\pi(\partial_\zeta)$. Under the isomorphisms described in §4, N acts on $\mathbb{C}[\zeta]$ by

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot f(\zeta) = f(\zeta - b).$$

The differential μ of this action is

$$\mu(E)f(\zeta) = -\partial_\zeta \cdot f(\zeta).$$

Since E spans $\mathfrak{n}$, this completely describes the $\mathfrak{n}$-action coming from the differential of the N-action. Therefore, for $k > 0$,

$$\pi(E)\zeta^k = -k\zeta^{k-1} + \eta(E)\zeta^k = \mu(E)\zeta^k + \eta(E)\zeta^k.$$

We conclude that for $\xi \in \mathfrak{n}$, $\pi(\xi) = \mu(\xi) + \eta(\xi)$, so $\mathcal{R}$ is an η-twisted Harish-Chandra sheaf on U_1. Additionally, we claim that $\mathcal{R}$ is irreducible. The proof of this fact is similar to the proof in §1 that N_t are irreducible $D_{\mathbb{C}^*}$-modules. The element $\partial_\zeta + \eta(E) \in D_{\mathbb{C}}$ acts by

$$\pi(\partial_\zeta + \eta(E))\zeta^k = k\zeta^{k-1} - \eta(e)\zeta^k + \eta(E)\zeta^k = k\zeta^{k-1},$$

so repeated action by $\partial_\zeta + \eta(E)$ and ζ moves ζ^j to ζ^i for any $i, j \in \mathbb{Z}_{\geq 0}$. Let $W \subset R$ be a $D_{\mathbb{C}}$-submodule, and let $D = (\partial_\zeta + \eta(E))\zeta \in D_{\mathbb{C}}$. Then ζ^k are eigenvectors for D:

$$\pi(D)\zeta^k = \pi(\partial_\zeta + \eta(E))\zeta^{k+1} = (k+1)\zeta^k.$$

We define a length function analogous to the length function in §1. If $w = \sum_{k \in \mathbb{Z}_{\geq 0}} \beta_k \zeta^k$, then $\ell(w) = \text{Card}\{n \in \mathbb{Z}_{\geq 0} | \beta_n \neq 0\}$. Let $w = \sum_{k \in \mathbb{Z}_{\geq 0}} \beta_k \zeta^k \in W$ be a non-zero element so that $\ell(w)$ is minimal. If $\ell(w) = 1$, then $w = \beta_k \zeta^k$ for some $k \in \mathbb{Z}_{\geq 0}$, and repeated action by $\partial_\zeta + \eta(E)$ and ζ on w generate all of R. In this case, R is irreducible, as desired. If $\ell(w) > 1$, then there exists some $n \in \mathbb{Z}_{\geq 0}$ such that $\beta_n \neq 0$ and $n + 1 \neq 0$. The element $\frac{1}{n+1}\pi(D)w - w \in W$, and $\ell(\frac{1}{n+1}\pi(D)w - w) < \ell(w)$, which contradicts the minimality of $\ell(w)$. Therefore, R is an irreducible $D_{\mathbb{C}}$-module.

Since $\mathbb{C}$ is affine, this in turn implies that $\mathcal{R}$ is irreducible as a $\mathcal{D}_{\mathbb{C}}$-module. Therefore, $\mathcal{R}$ is an irreducible η-twisted N-homogeneous connection on the open orbit U_1. Let $i_1 : U_1 \longrightarrow X$ be inclusion. Then $\mathcal{I}(U_1, \mathcal{R}, t) := i_{1+}(\mathcal{R})$ is the corresponding standard η-twisted Harish-Chandra sheaf. We claim this is an irreducible $\mathcal{D}_t$-module.

Since irreducibility of $\mathcal{D}_t$-modules is a local property and $\mathcal{I}(U_1, \mathcal{R}, t)|_{U_1} = \mathcal{R}$ is irreducible, it remains to show that $\mathcal{I}(U_1, \mathcal{R}, t)|_{U_0}$ is irreducible. We do so by using the relationship between the trivializations on the intersection V as we did in previous untwisted examples. Since U_1 is affine, we work with $\Gamma(U_1, \mathcal{I}(U_1, \mathcal{R}, t)|_{U_1}) = R$. As a $\mathbb{C}[\zeta]$-module, $R = \mathbb{C}[\zeta]$ since the η-twist in the module structure of R only comes from the ∂_ζ-action. Therefore, when we restrict R to $\mathbb{C}^*$, localization gives us the $D_{\mathbb{C}^*}$-module $R(\mathbb{C}^*) \otimes_{\mathbb{C}[\zeta]} R = R(\mathbb{C}^*)$ on the intersection. Now, the relationship

