Flag Varieties and the Bruhat Decomposition

MATH 7895 Lecture Notes

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For our first example, consider $V = \mathbb{C}^n$. Let $\{e_1, \dots e_n\}$ be the standard basis of $\mathbb{C}^n$, and fix the full flag

$$\mathcal{F}_\bullet = 0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \dots, e_{n-1} \rangle \subset V$$

Note that any other flag can be obtained from this one by acting with an element of $GL_n(\mathbb{C})$. Specifically, any flag $V_\bullet = 0 \subset V_1 \subset \cdots \subset V_n = V$ in $\mathbb{C}^n$ has the form

$$V_\bullet = 0 \subset \langle ge_1 \rangle \subset \langle ge_1, ge_2 \rangle \subset \cdots \subset \langle ge_1, \dots, ge_n \rangle = V$$

for some $g \in GL_n(\mathbb{C})$. From this we see that $GL_n(\mathbb{C})$ acts transitively on the set of full flags in $\mathbb{C}^n$. Under this action, the stabilizer of our standard flag $\mathcal{F}_\bullet$ is the Borel subgroup of upper triangular matrices in $GL_n(\mathbb{C})$. We would like make sense of the statement

$$GL_n(\mathbb{C})/B = \{\text{flags in } \mathbb{C}^n\},$$

geometrically. To do this, we have two tasks:

- Make sense of the quotient $GL_n(\mathbb{C})/B$ as a variety.
- Make sense of the set $\{\text{flags in } \mathbb{C}^n\}$ as a variety.

We will examine these tasks from two perspectives. First, we will return to our categorical perspective of algebraic groups as functors of points. Then, we will perform a more geometric construction.

1 The Functor of Points Perspective

Recall that we can apply the Yoneda embedding

$$\mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$$

to the category $\mathcal{C}$ of schemes over k and regard a variety or k-scheme X as its functor of points

$$h_X : k\text{-alg} \longrightarrow \mathrm{Set}.$$

We need to determine how we can use this functor of points perspective to think about quotients of algebraic groups. We begin with a cautionary example. A first naive guess as to how to define quotients in this perspective would be to define the following functor:

$$\begin{aligned} GL_n/B : k\text{-alg} &\longrightarrow \mathrm{Set} \\ R &\longmapsto GL_n(R)/B(R) \end{aligned}$$

To determine if this definition is what we're looking for, we must ask ourselves if this functor is the functor of points of a scheme. The following exercise hints that this might not be the case.

Exercise 1.1. Let $n = 2$, and $R = \mathbb{Z}[\sqrt{-5}]$ (notice that this is not a PID). Show that that the map

$$GL_2(\mathbb{Z}[\sqrt{-5}])/B(\mathbb{Z}[\sqrt{-5}]) \longrightarrow \mathbb{P}^1(\mathbb{Z}[\sqrt{-5}])$$

is not surjective.

This exercise illustrates that this functor does not give us the scheme we would expect (namely $\mathbb{P}^1$), so our definition of quotient needs some refinement. But first, we will define flag varieties in this categorical setting.

Definition 1.2. Let k be a field and V a vector space over k of dimension n. For each sequence of integers $(d) = d_1 \leq d_2 \leq \cdots \leq d_r = n$, we define the associated **flag variety** to be the functor

$$\begin{aligned} Fl_{(d)} : k\text{-alg} &\longrightarrow \text{Set} \\ R &\longmapsto \left\{ \begin{array}{c} \text{R -submodules } \; V_1 \subset V_2 \subset \cdots \subset V_r = V \otimes_k R \text{ with} \\ \text{each } V_i \text{ an } R\text{-module direct summand of rank } d_i \end{array} \right\} \end{aligned}$$

In the case where $d_i = i$ for $i = 1, \ldots, n$, we say $Fl_{(d)}$ is the **full flag variety**.

We can observe that

$$\begin{aligned} GL_n(k)/B(k) &\longrightarrow Fl_{(1,2,\cdots,n)}(k) \\ g &\longmapsto g \cdot \mathcal{F} \end{aligned}$$

where $\mathcal{F}$ is a fixed flag with $B = \mathrm{Stab}_{GL_n}(\mathcal{F})$, is an isomorphism. However, as our initial example reveals, this doesn't hold for general R; i.e.

$$GL_n(R)/B(R) \longrightarrow Fl(R)$$

is not always surjective. The problem here from a functorial perspective is that the naive presheaf quotient

$$\begin{aligned} (\mathrm{Aff}/k)^{\mathrm{op}} &\longrightarrow \text{Set} \\ R &\longmapsto GL_n(R)/B(R) \end{aligned}$$

"is not a sheaf." We explain what this means precisely in the following pages.

