Math 5750 - Topics in Applied Mathematics: Representation Theory Techniques in Quantum Physics

Course Description

May 1, 2017

1 Course Details

- **Course Number:** MATH 5750
- **Instructor:** Anna Macquarie Romanov
- **Semester:** Spring 2018
- **Course Title:** Topics in Applied Mathematics: Representation Theory Techniques in Quantum Physics
- **Credits:** 3
- **Prerequisites:** Grade of B+ or higher in MATH 2270 (Linear Algebra), PHYS 2210 (Calculus-Based Physics I), admission by instructor consent

2 Course Description

The predictive power of representation theory in quantum physics is one of the great successes of 20th century mathematics. Understanding this beautiful connection amounts to combining linear algebra with a little bit of group theory - mathematical materials readily accessible to junior- and senior-level undergraduate mathematics and physics students. This course will introduce the tools necessary to understand this interplay between mathematics and physics through a detailed exploration of the most basic example - the hydrogen atom. The course will develop all of the mathematics necessary to understand this application, starting with a review of linear algebra, and culminating with the representation theory of $SO(3)$ and $SU(2)$. Once the necessary mathematical background is established, we will use the spherical symmetry of the hydrogen atom to make predictions about its quantum states.

This course is primarily a mathematics course, with the application to quantum mechanics serving as motivation for the development of the representation theory of compact Lie groups. The mathematical content of the course will be the following:

- Linear Algebra over the Complex Numbers (review): complex vector spaces, dimension, linear transformations, kernels and images of linear transformations, linear operators, cartesian sums and tensor products
Hilbert Spaces: Lesbegue equivalence and $L^2(\mathbb{R}^3)$, complex scalar products, Euclidean geometry in Hilbert spaces, norms and approximations

Lie Groups: groups and Lie groups, $SO(3)$, $SU(2)$, $SO(4)$, the double cover of $SO(3)$ by $SU(2)$

Representation Theory of Lie Groups: representations, characters, Cartesian sums and tensor products of representations, unitary representations, sub-representations, dual representations, pullback and pushforward representations, representations in Hom spaces, Schur’s lemma, isotypic decompositions, representation theory of $SO(3)$, representation theory of $SU(2)$, spherical harmonics.

Further Topics (if time permits): Lie algebras, representations of Lie algebras, raising operators, lowering operators, the Casimir operator, representation theory of $su(2)$ and $so(4)$, projective representations and spin, independent events

We will develop the necessary physics machinery in parallel with the mathematics. Specifically, the physics content of the course will be the following.

Fundamental Assumptions of Quantum Mechanics: wave functions, probability distributions, observables, base states, self-adjoint linear operators, Pauli exclusion principle

The Hydrogen Atom: spectroscopy, quantum numbers, Schrödinger operator

Elementary States of Quantum Mechanical Systems: relationship of irreducible representations to base states of a quantum system

Further Topics (if time permits): bound states of the hydrogen atom, $SO(4)$ symmetry, Fock’s original article *On the theory of the hydrogen atom*, the qubit, projective unitary representations and spin, entanglement and quantum computing, the state space of a mobile spin-1/2 particle

This interdisciplinary course is intended for junior and senior mathematics and physics majors. A strong background in linear algebra is essential, but prior exposure to group theory, representation theory, or quantum mechanics is not necessary.