MODULES OF FINITE LENGTH AND FINITE PROJECTIVE DIMENSION

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1. Introduction: Intersection multiplicities for nonregular rings

The results in this paper have their origin in the question of generalizing intersection multiplicities to nonregular commutative rings. If A is a regular local ring of dimension d, and if M and N are two A-modules such that $M \otimes_A N$ has finite length, then Serre defined the intersection multiplicity of M and N to be

$$\chi(M, N) = \sum_{i=0}^{d} (-1)^i \text{length}\,(\text{Tor}\,_i^A(M, N)).$$

Among the properties of this intersection multiplicity is the *vanishing theorem*, which states that if $\dim(M) + \dim(N) < d$, then $\chi(M, N) = 0$. (This theorem was proven by Serre in the equicharacteristic case and later proven in general by Gillet and Soulé [4],[5] and Roberts [12]).

In investigating this property it was asked whether similar results might hold for nonregular rings. If A is not regular an additional assumption must be added, since the definition of intersection multiplicities requires that the alternating sum of Tor modules be finite. A natural condition to impose is that one or both of the modules M and N have finite projective dimension, and it was asked whether the vanishing theorem holds if M is assumed to have finite projective dimension.

This attempt to prove a more general result was brought to a halt by an example constructed by Dutta, Hochster, and McLaughlin [2]. They constructed a module of finite length and finite projective dimension over the ring $A = k[[X, Y, Z, W]]/(XY - ZW)$ such that $\chi(M, N) = -1$, where $N = A/(X, Z)$. Thus in this example we have $\dim(M) + \dim(N) = 0 + 2 < 3 = \dim(A)$ and $\chi(M, N) \neq 0$. We remark that there is no known counterexample when both M and N have finite projective dimension.

Combining this example with the theory of local Chern characters, it was possible to show that several other conjectured properties of modules of finite projective dimension did not hold. In turn, these results raised new questions on the behaviour of such modules. In addition, it was shown by Levine [8] that the existence of such a module was implied by a localization exact sequence in K-theory.

In this paper we discuss these questions and show that the localization exact sequence from K-theory of Thomason and Trobaugh [17], which is a generalization of the exact sequence used by Levine, can be used to provide more examples of modules of finite length and finite projective dimension, and, for some rings, to describe the intersection properties of such modules completely (see Theorem 1 below). Further, using the methods of Gillet and Soulé [5], it leads to a new proof

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that the vanishing theorem holds for pairs of modules M, N of finite projective dimension over a Cohen-Macaulay local ring R whose punctured spectrum is regular (*i.e.*, with an isolated singularity) – this is a consequence of Theorem 2 below.

In Theorem 1, we consider the situation in which the ring is the localization of a graded ring over a field at its maximal graded ideal. Let R be a graded ring over a field k, and assume that R is generated over $R_0 = k$ by finitely many elements of R_1. Let $\mathfrak{m}$ denote the maximal graded ideal of R. In addition to the ring R, we consider several schemes defined in terms of R. Let T be an indeterminate of degree 1, and let $X = \mathrm{Proj}\,(R[T])$. Let $Q = \mathrm{Proj}\,(R)$. Then Q can be embedded as a closed subscheme of X by identifying R with $R[T]/(T)$. Let p denote the point of X corresponding to the prime ideal $\mathfrak{m}R[T]$. By writing R as a graded quotient of a polynomial algebra over k, we may regard Q as a closed subscheme of some projective space $\mathbb{P}^N_k$, so that R is the homogeneous coordinate ring of Q in this embedding. We will further restrict our attention to the case when Q is non-singular, or equivalently, the singular locus of X consists of only the point p.

Let $CH^*(Q)$ denote the Chow ring of rational equivalence classes of algebraic cycles on Q *with rational coefficients* (this is a slight abuse of notation); let $h \in CH^1(Q)$ be the Chern class of the tautological invertible sheaf $\mathcal{O}_Q(1)$. Let $CH^*_{\mathrm{num}}(Q)$ be the quotient of $CH^*(Q)$ modulo numerical equivalence; this is a finite dimensional graded $\mathbb{Q}$-algebra (see [3], Chapter 19, and [11], Chapter 6, Theorem 11.7).

