Answers to Test 1 Practice Problems

1. Find the general solution to each of the following differential equations.

a. \( \frac{dy}{dx} = -2y. \)

Separating variables gives

\[
\frac{dy}{y} = -2dx
\]

\[
\int \frac{dy}{y} = \int -2dx,
\]

\[
\ln |y| = -2x + C.
\]

\[
|y| = C' e^{-2x}, \text{ where } C' > 0.
\]

If \( y \) is positive or negative, this becomes \( y = C'' e^{-2x} \) with \( C'' \) positive or negative respectively. If \( y \) is zero at some point, the solution is \( y = 0 \). Thus the general solution is \( y = C'' e^{-2x} \) with \( C'' \) an arbitrary constant.

b. \( \frac{dy}{dx} = y^2 \sin x. \)

Separating variables,

\[
\int \frac{dy}{y^2} = \int \sin xdx,
\]

\[
-\frac{1}{y} = -\cos x + C,
\]

\[
y = \frac{1}{\cos x + C'}.
\]

c. \( y' + 3x^2 y = 6x^2. \)

This is a linear first order equation. The integrating factor is

\[
e^{\int 3x^2 dx} = e^{x^3}.
\]

Multiplying by the integrating factor gives

\[
e^{x^3} y' + e^{x^3} 3x^2 y = 6x^2 e^{x^3},
\]

\[
(e^{x^3} y)' = 6x^2 e^{x^3},
\]

\[
e^{x^3} y = \int 6x^2 e^{x^3} dx = 2e^{x^3} + C,
\]

\[
y = 2 + C e^{-x^3}.
\]
2. Find the solution to each of the following initial value problems.

a. \( \frac{dy}{dx} = 6e^{2x-y}, y(0) = 0. \)

We can separate variables to get

\[
e^{y}dy = 6e^{2x}dx,
\]

\[
\int e^{y} dy = \int 6e^{2x} dx,
\]

\[
e^{y} = 3e^{2x} + C,
\]

\[
y = \ln(3e^{2x} + C).
\]

The initial condition says that we have

\[
0 = \ln(3 + C),
\]

so \( 3 + C = 1, \) or \( C = -2. \) Thus

\[
y = \ln(3e^{2x} - 2).
\]

b. \( xy' = 3y + x^4 \cos x, y(2\pi) = 0. \)

This is a linear first order equation. Write it

\[
y' - \frac{3}{x}y = x^{3}\cos x.
\]

The integrating factor is \( e^{\int -\frac{3}{x} dx} = e^{-3 \ln x} = 1/x^3. \)

The equation becomes

\[
\frac{1}{x^{3}}y' - \frac{3}{x^{4}}y = \cos x,
\]

\[
\left( \frac{y}{x^{3}}\right)' = \cos x,
\]

\[
\frac{y}{x^{3}} = \sin x + C,
\]

\[
y = x^{3}\sin x + Cx^{3}.
\]

Putting in the initial condition gives

\[
0 = 8\pi^{3} \cdot 0 + C \cdot 8\pi^{3},
\]

so \( C = 0 \) and

\[
y = x^{3}\sin x.
\]
3. For the population \( P(t) \), the rate at which births occur is \((0.1)P\), and the rate at which deaths occur is \((.002)P^2\).

- The differential equation is 
  \[
P' = (0.1)P - (0.002)P^2.
  \]

b. The equilibrium points are where the derivative \( P'(t) \) is zero. This means that \( P = 0 \) or \( 0.1 - (0.002)P = 0 \), so \( P = 50 \). Near 0, above 0 the slope is positive and below it it is negative, so it is an unstable equilibrium. Near 50, the slope is negative above 50 and positive below it, so the function tends to return to 50 and 50 is a stable equilibrium point.

c. The slope field should have positive slopes between 0 and 50 and negative slopes for \( P < 0 \) and \( P > 50 \).

d. The limiting population is 50, as is evident from the slope field.

4. The engine provides a constant 5 ft/sec² of acceleration, while the water resistance is \(-.2 \) ft/sec² for each ft/sec of velocity, or \(-0.2v\), where \( v \) is the velocity. The differential equation for \( v \) is thus
  \[
v' = 5 - 0.2v, \quad \text{or} \quad v' + 0.2v = 5.
  \]

This can be solved as a linear equation (or by separating variables); as a linear equation the integrating factor is \( e^{0.2t} \), and the solution is
  \[
v e^{0.2t} = \int 5e^{0.2t} \, dt = 25e^{0.2t} + C,
  \]
  \[
v = 25 + C e^{-0.2t}.
  \]

At time 0 the velocity is zero, so \( 0 = 25 + C \), and \( C = -25 \). The equation for \( v \) is therefore
  \[
v = 25 - 25e^{-0.2t}.
  \]

The limiting velocity is 25 ft/sec.

5. The cylindrical tank has cross-sectional area \( \pi r^2 = 9\pi \) square feet. The area of the hole from which water is draining is \( \pi (1/6)^2 = \pi /36 \) square feet. The general equation from Torricelli’s Law is
  \[
  A(y)y' = -a\sqrt{2gy};
  \]
here \( A(y) = 9\pi \) square feet, \( a = 9\pi /36 \) square feet, and \( g \) is of course 32 feet per second squared. The equation for this case is
  \[
  9\pi y' = -(\pi /36)\sqrt{64y}.
  \]
Separating variables, we get
\[
\int \frac{dy}{\sqrt{y}} = - \int 0.0247 \, dt,
\]
\[
2\sqrt{y} = -0.0247t + C.
\]
At time \( t = 0 \) we have \( y = 4 \), so
\[
2\sqrt{4} = C,
\]
\[
C = 4.
\]

Solving for \( y \), we get
\[
\sqrt{y} = 2 - .0123t,
\]
or
\[
y = (2 - .0123t)^2.
\]

6. The rate at which brine flows in is 150 gallons per hour, and the concentration of salt is 1 pound per 10 gallons, or 0.1 pounds per gallon, so salt is entering the tank at the rate of 15 pounds per hour.

The mixture is flowing out at 150 gallons per hour, and the concentration is \( x(t)/15000 \) pounds per gallon, so the rate at which salt is leaving the tank is \( 150x(t)/15000 = .01x(t) \) pounds per hour. The differential equation is therefore
\[
x'(t) = 15 - .01x(t).
\]
The initial condition is \( x(0) = 0 \).
We can solve this equation as a linear first order equation:
\[
x'(t) + .01x(t) = 15,
\]
\[
e^{0.01t}y' + e^{0.01t}(0.01)x(t) = 15e^{0.01t},
\]
\[
e^{0.01t}y = \int 15e^{0.01t} \, dt = 1500e^{0.01t} + C,
\]
\[
y = 1500 + Ce^{-0.01t}.
\]
Putting in the initial condition:
\[
0 = 1500 + C, \text{ so } C = -1500.
\]
The formula for \( x(t) \) is
\[
x(t) = 1500 - 1500e^{-0.01t}.
\]
The limiting amount of salt in the tank is 1500 pounds.