Algebraic D-modules: A First Example

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These are notes from a talk given in the algebraic geometry student seminar at the University of Utah in December 2016. In this talk, we discussed the basic definitions in the theory of algebraic D-modules, and we explored a family of simple examples. The intent of this talk was to give the flavor of the theory without many details. No results are proven in this document. For a detailed construction of the subject, see [1]. This talk also leans toward the viewpoint of representation theory, as the author is primarily interested in the use of algebraic D-modules to study the representation theory of complex semisimple Lie algebras.

1   The Minimal Quantity of Definitions

For this talk, we will be working over the field $\mathbb{C}$. Let $X \subset \mathbb{C}^n$ be a closed algebraic set, and $R(X)$ its ring of regular functions.

Definition 1.1. A *differential operator* on X is a $\mathbb{C}$-linear map $T : R(X) \longrightarrow R(X)$ such that there exists $p \in \mathbb{Z}_{>0}$ such that for any $f_0, f_1, \ldots f_p \in R(X)$,

$$[\ldots [[T, f_0], f_1], \ldots, f_p] = 0,$$

where $[A, B] = AB - BA$ is commutation.

If T satisfies the condition above for $p \in \mathbb{Z}_{>0}$, we say that T is a differential operator of *order* $\leq p$. Let $D_p(X)$ be the collection of all differential operators on X of order $\leq p$. We will give two examples of differential operators.

For our first example, let $g \in R(X)$. Then g defines a $\mathbb{C}$-linear automorphism of $R(X)$ by left multiplication:

$$g : R(X) \longrightarrow R(X)$$
$$f \longmapsto gf$$

If we denote by g and f the automorphisms of $R(X)$ corresponding to regular functions $g, f \in R(X)$, then $[g, f] = gf - fg = 0$, since regular functions commute. We see from this that regular functions are differential operators of order ≤ 0.

Our second example of a differential operator is a derivation ∂. Recall that a derivation is a $\mathbb{C}$-linear map $\partial : R(X) \longrightarrow R(X)$ satisfying the Leibnitz rule, $\partial(fg) = \partial(f)g + f\partial(g)$. Let f be the differential operator corresponding to $f \in R(X)$ as in the previous example. Then

$$[\partial, f](g) = \partial f(g) - f\partial(g) = \partial(fg) - f\partial(g) = \partial(f)g + f\partial(g) - f\partial(g) = \partial(f)g.$$

Therefore, as a differential operator, $[\partial, f] = \partial f$, which is a regular function on X. By the previous example, for any $g \in R(X)$,

$$[[\partial, f], g] = 0,$$

so derivations are differential operators of order ≤ 1.

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Definition 1.2. Let X be an affine variety over $\mathbb{C}$. The *ring of differential operators* on X is $D(X) = \cup D_p(X)$.

If $T \in D_p(X)$ and $S \in D_q(X)$, then it is an easy exercise to show that $T \circ S \in D_{p+q}(X)$. Therefore, $D_p(X)$ is an increasing filtration, and $D(X)$ is a noncommutative filtered ring.

In D-module theory, we study the category of modules over $D(X)$, which we denote $\mathcal{M}(D(X))$. This is a natural generalization of the category $\mathcal{M}(R(X))$ of modules over $R(X)$ which is studied in commutative algebra. For arbitrary affine varieties, the ring $D(X)$ does not have a nice structure. However, if we assume that X is smooth, then $D(X)$ is agreeable. In particular, it is left and right Noetherian, and is generated by regular functions and derivations. We can sheafify the notions above to study D-modules on arbitrary varieties. This is how we will proceed.

Let X be an arbitrary smooth variety over $\mathbb{C}$. There is a natural sheaf of functions on X, the structure sheaf $\mathcal{O}_X$. For an open affine set $U \subset X$, $\mathcal{O}_X(U)$ is the ring of regular functions on U. We can construct a sheaf of D-modules by taking differential operators on local regular functions. For affine open $U \subset X$, let $D_X(U) = D(U)$, the ring of differential operators on U defined previously. The sheaf D_X is the *sheaf of local differential operators on X*. Since regular functions are differential operators of order ≤ 0, we have an inclusion of sheaves $\mathcal{O}_X \hookrightarrow \mathcal{D}_X$. This allows us to make the following definition.

Definition 1.3. Let X be an arbitrary variety over $\mathbb{C}$. The *ring of differential operators* on X is $D(X) = \Gamma(X, \mathcal{D}_X)$.