We start by recalling the constructions of classical sheaf theory. For a topological space X, a *presheaf* on X is a functor

$$\mathcal{P} : (\mathrm{Open}_X)^{op} \longrightarrow \text{Set},$$

and a *sheaf* on X is a presheaf $\mathcal{P}$ that satisfies a glueing condition. We can define a similar notion of sheaf on the category of topological spaces, Top. A *presheaf* on Top is a function

$$\mathcal{P} : \mathrm{Top}^{\mathrm{op}} \longrightarrow \text{Set},$$

and a *sheaf* on Top is a presheaf $\mathcal{P}$ satisfying glueing conditions with respect to covering families of open immersions.

Notice that in this categorical setting, the notion of intersections no longer makes sense. The equivalent notion that we need is the fiber product; i.e. for open immersions $f_1 : U_2 \to X$ and $f_2 : U_2 \to X$, we define our gluing conditions on the fiber product $U_1 \times U_2$:

$$
\begin{tikzcd}
 & U_1 \arrow[r, "f_1"] & X \\
U_1 \times_X U_2 \arrow[ur] \arrow[dr] & & \\
 & U_2 \arrow[r, "f_2"] & X
\end{tikzcd}
$$

Examples of sheaves on Top are:

- The presheaf $\mathcal{P} = (X \mapsto \text{Continuous functions on } X)$ is a sheaf.
- For all objects $X \in \text{Top}$, the presheaf

$$
\begin{aligned}
h_X : \text{Top}^{\text{op}} &\longrightarrow \text{Set} \\
Y &\longmapsto Hom(Y, X)
\end{aligned}
$$

  is a sheaf.

Now, our goal is to replicate this structure in algebraic geometry. We repace the category Top with the category Aff/k. In this setting, a *presheaf* is a functor

$$(\text{Aff}/k)^{\text{op}} \longrightarrow \text{Set}.$$

For example, our functor $R \mapsto GL_n(R)/B(R)$ is a presheaf. To replicate the structure described above on Top, we must decide what kind of "covers" and "glueing conditions" we should impose to get a notion of a sheaf. There are a variety of ways we can do this. For instance, étale topology, fppf topology, or fpqc "topology" will give us a notion of sheaves on Aff/k. We will work with the fpqc ("fidelement plat quasi compact") topology.

In this setting, *covers* are jointly surjective familiies

$$\coprod_{i=1}^{d} \text{Spec} R_i \longrightarrow \text{Spec} R$$

with each $R \to R_i$ flat. This allows us to define a notion of a sheaf on the category Aff/k:

Definition 1.3. A *sheaf* for the fpqc topology is a presheaf

$$\mathcal{F} : (\text{Aff}/k)^{\text{op}} \longrightarrow \text{Set}$$

satisfying

(i) (locality) $\mathcal{F}(\prod_{i=1}^{n} R_i) \simeq \prod_{i=1}^{n} \mathcal{F}(R_i)$, and

(ii) (gluing) For all faithfully flat ring homomorphisms $R \to R'$,

$$\mathcal{F}(R) \to \mathcal{F}(R') \underset{q}{\overset{p}{\rightrightarrows}} \mathcal{F}(R' \otimes_R R')$$

is an equalizer diagram in the category Set; i.e. $\mathcal{F}(R)$ includes into $\mathcal{F}(R')$ as $\{x \in \mathcal{F}(R') | p(x) = q(x)\}$. Here, p is obtained from the ring homomorphism $- \otimes 1 : R' \to R' \otimes_R R'$ and q from $1 \otimes - : R' \longrightarrow R' \otimes_R R'$.

A follow-up exercise to the one at the beginning of this section is the following:

Exercise 1.4. Show that $R \mapsto GL_n(R)/B(R)$ is not a sheaf for the fpqc topology.

We have the following theorem (Grothendieck's theory of fpqc descent).

Theorem 1.5. *For all objects $X \in \mathit{Aff}/k$, the representable functor h_X is a sheaf in the fpqc topology.*

This theorem suggests that the correct notion of quotients of algebraic groups in our categorical setting is that GL_n/B should be the "sheafification[1]" of the presheaf $R \mapsto GL_n(R)/B(R)$. And indeed, the results of the following section will indicate that this is the correct construction. This ends our first perspective on the construction of the quotient. The philosophy that one should take away from this is that the correct approach is to "make the naive construction in group theory, then make sure it's a sheaf."

