Two Important Players

This project investigates an important interaction between two famous mathematical players: algebra and geometry. In particular, we hope to use geometry — especially questions of “tiling” — to understand the behaviour of certain algebraic functions.

How They Work Together

In [Smo18] and [CS18], J. Carvajal-Rojas and D. Smolkin developed a way of studying the zeroes of certain polynomials using geometry (and combinatorics). They let \( R \) denote some special polynomial sets and define a related set of points, called \( P_R \), in \( \mathbb{R}^2 \).

Then, they show that the intersection of \( P_R \) and \(-P_R\) (Figure 1) detects certain properties of the polynomials in \( R \). This amounts to determining if the parallelogram \( P \) tiles the plane by unit translations up, down, left, and right. When \( P \) does tile, we obtain information about \( R \). Hence, this project has investigated such questions of tiling to ultimately gain insight into the zeroes of certain polynomials.

Some Example Tilings

Using these observations, we conclude that the only parallelogram of interest has slope 0, height 1, and width 1: the unit square. Thus, we prove geometrically that the method developed by Carvajal-Rojas and Smolkin yields information — in two-dimensions — about a single set of polynomials \( R \), discussed next.

Algebraic Meaning

The corresponding \( R \) for the square with side lengths 1 is

\[
R = \{ \text{set of all polynomials in two variables } x \text{ and } y \} \setminus \{ \text{polynomials in } R \text{ which are zero on } \ell \text{ of order at least } h \}
\]

for any \( h \). In this case, \( h = 1 \). For more complicated examples where the polynomial sets have much interesting forms, we move to higher dimensions.

Geometric Results

A parallelogram is uniquely defined by its height, its width, and the slope of its lower edge. In particular, if a parallelogram has slope \( \frac{1}{2} \), we show that the smallest such parallelogram which tiles has an integer width (e.g. 1, 2, 3, etc.) and height that is a multiple of \( \frac{1}{2} \) (e.g. \( \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \text{ etc.} \)). Figure 3 depicts the strategy for proving these claims.

Higher Dimensions

So far, the poster has focused on two-dimensional tilings; we can in fact consider tilings in arbitrarily high dimensions. We use work from [Cho+16] to classify the three-dimensional cases which are of interest: there are only two, one of which the following figure depicts.

Acknowledgements

This research was conducted under the mentorship of Dr. Karl Schwede, Daniel Smolkin, and Marcus Robinson at the University of Utah, and funded by the Department of Mathematics, the Honors College, and the Office of Undergraduate Research at the same.

References


