Solutions to Practice Problems for the third midterm

1. a. Characteristic polynomial $r^2 - 7r + 6$ with roots 1, 6. General solution: $c_1 e^t + c_2 e^{6t}$.

b. Characteristic polynomial $r^2 + 6r + 10$ with roots $-3 \pm i$. General solution: $c_1 e^{-3 + it} + c_2 e^{-3 - it}$, or, better, $e^{-3t} (c_1 \cos t + c_2 \sin t)$.

c. The general solution to the homogeneous equation is $c_1 e^t + c_2 e^{2t}$. For the nonhomogeneous equation we use undetermined coefficients, looking for a solution of the form $x(t) = a \cos 2t + b \sin 2t$. Putting this into the differential equation gives

$$(-4a \cos 2t - 4b \sin 2t) + (6a \sin 2t - 6b \cos 2t) + (2a \cos 2t + 2b \sin 2t) = \cos 2t.$$ 

This gives equations $-2a - 6b = 2$ and $6a - 2b = 0$ with solution $a = -0.1$ and $b = -0.3$.

The general solution is

$$x(t) = -0.1 \cos 2t - 0.3 \sin 2t + c_1 e^t + c_2 e^{2t}.$$ 

d. The characteristic equation is $r^2 - 6r + 9 = (r - 3)^2$ with double root 9. The general solution to the homogeneous equation is $c_1 e^{3t} + c_2 t e^{3t}$.

We look for a solution to the nonhomogeneous equation of the form $at^2 + bt + c$. Substituting in the differential equation gives

$$2a + (-12at - 6b) + (9at^2 + 9bt + 9c) = 4t^2.$$ 

This gives $a = 4/9$, $b = 16/27$, and $c = 8/27$. The general solution is

$$x(t) = \frac{4}{9} t^2 + \frac{16}{27} t + \frac{8}{27} + c_1 e^{3t} + c_2 t e^{3t}.$$ 

2. The Wronskian of the functions $e^t, te^t, t$ is

$$\begin{vmatrix}
e^t & te^t & t \\
e^t & e^t + te^t & 1 \\
e^t & 2e^t + te^t & 0 \\
\end{vmatrix}.$$ 

It is not hard to evaluate this matrix, but to show the functions are linearly independent it suffices to show the determinant is not zero for any given value of $t$; we choose $t = 0$. The determinant becomes

$$\begin{vmatrix}
1 & 0 & 0 \\
1 & 1 & 1 \\
1 & 2 & 0 \\
\end{vmatrix},$$ 

which is clearly equal to $-2 \neq 0$. 

3. The characteristic equation of the homogeneous equation is \( r^2 + 1 = 0 \) with roots \( \pm i \). The solution of the homogeneous equation is \( c_1 \cos x + c_2 \sin x \). For the nonhomogeneous equation we look for a solution \( ax + b \) and we get that \( x \) is a solution. The general solution is therefore \( x + c_1 \cos x + c_2 \sin x \). The condition \( x(0) = 0 \) implies that \( c_1 = 1 \), and the condition \( x'(0) = 0 \) implies that \( c_2 = -1 \). The solution is \( y(x) = x - \sin x \).

4. The differential equation is
\[
4x'' + 2x' + 20x = 0.
\]
The roots of the characteristic equation are
\[
-2 \pm \sqrt{4 - 320} = -2.25 \pm 2.22i.
\]
The general solution is \( e^{-2.25t}(c_1 \cos 2.22t + c_2 \sin 2.22t) \). The initial conditions give equations \( c_1 = .1 \) and \( -2.25c_1 + 2.22c_2 = 0 \), so \( c_2 = .0113 \). The solution is
\[
x(t) = e^{-2.25t}(.1 \cos 2.22t + .0113 \sin 2.22t).
\]

5. The spring constant is \( 10/4 = 2.5 \) Newtons per meter. The differential equation is
\[
10x'' + 2.5x = 0
\]
with general solution \( c_1 \cos 0.5t + c_2 \sin 0.5t \). The initial conditions give equations \( c_1 = 1 \) and \( 2c_2 = 2 \), so the solution is \( x(t) = \cos 0.5t + \sin 0.5t \). This can be written \( x(t) = \sqrt{2}(\cos(0.5t - \pi/4)) \).

6. The characteristic equation for the homogeneous equation is
\[
r^4 - 8r^2 + 16 = 0 = (r^2 - 4)^2.
\]
The roots are \( 2, 2, -2, -2 \).

The nonhomogeneous equation is solved in two parts, for \( 2x \) and for \( -e^{3x} \). For the first, take a solution of the form \( ax + b \) and get \( (1/8)x \). For the second, take a solution of the form \( ae^{3x} \) and get the equation
\[
81a - 72a + 16a = -1, \quad \text{so} \quad a = -25/72.
\]
The general solution is
\[
y(x) = (1/8)x - (25/72)e^{3x} + c_1 e^{2x} + c_2 xe^{2x} + c_3 e^{-2x} + c_4 xe^{-2x}.
\]

7. The roots of the characteristic equation are \(-1\) and \(-3\). We find a solution of the form \( a \cos t + b \sin t \). Substituting this into the differential equation gives the equations \( 2a + 4b = 10, -4a + 2b = 0 \), with solution \( a = 1, b = 2 \). The general solution is
\[
x_t = \cos t + 2 \sin t + c_1 e^{-t} + c_2 e^{-3t}.
\]
We have \( x(0) = 1 + c_1 = 2 \), so \( c_1 = -2 \). We have \( x'(0) = 2 - c_1 - 3c_2 = 4 - 3c_2 = 0 \), so \( c_2 = 4/3 \). The solution is 

\[ x(t) = \cos t + 2 \sin t - 2e^{-t} + (4/3)e^{-3t}. \]

The steady state part is \( \cos t + 2 \sin t \) and the transient part is \(-2e^{-t} + (4/3)e^{-3t}\).