We have a natural filtration $F_p \mathcal{D}_X$ of this sheaf of rings by order, so we can construct a graded sheaf of rings:

$$Gr\mathcal{D}_X = \bigoplus F_p \mathcal{D}_X / F_{p-1} \mathcal{D}_X$$

This is a commutative sheaf of rings. We study the category $\mathcal{M}(\mathcal{D}_X)$ of sheaves of $\mathcal{D}_X$-modules. Some important properties of this category are the following.

- If X is affine, the global sections functor $\Gamma : \mathcal{M}_{coh}(\mathcal{D}_X) \longrightarrow \mathcal{M}_{fg}(\mathcal{D}_X)$ gives an equivalence of categories between the category of sheaves of coherent $\mathcal{D}_X$-modules and the category of finitely generated $D(X)$-modules. This is analogous to the theorem of Serre establishing the equivalence between coherent $\mathcal{O}_X$-modules and finitely generated $R(X)$-modules which allows us to use commutative algebra to study algebraic geometry.

- Let $\pi : T^*(X) \longrightarrow X$ be the cotangent bundle. Then $Gr\mathcal{D}_X = \pi_\bullet(\mathcal{O}_{T^*(X)})$.

- There is a natural way to attach a variety to a $\mathcal{D}_X$-module. If $\mathcal{V}$ is a coherent $\mathcal{D}_X$-module, then it has a "good" filtration (for precise definition see [1]) which allows us to construct the graded ring $Gr\mathcal{V}$. Then by the previous remark, $Gr\mathcal{V}$ is a coherent $\pi_\bullet(\mathcal{O}_{T^*(X)})$-module. Let $\mathcal{I}$ be the annihilator of $Gr\mathcal{V}$ in $\pi_\bullet(\mathcal{O}_{T^*(X)})$. The *characteristic variety $Ch\mathcal{V}$* of $\mathcal{V}$ is the zero set of $\mathcal{I}$ in $T^*(X)$.

This last remark leads us to the first major result of $\mathcal{D}$-module theory.

Theorem 1.4. *(Bernstein) Let $\mathcal{V}$ be a non-zero coherent $\mathcal{D}_X$-module. Then*

$$2 dim X \geq dim Ch\mathcal{V} \geq dim X.$$

This theorem shows us that there is a minimal possible dimension of the characteristic variety. This allows us to make the following definition.

Definition 1.5. A coherent $\mathcal{D}_X$-module $\mathcal{V}$ is *holonomic* if $\mathcal{V} = 0$ or $\dim Ch\mathcal{V} = \dim X$.

Holonomic $\mathcal{D}$-modules are a particularly nice category. They form an abelian subcategory of all $\mathcal{D}_X$-modules, they are closed under submodules, quotients, and extensions, and they have finite length.

2   A Family of Examples

In this section, we introduce a family of $D(\mathbb{C}^*)$-modules on $\mathbb{C}^*$. Then, we push these modules forward to $D(\mathbb{C})$-modules on $\mathbb{C}$ using the D-module direct image functor, and explore the extra structure that arises in the family on this larger variety. Since both of the varieties discussed in this example are affine, we can do our calculations in the category $\mathcal{M}(D(X))$ instead of the sheafified version $\mathcal{M}(\mathcal{D}_X)$ by the first bullet point on page 2. This makes our calculations more straightforward.

Let $\alpha \in \mathbb{C}$. Since $\mathbb{C}^*$ is affine and smooth, the ring $D(\mathbb{C}^*)$ is generated by $R(\mathbb{C}^*)$ and differentiation with respect to z, which we denote by ∂. We define a $D(\mathbb{C}^*)$-module N_α to be the $D(\mathbb{C}^*)$-module spanned as a vector space by $\{e_k\}, k \in \mathbb{Z}$ with actions

$$\partial \cdot e_k = (k - \alpha)e_{k-1} \qquad z \cdot e_k = e_{k+1} \qquad z^{-1} \cdot e_k = e_{k-1}.$$