2 The Geometric Perspective

Definition 2.1. Let G be an algebraic group over k. Let $H \hookrightarrow G$ be a Zariski closed subgroup of G. Then a *quotient* $X = G/H$ is a pair $(X, \pi : G \to X)$, where X is a scheme over k, satisfying

(i) π is faithfully flat,

(ii) π is right-H-invariant, and

(iii) the map

$$\begin{aligned} G \times H &\longrightarrow G \times_X G \\ (g,h) &\longmapsto (g, gh) \end{aligned}$$

is an isomorphism; i.e. the non-empty fibers of $G(R) \to X(R)$ are just $H(R)$-orbits for all k-algebras R.

A consequence of this definition is that (X, π) is *initial* among H-equivariant maps from G to any k-scheme Y with a trivial H-action. More precisely, for any k-scheme Y with trivial H-action and map $f : G \to Y$ such that $f(gh) = f(g)$, there exists a unique factorization $\bar{f}$ so that the diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "\pi"'] & Y \\ & G/H \arrow[u, dashed, "\bar{f}"'] \end{tikzcd}$$

commutes. The proof of this universal property uses fpqc descent. Morever, it shows that this geometric construction agrees with the functorial one from the previous section.

Next we explore why such a constuction exists.

Theorem 2.2. *Let G be a smooth affine algebraic group over k, and $H \hookrightarrow G$ a Zariski closed subgroup. Then the quotient $G \to G/H$ exists and is a smooth quasiprojective variety over k. If B is a Borel subgroup of G then G/B is a projective variety.*

[1] This is not technically well-defined. There is no sheafification functor for the fpqc topology. However, as long as the presheaves are "small enough," sheafification will be possible, and the presheaves that we are interested are "small enough." For a precise statement of what it means for a presheaf to be "small enough" see William Charles Waterhouses article *Basically Bounded Functors and Flat Sheaves.*

Proof. We will use the following theorem of Chevalley.

Theorem 2.3. *(Chevalley) If G is a smooth affine algebraic group and $H \subset G$ a closed subgroup as in the statement of the theorem, then there exists a finite dimensional representation $r : G \to GL(V)$ and a line $L \subset V$ such that $H = Stab_G(L)$. (More precisely, H represents the functor $R \mapsto Stab_{G(R)}(L \otimes_k R)$.)*

Let L, V be the line and vector space given by the theorem. We have an action map:

$$\begin{aligned} G &\longrightarrow \mathbb{P}(V) \\ g &\longmapsto g \cdot L \end{aligned}$$

Let $G \cdot L$ be the orbit of G acting on L; i.e. the set-theoretic image of this action map.

Fact 1. $G \cdot L \subset \mathbb{P}(V)$ is a locally closed subset in the Zariski topology on $\mathbb{P}(V)$.

We can use this fact to endow $G \cdot L$ with a scheme structure: $\overline{G \cdot L}$ inherits the reduced closed subscheme structure, and since $G \cdot L$ is locally closed, $G \cdot L$ is open in $\overline{G \cdot L}$ and inherits a scheme structure as an open subset. Therefore, $G \cdot L$ is a quasiprojective variety and one can check that the map $G \to G \cdot L$ is a quotient map in the sense defined at the beginning of this section. This completes the proof of the first part of the theorem.

Before proving that G/B is projective, we will elaborate on Fact 1. Generally, the image $\mathcal{O}$ of the action map contains some open subset U of its closure $\overline{\mathcal{O}}$. Since G-translates gU are in $\mathcal{O}$, we see that $\mathcal{O}$ is open in $\overline{\mathcal{O}}$. In particular, the complement $\overline{\mathcal{O}} \backslash \mathcal{O}$ is a union of orbits of strictly smaller dimension. We conclude that *orbits of minimal dimension are necessarily closed.*

Now we prove the second part of the theorem. Again, let L, V be the line and finite dimensional vector space guaranteed by Chevalley's theorem, so $B = \mathrm{Stab}_G(L)$ and G/B is constructed as the image of the orbit map $g \mapsto g \cdot L$ as before. Since B stabilizes L, B acts on V/L, and since B is connected and solvable, it stabilizes a full flag $\overline{V_\bullet} = 0 \subset \overline{V_1} \subset \overline{V_2} \subset \cdots \subset \overline{V_{\mathrm{dim} V/L}} = V/L$ in V/L. (This follows from the Lie-Kolchin Theorem.) We can lift this flag to V, and we obtain a B-stable full flag $V = 0 \subset L \subset V_1 \subset V_2 \subset \cdots \subset V$, where V_i is the preimage of $\overline{V_i}$ under the quotient map. Since $B = \mathrm{Stab}_G(L)$, and $V_\bullet$ is B-stable, we have that $B = \mathrm{Stab}_G(V_\bullet)$. This gives us a new action of G on the full flag variety of $GL(V)$,

$$\begin{aligned} G &\to Fl \\ g &\mapsto g \cdot (0 \subset L \subset V_1 \subset \cdots \subset V) \end{aligned}$$

