Solve the one-dimensional wave equation
\[ \frac{\partial^2 u}{\partial t^2} = 9 \frac{\partial^2 u}{\partial x^2} \]
for \(0 \leq x \leq 1\) with boundary conditions \(u(0, t) = 0\) and \(u(1, t) = 0\) and with initial conditions
\[ u(x, 0) = \sin 2\pi x + 5 \sin 3\pi x \quad \text{and} \quad \frac{\partial u}{\partial t}(x, 0) = 3 \sin 5\pi x. \]

**Solution:** Notice that \(c = 3\).
\[ u(x, t) = \sin 2\pi x \cos 6\pi t + 5 \sin 3\pi x \cos 9\pi t + \frac{1}{5\pi} \sin 5\pi x \sin 15\pi t. \]

2. Solve the heat equation
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \]
\((c = 1)\) for \(0 \leq x \leq L\) with boundary conditions \(u(0, t) = 0\) and \(u(L, t) = 50\) and initial temperature \(u(x, 0) = 0\).

**Solution:** First find the steady-state solution
\[ u_0(x, t) = \frac{50x}{L}. \]
Next solve using Fourier sine series for \(u(x, t) - \frac{50x}{L}\) with boundary conditions 0 at 0 and \(L\) and initial condition \(u(x, 0) = -\frac{50x}{L}\). We get
\[ b_n = \frac{2}{L} \int_0^L -\frac{50x}{L} \sin \frac{n\pi x}{L} dx = \]
\[ = \frac{-50}{L} \left( \frac{L^2}{n^2\pi^2} \sin \frac{n\pi x}{L} - \frac{xL}{n\pi} \cos \frac{n\pi x}{L} \right) \bigg|_0^L \]
\[ = \frac{50}{L} \frac{L}{n\pi} \cos n\pi = \frac{50}{n\pi} (-1)^n. \]
The solution is
\[ u(x, t) = \frac{100}{\pi} \sum_{n=1}^\infty \frac{(-1)^n}{n} \sin \frac{n\pi x}{L} e^{-\frac{n^2\pi^2}{L^2} t} + \frac{50x}{L}. \]
3. Find the general product solution of the heat equation
\[
\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}
\]
on an interval from 0 to \(L\) subject to the boundary conditions that the temperature at 0 is kept at 0 and the end at \(L\) is insulated. Examine all possibilities for the separation constant \(k\).

**Solution:** The boundary conditions are
\[
u(0, t) = 0, \quad \frac{\partial u}{\partial x}(L, t) = 0.
\]
Separating variables gives
\[
\frac{T'}{c^2 T} = \frac{X''}{X} = k.
\]
If \(k = \lambda^2 > 0\), the solution is
\[
X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}.
\]
Since \(X(0) = 0\), we have \(c_1 = -c_2\) and \(X(x)\) is a multiple of \(\sin x\). But then \(X'(x)\) is a multiple of \(\cosh x\) which has no zeros, so it cannot be zero at \(L\). Thus this case cannot satisfy the boundary conditions unless it is constantly equal to zero.

If \(k = 0\), \(X(x) = ax + b; b = 0\) since \(X(0) = 0\) and \(a = 0\) since \(X'(L) = 0\). Again this only gives the zero solution.

If \(k = -\lambda^2 < 0\), the solution is of the form
\[
c_1 \cos \lambda x + c_2 \sin \lambda x.
\]
Since \(X(0) = 0\), we have \(c_1 = 0\). The derivative of \(\sin \lambda x\) is \(\lambda \cos \lambda x\), so the condition that \(X'(L) = 0\) says that \(\cos(\lambda L) = 0\), which means that \(\lambda L\) is an odd multiple of \(\pi/2\). Thus the general product solution is a constant times
\[
\sin \left(\frac{2k+1}{2L} \pi x\right) e^{-\frac{c^2 \pi^2 (2k+1)^2 t}{4L^2}}.
\]

4. Find the solution to the two-dimensional wave equation
\[
\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]
in a rectangular region \(0 \leq x \leq \pi\) and \(0 \leq y \leq \pi\) with the boundary kept at zero and initial conditions
\[
u(x, y, 0) = \sin x \sin 3y \quad \text{and} \quad \frac{\partial u}{\partial t}(x, y, 0) = 2 \sin 2x \sin 5y.
\]
Solution:

\[ u(x, y, t) = \sin x \sin 3y \cos \sqrt{10}t + \frac{2}{\sqrt{29}} \sin 2x \sin 5y \sin \sqrt{29}t. \]

5. Use separation of variables to find the general product solution of the equation

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \]

(Don’t worry about boundary conditions.)

Solution: Separating variables gives

\[ X''Y + XY''' = 0 \]

or

\[ \frac{X''}{X} = -\frac{Y''}{Y} = k. \]

If \( k = \mu^2 > 0 \), the solution is

\[ u(x, y) = (c_1 e^{\mu x} + c_2 e^{-\mu x})(d_1 \cos \mu y + d_2 \sin \mu y). \]

If \( k < 0 \) we get a similar solution with

\[ u(x, y) = (d_1 \cos \mu x + d_2 \sin \mu x)(c_1 e^{\mu y} + c_2 e^{-\mu y}). \]

If \( k = 0 \) we get

\[ u(x, y) = (ax + b)(cy + d) = c_1 xy + c_2 x + c_3 y + c_4. \]