This $D(\mathbb{C}^*)$-module is irreducible since repeated action by z or z^{-1} can move e_k to e_j for any $k, j \in \mathbb{Z}$. Also, for any $p \in \mathbb{Z}$, the map

$$\psi_p : N_\alpha \longrightarrow N_{\alpha+p}$$
$$e_k \longmapsto f_{k+p}$$

is an isomorphism of $D(\mathbb{C}^*)$-modules. Here $\{e_k\}$ is the spanning set for N_α and $\{f_k\}$ is the spanning set for $N_{\alpha+p}$. Therefore, these N_α form a family of irreducible $D(\mathbb{C}^*)$-modules with isomorphism classes parameterized by $\mathbb{C}/\mathbb{Z}$. Notice that if $\alpha = 0$, this module is isomorphic to the $D(\mathbb{C}^*)$-module of regular functions on $\mathbb{C}^*$ with a pole at 0, which is spanned by $\{z^k\}$ for $k \in \mathbb{Z}$.

Let $i : \mathbb{C}^* \longrightarrow \mathbb{C}$ be inclusion of varietes. There is a D-module direct image functor $i_+ : \mathcal{M}(D(\mathbb{C}^*)) \longrightarrow \mathcal{M}(D(\mathbb{C}))$. (For a precise construction of this function in full generality, see [1].) In this case, the image of N_α under this functor is very simple. Since $D(\mathbb{C}) \subset D(\mathbb{C}^*)$, $i_+(N_\alpha)$ is the $D(\mathbb{C})$-module which is isomorphic to N_α as a vector space, with the restricted action of $D(\mathbb{C})$ as a subring of $D(\mathbb{C}^*)$. The key difference between the module $i_+(N_\alpha)$ and our original module N_α is that $z^{-1} \notin D(\mathbb{C})$, so reducibility of $i_+(N_\alpha)$ depends on the integrality of α. We have two cases.

Our first case is when $\alpha \notin \mathbb{Z}$. In this case, $\partial \cdot e_k = (k - \alpha)e_{k-1} \neq 0$ for any $k \in Z$, so repeated action by ∂ and z (along with some scaling by $\mathbb{C}$) can move e_k to e_j for any $k, j \in \mathbb{Z}$. Therefore, $i_+(N_\alpha)$ is an irreducible $D(\mathbb{C})$-module.

Next we will calculate the characteristic variety of $i_+(N_\alpha)$. We can define a filtration $F_p i_+(N_\alpha)$ of $i_+(N_\alpha)$ in the following way. Let

$$F_p i_+(N_\alpha) = 0 \text{ if } p < 0, \text{ and}$$
$$F_p i_+(N_\alpha) = \text{span}\{e_k; k \geq -p\} \text{ if } p \geq 0.$$

The associated graded ring is

$$Gr i_+(N_\alpha) = \bigoplus_{p \in \mathbb{Z}} Gr^p i_+(N_\alpha),$$

where $Gr^p i_+(N_\alpha) = 0$ for $p < 0$, $Gr^0 i_+(N_\alpha) = \text{span}\{e_k; k \geq 0\}$, and $Gr^p i_+(N_\alpha) = \text{span}\{e_k; k \geq -p\}/\text{span}\{e_k; k > -p\}$. Also, we can recognize $GrD(\mathbb{C}) \simeq \mathbb{C}[z, \xi]$ as a polynomial ring in two variables. (This association falls from the "symbol" map, see [1].) We wish to find the annihilator of $Gr i_+(N_\alpha)$ in $GrD(\mathbb{C})$. We can see that z annihilates $Gr^p i_+(N_\alpha)$ for $p \neq 0$, and ξ annihilates $Gr^0 i_+(N_\alpha)$ and maps $Gr^p i_+(N_\alpha)$ into $Gr^{p+1} i_+(N_\alpha)$. Therefore, the annihilator of $Gr i_+(N_\alpha)$ is the ideal generated by ξz in $\mathbb{C}[z, \xi]$, and the characteristic variety of $i_+(N_\alpha)$ is $\{z = 0\} \cup \{\xi = 0\} \subset \mathbb{C}^2$. The characteristic variety is one dimensional, so $i_+(N_\alpha)$ is a holonomic $D(\mathbb{C})$-module.