By a result of Kurano [7], the (rational) Chow group $CH_*(\mathrm{Spec}\,R_\mathfrak{m})$ of the local ring $R_\mathfrak{m}$ is identified (up to reindexing by codimension) with the quotient graded ring $CH^*(Q)/hCH^*(Q)$. Hence there is a natural graded surjection from $CH_*(\mathrm{Spec}\,R_\mathfrak{m})$ onto $CH^*_{\mathrm{num}}(Q)/hCH^*_{\mathrm{num}}(Q)$, which is a finite dimensional $\mathbb{Q}$-vector space.

Let $\mathcal{C}$ denote the triangulated category of perfect complexes of $R_\mathfrak{m}$-modules whose homology has finite length, and let $K_0(\mathcal{C})$ denote its Grothendieck group. If $R_\mathfrak{m}$ is Cohen-Macaulay, $K_0(\mathcal{C})$ coincides with the Grothendieck group of the exact category of $R_\mathfrak{m}$-modules of finite length and finite projective dimension. Using the localized Chern character, there is a pairing

$$K_0(\mathcal{C}) \otimes_\mathbb{Z} CH_*(R_\mathfrak{m}) \to \mathbb{Q}$$

and hence a $\mathbb{Q}$-linear map

$$K_0(\mathcal{C}) \otimes \mathbb{Q} \to CH_*(R_\mathfrak{m})^\vee \tag{1}$$

(for a $\mathbb{Q}$-vector space V, we let $V^\vee = \mathrm{Hom}_\mathbb{Q}(V, \mathbb{Q})$ be the dual space).

Let $W \subset CH^*_{\mathrm{num}}(Q)$ be the image of $\ker CH^*(Q) \xrightarrow{\cdot h} CH^*(Q)$, the annihilator of h in $CH^*(Q)$. Clearly W is contained in the annihilator of h in $CH^*_{\mathrm{num}}(Q)$. We consider the composition of the inclusion $W \subset CH^*_{\mathrm{num}}(Q)$ with the isomorphism $CH^*_{\mathrm{num}}(Q) \to CH^*_{\mathrm{num}}(Q)^\vee$ defined by the intersection pairing. Again the image of this composition is contained in the annihilator of h on $CH^*_{\mathrm{num}}(Q)^\vee$, which is $(CH^*_{\mathrm{num}}(Q)/hCH^*_{\mathrm{num}}(Q))^\vee$. We can now state our first result.

Theorem 1. *The image of the map* (1) *above is the image of the composition*

$$W \to (CH^*_{\mathrm{num}}(Q)/hCH^*_{\mathrm{num}}(Q))^\vee \subset CH_*(R_\mathfrak{m})^\vee,$$

This image coincides with $(CH^*_{\mathrm{num}}(Q)/hCH^*_{\mathrm{num}}(Q))^\vee$ precisely when W equals the annihilator of h in $CH^*_{\mathrm{num}}(Q)$.

Corollary 1. *If in the above situation, $CH^*(Q) = CH^*_{\mathrm{num}}(Q)$, then the image of* (1) *is $(CH^*(Q)/hCH^*(Q))^\vee = CH^*(Spec\,R_\mathbf{m})^\vee$.*

Corollary 2. *The following are equivalent.*

(a) *We have $\chi(A_\bullet, B_\bullet) = 0$ whenever $A_\bullet$ is a perfect complex with punctual support, and $B_\bullet$ is a complex of coherent sheaves which is exact on some non-empty open subscheme of Spec $R_\mathfrak{m}$.*

(b) $W = CH^n_{num}(Q)$, *where* $n = \dim Q$.

In general, the question of whether W coincides with the annihilator of h in $CH^*_{\mathrm{num}}(Q)$ seems to be a rather subtle problem in the theory of algebraic cycles. We give examples of Esnault and Viehweg of complex surfaces Q for which the question has a negative answer. However, if Q and h are defined over $\bar{\mathbb{Q}}$ or $\bar{\mathbb{F}}_p$, we sketch a proof that the question has a positive answer, if we assume certain well-known conjectures (Grothendieck's Standard Conjectures, and the Bloch-Beilinson conjectures).

