General Session on Algebra:
Searching for Toric Rings with USTP

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Ordinary Power of an Ideal

Let $R$ be a Noetherian ring.

Definition

Let $\mathfrak{p}$ be an ideal of $R$. The $n^{th}$ ordinary power of $\mathfrak{p}$, denoted $\mathfrak{p}^n$, is the ideal generated by all products of $n$ elements of $\mathfrak{p}$.

Remark

When $i > j$, we have $\mathfrak{p}^i \subseteq \mathfrak{p}^j$.

Example

Let $R := k[x, y]$ and $\mathfrak{p} := (x, y)$.

Then $\mathfrak{p}^2 = (x^2, xy, y^2)$ and $\mathfrak{p}^3 = (x^3, xy^2, x^2y, y^3)$. 
Symbolic Power of a Prime Ideal

Let $R$ be a Noetherian ring.

**Definition**

Let $\mathfrak{p}$ be a prime ideal of $R$. The $n^{th}$ **symbolic power** of $\mathfrak{p}$, denoted $\mathfrak{p}^{(n)}$, is the ideal

$$\mathfrak{p}^{(n)} := \{ x \in R \mid \exists s \in (R \setminus \mathfrak{p}), xs \in \mathfrak{p}^n \}.$$ 

**Remark**

- When $i > j$, we have $\mathfrak{p}^{(i)} \subseteq \mathfrak{p}^{(j)}$.
- For fixed $i$, we have $\mathfrak{p}^i \subseteq \mathfrak{p}^{(i)}$. 
Symbolic Power of a Prime Ideal (continued)

Example
Let $R := k[x, y]$ and $p := (x, y)$.
Then $p^{(2)} = (x^2, xy, y^2)$ and $p^{(3)} = (x^3, xy^2, x^2y, y^3)$.

Example
Let $R = \frac{k[x,y,z]}{(xy-z^2)}$ and $p := (x, z)$.
Then $p^{(2)} = (x) \supsetneq (x^2, xz, z^2) = p^2$. 
Uniform Symbolic Topology Property (USTP)

Question

How far from equality does the containment $p^n \subseteq p^{(n)}$ lie across the ideals of some ring $R$?

Definition

A ring $R$ is said to have the **Uniform Symbolic Topology Property** if there exists $h$ such that for all prime ideals $p$ of $R$ and all $n > 0$,

$$p^{(hn)} \subseteq p^n.$$ 

In some sense, this says that the “difference” between $p^{(n)}$ and $p^n$ varies *uniformly* among all the different ideals $p \subset R$, where that uniform difference is captured by the value $h$. 
Toric Rings

Let $v_i \in \mathbb{Z}^n$ be a finite collection of vectors and $k$ a field.

**Definition**

Let $R = k[x_1^{\pm1}, \ldots, x_n^{\pm1}]$. Then the toric ring associated to the vectors $v_i$ is the subring of $R$ generated by all monomials $x_1^{\lambda_1} \cdots x_n^{\lambda_n} = x^\lambda$ for which $\langle \lambda, v_i \rangle \geq 0$ for all $i$.

**Example**

For $(1,0)$ and $(0,1)$, the associated toric ring is $k[x,y]$. 
Toric Rings (continued)

Example

For $(1, 0)$ and $(-1, 2)$, the associated toric ring is $k[y, xy, x^2y]$. 
The Sets

Let $R$ be a toric ring associated to the vectors $v_i$. Work in [Smo18] and [CS18] uses the Frobenius endomorphism of the $R$ to define a set $\mathcal{D}(m)$ which detects when $R$ has USTP.

Specifically, they show that if these sets $\mathcal{D}(m)$ are “sufficiently large” for all $m \geq 2$, then $R$ has USTP.

**Definition**

We say that $\mathcal{D}(m)$ is **sufficiently large** if $\mathcal{D}(m)$ contains an epsilon ball centered around the origin.
Results in Two Dimensions

Using work from [CS18], we show

Proposition ([J19])

The only two-dimensional toric ring for which $\mathcal{D}^{(2)}$ is sufficiently large is the toric ring associated to $e_1$ and $e_2$.

Remark

Let $R$ be the toric ring associated to $e_1$ and $e_2$. Then $\mathcal{D}^{(2)} = \mathcal{D}^{(m)}$, $m \geq 3$. Thus, the sets $\mathcal{D}^{(m)}$ are sufficiently large for all $m$, so $R$ has USTP.
Results in Three Dimensions

Using work from [CHP⁺16], we show

Proposition ([J19])

There are only 2 three-dimensional toric rings for which $D^{(2)}$ is sufficiently large: the toric ring associated to $e_1$, $e_2$, and $e_3$ and the toric ring associated to $e_1$, $e_2$, $e_3$, and $(-1,1,1)$.

Proposition ([CS18;J19])

In the case of $R$ associated to $e_1$, $e_2$, $e_3$, and $(-1,1,1)$, the sets $D^{(m)}$ are sufficiently large for all $m$, so $R$ has USTP.

Remark

In the case of $R$ associated to $e_1$, $e_2$, and $e_3$, we have $D^{(2)} = D^{(m)}$, $m \geq 3$. Thus, the sets $D^{(m)}$ are sufficiently large for all $m$, so $R$ has USTP.
Results in Higher Dimensions

Let $R$ be the toric ring associated to the vectors

\[
\begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
-1 \\
0 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
0 \\
-1 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
-1 \\
0 \\
0 \\
1
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
-1 \\
1
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
0 \\
-1
\end{pmatrix}, \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}.
\]

By work in [PST18], the set $\mathcal{D}^{(2)}$ is sufficiently large.

**Proposition ([J19])**

*The set $\mathcal{D}^{(3)}$ is sufficiently large. Thus, there exists $h \leq \dim R = 5$ such that for all prime ideals $p$ of $R$,

\[p^{(3h)} \subset p^3.\]*
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References


Working with $\mathcal{D}^{(2)}$ in $\mathbb{R}^2$

Thus, to understand $\mathcal{D}^{(2)}$ we need to understand when $P_R \cap (v - P_R)$ tiles $\mathbb{R}^2$ by integer translations.

**Remark**

In two-dimensions, the intersections $P_R \cap (v - P_R)$ are all parallelograms.
Tiling in $\mathbb{R}^3$

To show that $P_R \cap (v - P_R)$ can tile $\mathbb{R}^3$, we consider the cross sections of level planes. These cross sections are 2-dimensional.

We can prove that this polytope tiles $\mathbb{R}^3$ by showing that all cross sections of level planes in some unit interval -- say, from $-\frac{1}{2}$ to $\frac{1}{2}$ -- can tile $\mathbb{R}^2$. 