$$\begin{aligned} \psi : D_{\mathbb{C}^*} &\longrightarrow D_{C^*} \\ \partial_\zeta &\longmapsto \partial_\zeta + \frac{t+1}{\zeta} \end{aligned}$$

between our trivializations gives us another module structure on $R(\mathbb{C}^*)$, as in the untwisted case. Let S be the $D_{\mathbb{C}^*}$-module which is ismorphic to $R(\mathbb{C}^*)$ as a vector space, with $\mathcal{D}_{\mathbb{C}^*}$-module action ν given by

$$\begin{aligned} \nu(\partial_\zeta)\zeta^k &= \psi \circ \pi(\partial_\zeta)\zeta^k = k\zeta^{k-1} + (t+1)\zeta^{k-1} - \eta(E)\zeta^k \text{ for } k \neq 0, \\ \nu(\partial_\zeta)1 &= (t+1)\zeta^{-1} - \eta(E), \text{ and} \\ \nu(\zeta)\zeta^k &= \zeta^{k+1} \text{ for } k \in \mathbb{Z}. \end{aligned}$$

Using the relationship $z = \frac{1}{\zeta}$, $\partial = -\zeta^2\partial_\zeta$ between our coordinate systems on the intersection V, we can express this action in U_0-coordinates as

$$\begin{aligned} \nu(\partial)z^k &= -\zeta^2(-k\zeta^{-k-1} + (t+1)\zeta^{-k-1} - \eta(E)\zeta^{-k}) = (k - (t+1))z^{k-1} + \eta(E)z^{k-2} \text{ for } k \neq 0, \\ \nu(\partial)1 &= \zeta^2(\partial_\zeta + (t+1)\zeta^{-1} - \eta(E)) \cdot 1 = -(t+1)z^{-1} + \eta(E)z^{-2}, \text{ and} \\ \nu(z)z^k &= z^{k+1}. \end{aligned}$$

We claim that S is an irreducible $D_{\mathbb{C}^*}$-module. Again, we prove this fact analogously to the proof in §1. Since both $z, z^{-1} \in D_{\mathbb{C}^*}$, we can move z^j to z^i for any $i, j \in \mathbb{Z}$ by repeated action by z and z^{-1}. Let $W \subset S$ be a $D_{\mathbb{C}^*}$-submodule. Let $T = \partial z^2 - \eta(E) \in D_{\mathbb{C}^*}$. Then W is T-stable, and T acts on spanning set elements by

$$\alpha(T)z^k = \alpha(\partial)z^{k+2} - \eta(E)z^k = ((k+2)-(t+1))z^{k+1} + \eta(E)z^k - \eta(E)z^k = (k+1-t)z^{k+1}.$$

As before, define length of $w = \sum_{k\in\mathbb{Z}} \beta_k z^k$ to be $\ell(w) = \mathrm{Card}\{k \in \mathbb{Z} | \beta_k \neq 0\}$. Let $w \in W$ be a nonzero element of minimal length. If $\ell(w) = 1$, then $w = \beta_n z^n$ for some $n \in \mathbb{Z}$, so repeated action by z and z^{-1} will generate all of S, and our module is irreducible as claimed. Assume $\ell(w) > 1$. If $w = \sum_{k\in\mathbb{Z}} \beta_k z^k$, then

$$\alpha(T)w = \sum_{k\in\mathbb{Z}} (k+1-t)\beta_k z^{k+1},$$

so $\ell(\alpha(T)w) \leq \ell(w)$. If $t = k+1 \in \mathbb{Z}$ and $\beta_k \neq 0$, then $\ell(\alpha(T)w) < \ell(w)$, which is a contradiction. If $t = k+1 \in \mathbb{Z}$ and $\beta_k = 0$, or if $t \notin \mathbb{Z}$, then there exists $n \in \mathbb{Z}$ such that $\beta_n \neq 0$ and $n+1-t \neq 0$. In this case, $\ell(\frac{1}{n+1-t}\alpha(T)w - zw) < \ell(w)$, which is again a contradiction. We conclude that S is irreducible.