Now, G/B is also the image of this action map. Furthermore, since B is a *maximal* connected solvable subgroup of G, $G \cdot (0 \subset L \subset V_1 \subset \cdots \subset V)$ is an orbit of minimal dimension. Indeed, if $\mathcal{F}_\bullet$ is any flag and $\mathcal{S} = \mathrm{Stab}_G(\mathcal{F}_\bullet)$, then the connected component of the identity $\mathcal{S}^\circ$ is connected and solvable, so it is contained in a maximal connected solvable subgroup of G; i.e. a Borel subgroup B'. Since all Borel subgroups are conjugate, this implies

$$\mathrm{dim}\mathcal{S}^\circ \leq \mathrm{dim} B, \text{ and thus } \mathrm{dim} G/B \leq \mathrm{dim} G/\mathcal{S}^\circ = \mathrm{dim} G/\mathcal{S}.$$

Therefore, G/B is an orbit of minimal dimension, so it is necessarily closed in Fl, which is a projective variety. This completes the proof of the theorem. □

Now we have two ways of thinking about quotients and we have found a family of projective varieties G/B. Our next step is to investigate the structure of these particular flag varieties G/B.

3 The Bruhat Decomposition

We begin this section by recalling the structure of root subspaces of a reductive algebraic group. To a pair (G, T), where G is a connected reductive group and T is a maximal torus, we can associate a collection of roots $\Phi(G, T)$ by decomposing G into simultaneous eigenspaces under the Adjoint action of T, as described previously. Recall one of our key structural results of reductive algebraic groups: For $\alpha \in \Phi(G, T)$, there exists a homomorphism

$$\psi_\alpha : SL_2 \longrightarrow G,$$

obtained from the Lie algebra isomorphism $\mathfrak{sl}_2 \longrightarrow \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. We define *root subgroups* $U_\alpha \subset G$ to be

$$U_\alpha := \psi_\alpha\left(\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right) = exp(\mathfrak{g}_\alpha).$$

Let s_α be the image of $\psi_\alpha\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$ in the Weyl group $W = W(G, T) = N_G(T)/T$.

Fact 2. The set $\{s_\alpha\}_{\alpha \in \Delta(G,T,B)}$ generates the Weyl group.

Now we turn our attention to the Bruhat decomposition. Let G be a connected reductive group over an algebraically closed field $k = \bar{k}$. Let $T \subset B \subset G$ be a choice of a torus and a Borel subgroup of G. Then $B \times B$ acts on G by $(b_1, b_2) \cdot g = b_1 g b_2^{-1}$. The orbits BgB are locally closed subvarieties of G. (Note that these orbits are not always closed. For example, if $g \in B$, then the orbit $BgB = B \hookrightarrow G$ is a closed subvariety. But if $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2$, then BgB is open in SL_2.) For each $w \in W$, let $\dot{w} \in N_G(T)$ be a choice of lift. Set $C(w) = B\dot{w}B$. Any choice of lift will give the same $C(w)$, so this is well defined. Our goal of this section is to study the structure of these $C(w)$.

Let $N = R_u(B)$ be the unipotent radical of B. (For example, if B is the upper triangular Borel subgroup in GL_n, N is the subgroup of upper triangular matrices with 1's on the diagonal.) The following lemma reveals much about the structure of $C(w)$.

Lemma 3.1. *For $w \in W$, let $R(w) = \Phi^+ \cap (w\Phi^-)$ be the collection of positive roots that are sent to negative roots under action by w^{-1}. Let $N_{R(w)}$ be the subgroup of N generated by $\{U_\alpha | \alpha \in R(w)\}$. Then the multiplication map*

$$N_{R(w)}\dot{w} \times B \longrightarrow C(w)$$

is an isomorphism of varieties. Moreover, the multiplication map

$$\prod_{\alpha \in R(w)} U_\alpha \longrightarrow N_{R(w)}$$

is an isomorphism of varieties.[2]

Note that the space $\prod_{\alpha \in R(w)} U_\alpha$ is a product of affine spaces. The main result of this section is the following theorem.

Theorem 3.2. *(Bruhat Decomposition)*

$$G = \bigsqcup_{w \in W} C(w)$$

[2]Note that this is not an isomorphism of groups.