Theorem 1 will be obtained by investigating the relations between Chow groups and K-groups over Spec (R), X, Q, the blow-up $\widetilde{X}$ of X at p, and $X - p$. This is the same setup which was used by Levine [8], whose results we are generalizing here, and also by Kurano [7], whose results on the Chow group of $R_\mathfrak{m}$ we cited above.

The second result obtained in this paper is the following vanishing theorem, proved earlier by the first author in [12] using different methods. Here it is obtained by adapting the argument of Gillet and Soulé. Recall that the support of a perfect complex is the complement of the largest open subscheme on which it is exact.

Theorem 2. *Let R be a Noetherian local ring whose punctured spectrum is regular. Let $A_\bullet$, $B_\bullet$ be two perfect complexes on Spec R such that dim supp $A_\bullet$ + dim supp $B_\bullet$ < dim R, while $A_\bullet \otimes B_\bullet$ is supported at the closed point of Spec R. Then*

$$\chi(A_\bullet, B_\bullet) = \sum_i (-1)^i \ell \left(H_i(A_\bullet \overset{\mathbb{L}}{\otimes} B_\bullet) \right) = 0.$$

Corollary 3. *If R is a Noetherian local ring whose punctured spectrum is regular, then the Vanishing Theorem holds for pairs of modules of finite projective dimension.*

2. Chow groups and local Chern characters

In this section we summarize the definition of the Chow group and the properties of local Chern characters which we need and describe their relation with perfect complexes and modules of finite projective dimension. For further details we refer to Fulton [3], Chapter 18, Roberts [14], Chapters 11 and 12, and the paper of Thomason and Trobaugh [17], Section 2.

Let Y be a Noetherian scheme. For technical reasons (which are explained in [3], [17] and [14]) we assume that Y is quasi-projective over a Noetherian regular scheme; in the situation of Theorem 1, Y will be one of the schemes described near the end of the previous section, which clearly have this property. The *rational Chow group* of Y is defined as follows: we let $Z_*(Y)$ be the vector space over $\mathbb{Q}$ with basis equal to the set of reduced and irreducible subschemes of Y (if $Y = \mathrm{Spec}\,(A)$ for a ring A, $Z_*(Y)$ is the vector space with basis the set of prime ideals of A). Then $Z_*(Y)$ is a $\mathbb{Q}$-vector space graded by dimension called the (rational) group of cycles on Y. For each irreducible closed subset Z of Y of dimension $i+1$ and each rational function f on Z, we let div (f, Z) denote the cycle of dimension i defined as the difference of zeros and poles of f counted with multiplicity. The rational Chow group is then defined to be the quotient of $Z_*(Y)$ by the subspace generated by $\operatorname{div}(f, Z)$ for all such Z and f. The Chow group is usually defined over the integers rather than the rational numbers; however, we only need to consider the rational Chow group so we will refer to it simply as the Chow group of Y. The Chow group is also graded by dimension, and we let $CH_i(X)$ denote the component generated by reduced and irreducible subschemes of dimension i. If Z is such an irreducible closed subscheme, the class it defines in the Chow group is denoted $[Z]$; in the case of a ring A, we often denote the generator corresponding to $\mathfrak{p}$ by $[A/\mathfrak{p}]$.

Let $\mathcal{F}_\bullet$ be a bounded complex of quasi-coherent sheaves of $\mathcal{O}_Y$-modules on Y which, Zariski locally on Y, is quasi-isomorphic to a bounded complex of locally free sheaves of finite rank. We call such a complex a *perfect complex* on Y. A bounded complex of locally free $\mathcal{O}_Y$-modules of finite rank is called a *strict perfect complex*. This terminology is different from that of [14], though more or less consistent with [17], Section 2 (in [14], perfect complexes are defined to be what are here called strict perfect complexes).

The *support* of a complex on Y is the set of points of Y where the complex is not exact. If $Z \subset Y$ is a closed subscheme, the Grothendieck group $K_0^Z(Y)$ is the quotient of the free abelian group on isomorphism classes of perfect complexes on Y which are supported within Z (i.e. are exact on $Y \setminus Z$), modulo two types of relations.