Our final step is to push this module forward to U_0. Let $j_0 : V \longrightarrow U_0$ be inclusion. Then $j_{0+}(S)$ is the $D_{\mathbb{C}}$-module which is isomorphic to $R(\mathbb{C}^*)$ as a vector space with $D_{\mathbb{C}}$-action α given by

$$\begin{aligned}
\alpha(\partial)z^k &= (k-(t+1))z^{k-1} + \eta(E)z^{k-2} \text{ for } k \neq 0,\\
\alpha(\partial)1 &= -(t+1)z^{-1} + \eta(E)z^{-2}, \text{ and}\\
\alpha(z)z^k &= z^{k+1}.
\end{aligned}$$

For $k \in \mathbb{Z}_{\neq 0}$, define $T_k = \partial z - (k-t) \in D_{\mathbb{C}}$. Then,

$$\alpha(T_k)z^k = \alpha(\partial)z^{k+1} - (k-t)z^k = ((k+1)-(t+1))z^k + \eta(E)z^{k-1} - (k-t)z^k = \eta(E)z^{k-1}.$$

Define $T_0 = \partial z + t \in D_{\mathbb{C}}$. Then

$$\alpha(T_0)1 = \alpha(\partial)z + t = (1-(t+1)) + \eta(E)z^{-1} + t = \eta(E)z^{-1}.$$

As a vector space, $j_{0+}(S)$ is spanned by $\{z^k\}_{k\in\mathbb{Z}}$, and we have found a collection of operators $T_k \in D_{\mathbb{C}}$ so that repeated action by these operators maps z^j to z^i for any $i, j \in \mathbb{Z}$. Since the operator $T = \partial z^2 - \eta(E)$ is in $D_{\mathbb{C}}$ as well as $D_{\mathbb{C}^*}$, an analogous argument to the argument that S is irreducible shows that $j_{0+}(S)$ is irreducible as a $D_{\mathbb{C}}$-module. We conclude that $\mathcal{I}(U_1, \mathcal{R}, t)|_{U_0} = j_{0+}(S)$ is irreducible, so our standard module $\mathcal{I}(U_1, \mathcal{R}, t)$ is an irreducible $\mathcal{D}_t$-module.

We finish this section by arguing that $\mathcal{I}(U_1, \mathcal{R}, t)$ is the only standard η-twisted Harish-Chandra sheaf on X. To do this, we must show that $\mathcal{R}$ is the unique irreducible η-twisted N-homogenous connection on U_1. Under the ismorphism $U_1 \simeq \mathbb{C}$ described in §4, the action of N on U_1 corresponds to the action of N on $\mathbb{C}$ via

$$\begin{aligned}
N \times \mathbb{C} &\longrightarrow \mathbb{C}\\
(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \zeta) &\longmapsto \zeta + b
\end{aligned}$$

This induces an action ρ of N on $R(\mathbb{C}) \simeq \mathbb{C}[\zeta]$ given by $\rho\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) f(\zeta) = f(\zeta - b)$. Let τ be an irreducible N-homogeneous connection on $\mathbb{C}$. Then $M = \Gamma(\mathbb{C}, \tau)$ is an irreducible N-equivariant

$R(\mathbb{C})$-module. What this means precisely is the following. If ν is the action of N on M giving it the structure of a N-module and ρ is the action of N on $R(\mathbb{C})$ induced from the action of N on $\mathbb{C}$ as described above, then the $R(\mathbb{C})$-module action map

$$R(\mathbb{C}) \times M \longrightarrow M$$

is N-equivariant; i.e. for $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in N$, $\nu\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)(f \cdot m) = (\rho\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) f) \cdot \nu\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) m.$

We claim that any $R(\mathbb{C})$-invariant subspace of M must contain an N-fixed vector. Since N is abelian, irreducible representations of N are one-dimensional. Therefore, since M is semisimple as an N-module, there exists an element $m \in M$ such that the N-action is given by

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot m = \varphi(b)m$$

for $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ an irreducible representation of $\mathbb{C}$. This action must satisfy

$$\varphi(a+b)m = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix} \cdot m = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot m = \varphi(a)\varphi(b)m,$$

so $\varphi(b) = e^{\lambda b}$ for some $\lambda \in \mathbb{C}$. However, we are working in the category of algebraic representations, so the only choice of λ that makes φ an algebraic representation is $\lambda = 0$. Therefore, there exists $m \in \mathcal{M}$ such that $N \cdot m = m$.