8. a. We have

\[
\begin{vmatrix}
2 - \lambda & 1 \\
3 & 4 - \lambda
\end{vmatrix} = \lambda^2 - 6\lambda + 8 - 3 = \lambda^2 - 6\lambda + 5.
\]

The roots of the characteristic polynomial are 1 and 5. For eigenvalue 1 we solve \((2 - 1)x + 1y = 0\), and we get the eigenvector \([1, -1]\). For eigenvalue 5 we solve \(-3x + y = 0\) and get the eigenvector \([1, -1]\).

b. Computing the characteristic polynomial reduces to multiplying the diagonal elements, and the eigenvalues are 2, 3, \(-1\). Solving for 2 gives \([1, 0, 0]\). For 3 we solve the system

\[
\begin{bmatrix}
-1 & 2 & 6 \\
0 & 0 & 0 \\
0 & 5 & -4
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]

which gives the vector \([38, 4, 5]\). For \(-1\) an eigenvector is \([-2, 0, 1]\).

9. a. \( x' = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}x \).

b. We have

\[
\begin{vmatrix}
-1 - \lambda & 2 \\
1 & -2 - \lambda
\end{vmatrix} = \lambda^2 + 3\lambda.
\]

The eigenvalues are 0 and \(-3\). For 0 an eigenvector is \([2, 1]\). For \(-3\) and eigenvector is \([1, -1]\).

The general solution is

\[ x(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} e^{-3t} \\ -e^{-3t} \end{bmatrix}. \]
or
\[ x(t) = 2c_1 + c_2 e^{-3t}, \quad y(t) = c_1 - c_2 e^{-3t}. \]

10. Using the notation of the book, we have
\[
K = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 4 & -32 \end{bmatrix}
\]
and
\[
M = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.
\]
Letting \( A = M^{-1}K \), we get the equation \( x'' = Ax \), where
\[
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} -5 & 4 \\ 1 & -8 \end{bmatrix}.
\]

The characteristic polynomial of \( A \) is \( \lambda^2 + 13\lambda + 36 \), with roots \(-4\) and \(-9\). The corresponding eigenvectors are \( \begin{bmatrix} 4 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

The natural angular frequencies are 2 and 3, and the natural modes of vibration are given by
\[
\begin{align*}
x(t) &= (c_1 \cos 2t + c_2 \sin 2t) \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \text{and} \quad x(t) = (c_3 \cos 3t + c_4 \sin 3t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\end{align*}
\]

The first of these has the two masses oscillating together, with \( m_1 \) moving four times as far as \( m_2 \), and the second has them oscillating in opposite directions at the same speed.

11. In this situation we are looking for a solution \( x \) of
\[
x'' - Ax = \begin{bmatrix} 2 \cos t \\ 0 \end{bmatrix}.
\]
We look for a solution of the form \( x(t) = \cos t v \) for some vector \( v \). If \( x \) is of this form, then \( x'' \) is \(-\cos t v\), so the equation becomes
\[
-\cos t v - A \cos t v = \begin{bmatrix} 2 \cos t \\ 0 \end{bmatrix},
\]
which reduces to the matrix equation
\[
(A + I)v = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.
\]
(Here $I$ is the identity matrix.) The matrix $A + I$ is $\begin{bmatrix} -4 & 4 \\ 1 & -7 \end{bmatrix}$, and its inverse is 
$\frac{1}{24} \begin{bmatrix} -7 & -4 \\ -1 & -4 \end{bmatrix}$. Solving the above equation by multiplying by the inverse gives $\begin{bmatrix} 7/12 \\ 1/12 \end{bmatrix}$.

The particular solution is thus $x_1(t) = 7/12 \cos t, x_2(t) = 1/12 \cos t$.

We now solve the initial conditions. The general solution is

$$x(t) = \begin{bmatrix} 7/12 \cos t \\ 1/12 \cos t \end{bmatrix} + (c_1 \cos 2t + c_2 \sin 2t) \begin{bmatrix} 4 \\ 1 \end{bmatrix} + (c_3 \cos 3t + c_4 \sin 3t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The initial conditions $x_1(0) = 0 = x_2(0)$ give

$$\begin{bmatrix} 7/12 \\ 1/12 \end{bmatrix} + c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which can be solved to give $c_1 = -2/15$ and $c_3 = -1/20$. The initial conditions $x_1'(0) = 0 = x_2'(0)$ give $c_2 = c_4 = 0$. The solution is

$$x_1(t) = 7/12 \cos t - 8/15 \cos 2t - 1/20 \cos 3t,$$

$$x_2(t) = 1/12 \cos t - 2/15 \cos 2t + 1/20 \cos 3t.$$