Exercise 3.3. Let $G = SL_2$ and $W = \left\{1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$. Show directly that the theorem holds in this case; i.e. that

$$SL_2 = B \sqcup B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B,$$

where B is the Borel subgroup of upper triangular matrices.

The theorem immediately implies our main structural result about the projective varieties G/B. Let $Y(w) = C(w)/B$. Such $Y(w)$ are referred to as *Bruhat cells.* By lemma 3.1, $Y(w)$ is an affine variety; i.e. $Y(w) \simeq N_{R(w)}\dot{w} \times B/B \simeq \mathbb{A}^{\#R(w)}$.

Corollary 3.4. *The flag variety G/B is the disjoint union of Bruhat cells*

$$G/B = \bigsqcup_{w \in W} Y(w).$$

This gives us an affine stratification of the flag variety.

Remark 3.5. The Bruhat cells $Y(w)$ are locally closed subvarieties of G/B. Their closure, $X(w) := \overline{Y(w)}$ is called the *Schubert variety* of w. These are complete, but may be singular. The singularities of Schubert varieties are of great representation-theoretic significance.

Example 3.6. For any G, there exists a unique element $w_0 \in W$ such that $\#R(w_0)$ is maximal. The number $\#R(w)$ is called the *length* of the element $w \in W$,[3] and defines a *length function* $\ell : W \to \mathbb{Z}$ by $\ell(w) = \#R(w)$. This unique element w_0 is often referred to as the *longest element* of the Weyl group since its length is maximal. The associated stratum $Y(w_0)$ is open and dense in G/B. This longest element w_0 has the property that $w_0(\Phi^-) = \Phi^+$, so $R(w_0) = \Phi^+ \cap (w_0\Phi^-) = \Phi^+$, and therefore $N_{R(w_0)} = N$.

Remark 3.7. There is a partial order on W characterized by $v \leq w$ if $Y(v) \subset \overline{Y(w)} = X(w)$.[4]

Now we give the idea of the proof of the theorem.

Proof. The proof relies on the following combinatorial result, which we will not prove.

Lemma 3.8. *For all simple reflections s_α, $\alpha \in \Delta$, and all $w \in W$,*

$$C(s_\alpha) \cdot C(w) = \begin{cases} C(s_\alpha w) & \text{if } \ell(s_\alpha w) = \ell(w) + 1 \\ C(s_\alpha w) \cup C(w) & \text{if } \ell(s_\alpha w) = \ell(w) - 1 \end{cases}$$

Exercise 3.9. Check this for SL_2.

Granted this lemma, we can deduce the decomposition $G = \sqcup_{w \in W} C(w)$ as follows:

1. Let $H = \cup_{w \in W} C(w)$. We wish to show that $H = G$. The lemma implies that $C(s_\alpha) \cdot H \subset H$ for all $\alpha \in \Delta$. But it can be shown that $\langle C(s_\alpha) \rangle_{\alpha \in \Delta} = G$, so $G \cdot H \subset H$. We conclude that $H = G$.

[3] The number $\#R(w)$ can also be described combinatorially by expressing an element $w \in W$ as a product of simple reflections. The minimal number of simple reflections needed to express w is a well-defined invariant and it agrees with the cardinality of $R(w)$. This explains the nomenclature "length." For more information on the combinatorics of Coxeter groups, see James E. Humphreys *Reflection Groups and Coxeter Groups.*

[4] This partial order can also be realized combinatorially in terms of the length.

2. (Disjointness) Let $w, w' \in W$ and assume that $C(w) \cap C(w') \neq \varnothing$. Since both $C(w)$ and $C(w')$ are $B \times B$-orbits, this can only happen if $C(w) = C(w')$. In particular, $\ell(w) = \ell(w')$. We use induction in $\ell(w)$ to show that $w = w'$. If $\ell(w) = \ell(w') = 0$, then $w = 1 = w'$ and we are done, so we assume $\ell(w) > 0$. Then there exists s_α for $\alpha \in \Delta$ such that $\ell(s_\alpha w) = \ell(w) - 1$. By the lemma,

$$C(s_\alpha w) \subset C(s_\alpha) \cdot C(w) = C(s_\alpha) \cdot C(w') \subset C(s_\alpha w') \cup C(w').$$

This implies that $C(s_\alpha w) = C(s_\alpha w')$ or $C(s_\alpha w) = C(w')$. Since $\dim C(w') = \dim C(s_\alpha w) + 1$, the latter is impossible. Therefore, we can imply the induction hypothesis to $C(s_\alpha w) = C(s_\alpha w')$, and we are done.

$\square$