Let $v_0 \in M$ be an N-fixed vector. Since M is irreducible as an $R(\mathbb{C})$-module, then the submodule $\operatorname{span}\{v_n := \zeta^n v_0\}_{n \in \mathbb{Z}_{\geq 0}} \subset M$ generated by v_0 must be all of M. Let μ be the action of $\mathfrak{n}$ on M obtained by differentiating the action ν of N on M. Since the N-action by ν fixes v_0, the $\mathfrak{n}$-action by μ must annihilate v_0. Since $\mathfrak{n} = \operatorname{span}\{E\}$, this implies that $\mu(E)v_0 = 0$.

So far, we have only described the $R(\mathbb{C})$-module structure of M. If we want to give M the additional structure of an η-twisted $D_{\mathbb{C}}$-module (i.e. $\pi(\xi) = \mu(\xi) + \eta(\xi)$ for all $\xi \in \mathfrak{n}$, where π is the action of $\mathfrak{n}$ on M induced from the $D_{\mathbb{C}}$-module structure), then there is only one possibility for the $D_{\mathbb{C}}$-action on v_0:

$$\pi(E)v_0 = \mu(E)v_0 + \eta(E)v_0 = \eta(E)v_0$$

Since v_0 generates M as an $R(\mathbb{C})$-module, this determines the action of $D_{\mathbb{C}}$ on all of M. Indeed, since $E = -\partial_\zeta$ as a differential operator on U_1, the following calculation

$$\pi(E)v_1 = \pi(-\partial_\zeta)\zeta v_0 = -v_0 + \eta(E)\zeta v_0 = -v_0 + \eta(E)v_1$$

lets us determine that $\pi(E)v_n = -nv_{n-1} + \eta(E)v_n$, and thus $\pi(\partial_\zeta)v_n = nv_{n-1} - \eta(E)v_n$.

Let us review the conclusions of the preceeding paragraphs. We have determined that any irreducible N-equivariant $R(\mathbb{C})$-module must be spanned by vectors of the form $\{v_n := \zeta^n v_0\}_{n \in \mathbb{Z}_{\geq 0}}$, where v_0 is a vector fixed by N. If we want to give this module the additional structure of an η-twisted $D_{\mathbb{C}}$-module, then the $D_{\mathbb{C}}$-action must be given by $\pi(\partial_\zeta)v_n = nv_{n-1} - \eta(E)v_n$. We claim that any such module must be isomorphic to the module R described in the beginning of this section. What remains is to show that the spanning set $\{v_n\}$ is linearly independent.

Let $\sum_{n \in \mathbb{Z}_{\geq 0}} \beta_n v_n = 0$ be a linear combination of spanning set elements with non-zero coefficients. Then since $v_n = \zeta^n v_0$,

$$\sum_{n \in \mathbb{Z}_{\geq 0}} \beta_n v_n = \sum_{n \in \mathbb{Z}_{\geq 0}} \beta_n \zeta^n v_0 = P(\zeta)v_0 = 0,$$

where $P(\zeta) \in R(\mathbb{C})$ is a polynomial in ζ that annihilates v_0. Consider the action of ∂_ζ on this element:

$$0 = \pi(\partial_\zeta)(0) = \pi(\partial_\zeta)P(\zeta)v_0 = P'(\zeta)v_0 + \eta(E)P(\zeta)v_0 = P'(\zeta)v_0.$$

We see that the derivative of $P(\zeta)$ also annihilates v_0. We conclude that all derivatives of $P(\zeta)$ annihilate v_0. Let $m = \deg(P)$. Then the m-th derivative annihilates v_0, $P^{(m)}(\zeta)v_0 = \beta_m m! v_0 = 0$. This implies that $\beta_m = 0$, and thus all $\beta_i = 0$.

Therefore, the set $\{v_n\}$ consists of linearly independent vectors in M, and the map sending $v_n \mapsto \zeta^n$ is a $D_{\mathbb{C}}$-module isomorphism of M and R. We conclude that $\mathcal{R}$ is the unique irreducible η-twisted $\mathcal{N}$-homogeneous connection on U_1, and for non-zero η, there is a single standard object in the category $\mathcal{M}(\mathcal{D}_t, N, \eta)$ of η-twisted Harish-Chandra sheaves on X. Note that if we specialize to $\eta = 0$, the argument above shows that there is a unique irreducible N-homogeneous connection on U_1, which was stated without proof in §4.1.

References

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