Our second case is when $\alpha \in \mathbb{Z}$. In this case, $\partial \cdot e_\alpha = (\alpha - \alpha)e_{\alpha-1} = 0$, so $i_+(N_\alpha)$ has a submodule M_α spanned by $\{e_k; k \geq \alpha\}$. This submodule is isomorphic to $\mathbb{C}[z]$ by the isomorphism sending $e_k \mapsto z^{k-\alpha}$. The quotient $i_+(N_\alpha)/M_\alpha$ is ismorphic to the $D(\mathbb{C})$-module of truncated Laurent series at 0, $Q = \text{span}\{z^k; k \in \mathbb{Z}\}/\mathbb{C}[z]$. This gives us a short exact sequence

$$0 \longrightarrow \mathbb{C}[z] \longrightarrow i_+(N_\alpha) \longrightarrow Q \longrightarrow 0.$$

Therefore, since characteristic varieties sum over short exact sequences, $Ch(i_+(N_\alpha)) = Ch(\mathbb{C}[z]) \cup Ch(Q)$. The annihilator of $\mathbb{C}[z]$ in $\mathbb{C}[z, \xi]$ is the ideal generated by ξ and the annihilator of Q in $\mathbb{C}[z, \xi]$ is the ideal generated by z, so we conclude that $Ch(i_+(N_\alpha)) = \{z = 0\} \cup \{\xi = 0\}$ in this case as well.

These two simple examples illustrate some properties of characteristic varieties. First, notice that in the nonintegral case, the $D(\mathbb{C})$-module is irreducible, but the characteristic variety is reducible, so reducibility of the characteristic variety does not determine reducibility of the $D(\mathbb{C})$-module. Also, we see that two $D(\mathbb{C})$-modules with the same characteristic variety are not necessarily isomorphic.

In more complicated examples, this process of pushing and pulling D-modules on affine subvarieties allows us to construct standard objects in certain categories of holonomic $\mathcal{D}$-modules on arbitrary smooth varieties. The local structure of $\mathcal{D}$-modules is what makes them a useful tool in studying representation theory.

3 A Word on How Representation Theory Enters the Picture

Algebraic $\mathcal{D}$-modules are a powerful tool in geometric representation theory. This is the reason why I am interested in them. They arise in representation theory in the following way:

Let G be a complex semisimple algebraic Lie group. The tangent space at the identity, $T_1(G)$, has the structure of a complex semisimple Lie algebra. We call this the Lie algebra $\mathfrak{g}$ associated to G. A maximal closed connected solvable subgroup of G is called a Borel subgroup. Let B be a Borel subgroup of G. Then the quotient G/B has the structure of a flag variety. This is the flag variety X of G. A foundational theorem of Beilinson and Bernstein realizes sheaves of $\mathcal{D}$-modules on the flag variety as representations of the Lie algebra $\mathfrak{g}$. More precisely, Beilinson and Bernsteirn proved that there is an equivalence of categories between the category of finitely generated representations of $\mathfrak{g}$ with trivial infinitesimal character and the category of sheaves of holonomic $\mathcal{D}_X$-modules. If we expand our $\mathcal{D}$-module category to include sheaves of modules over sheaves of *twisted* differential operators on the flag variety, Beilinson and Bernstein's theorem captures all finitely generated representations of $\mathfrak{g}$ with any infinitesimal character.

The family of examples we explored in the previous section describe the representation theory of $\mathfrak{sl}(2, \mathbb{C})$. For $G = SL(2, \mathbb{C})$, the flag variety X is isomorphic to $\mathbb{P}^1$. We can cover $\mathbb{P}^1$ with two open affine sets which are each isomorphic to $\mathbb{C}$. Their intersection is isomorphic to $\mathbb{C}^*$. The process described in section 2 of pushing $D(\mathbb{C}^*)$-modules to $\mathbb{C}$ can be used to build "standard

Harish-Chandra sheaves," which correspond to a well-studied class of representations of $\mathfrak{sl}(2,\mathbb{C})$ called Verma modules. The interested reader can find full details of this construction in [2].

The power of this approach lies in the local properties of algebraic geometry. Representation theory is the algebraic study of vector fields, which come with the rigidity of linear algebra. In contrast, studying sheaves of modules on varieties gives us the freedom to "zoom in" and work locally with simpler objects. The local-global freedom of sheaf theory give us additional machinery to unveil deeper structure in the representation theory.

References

[1] D. Miličić, *Lectures on Algebraic Theory of D-modules*. Unpublished manuscript, 1999. Available at `www.math.utah.edu/~milicic`

[2] A. Romanov, *Twisted Sheaves on Flag Varieties:* $\mathfrak{sl}(2,\mathbb{C})$ *Calculations*. Unpublished notes, 2016. Available at `www.math.utah.edu/~romanova/talks`