(i) Quasi-isomorphic complexes have the same class in K_0.
(ii) If $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is an exact sequence of perfect complexes which is term-wise split exact, then
$$[A_\bullet] - [B_\bullet] + [C_\bullet] = 0 \text{ in } K_0^Z(Y).$$

The group $K_0^Z(Y)$ is denoted by $K_0(Y \text{ on } Z)$ in [17], Section 3, and it is shown there that this group coincides with the corresponding Grothendieck group formed using only strict perfect complexes. The following facts are easy consequences of the definitions.

(i) For any complex $A_\bullet$, there is a relation $[A_\bullet[n]] = (-1)^n[A_\bullet]$ in K_0, where $A_\bullet[n]$ denotes the shifted complex defined by $(A_\bullet[n])_i = A_{i+n}$.
(ii) The Grothendieck group may be equivalently defined as the free abelian group on isomorphism classes in the corresponding bounded derived category of perfect complexes, modulo relations $[B] = [A] + [C]$ for each exact triangle $A \to B \to C \to A[1]$.
(iii) If $Y' \subset Y$ is a local complete intersection closed subscheme, then the direct image on Y of any perfect complex on Y' is a perfect complex; hence there is a well-defined homomorphism $K_0^Z(Y') \to K_0^Z(Y)$, for any closed subscheme $Z \subset Y'$.

If $Z = Y$, we use the notation $K_0(Y)$ in place of $K_0^Y(Y)$. If Y is quasi-projective over a Noetherian ring, so that Y supports an ample family of line bundles (as described in [17], 2.1.1), then $K_0(Y)$ (as defined here using perfect complexes) coincides with the "usual" Grothendieck group of locally free $\mathcal{O}_Y$-modules (vector bundles). This is a (very) special case of [17], Prop. 3.10. More generally, suppose $Z \subset Y$ is a closed, local complete intersection subscheme, where Y is quasi-projective over a Noetherian ring (or Y supports an ample family of line bundles as in *loc. cit*). Then $K_0^Z(Y)$ coincides with the "usual" Grothendieck group of the exact category of coherent $\mathcal{O}_Y$-modules which have finite $\mathcal{O}_Y$-homological dimension and vanish on $Y \setminus Z$ (see [17], Ex. 5.7, as well as Proposition 2 below).

For each perfect complex $\mathcal{F}_\bullet$ on Y with support in Z, and for each closed subscheme $W \subset Y$, there is a *local Chern character* denoted $\mathrm{ch}(\mathcal{F}_\bullet)$, which determines a homomorphism of Chow groups $CH_*(W) \to CH_*(W \cap Z)$ (which is functorial for inclusions $W \subset W'$ of closed subschemes). More precisely, $\mathrm{ch}(\mathcal{F}_\bullet)$ is a sum

$$\mathrm{ch}(\mathcal{F}_\bullet) = \mathrm{ch}_0(\mathcal{F}_\bullet) + \mathrm{ch}_1(\mathcal{F}_\bullet) + \dots$$

where for each i, if α is an element of $CH_j(W)$, then $\mathrm{ch}_i(\mathcal{F}_\bullet)(\alpha)$ is an element of $CH_{j-i}(W \cap Z)$.

In the case of most interest in this paper, $Y = \mathrm{Spec}(A)$, where A is a local ring, and $\mathcal{F}_\bullet$ is a bounded complex of finitely generated free modules with support at the maximal ideal of A, or equivalently, with homology of finite length. In this case, if $\alpha = [A/\mathfrak{p}]$ for some prime ideal $\mathfrak{p}$, then $\mathrm{ch}(\mathcal{F}_\bullet)([A/\mathfrak{p}])$ defines an element of $CH_0(\mathrm{Spec}(A/\mathfrak{m}_A)) \cong \mathbb{Q}$. This number is related to the Euler characteristic of $\mathcal{F}_\bullet \otimes_A A/\mathfrak{p}$ in a manner which will be outlined below.

The connection between Euler characteristics and local Chern characters is expressed using the *Todd class* of a finitely generated module, or, more generally, of a bounded complex of coherent sheaves. Let $\mathcal{M}_\bullet$ be a bounded complex of coherent $\mathcal{O}_Y$-modules. Then there is a class $\tau(\mathcal{M}_\bullet)$ in $CH_*(Z)$, where Z is the support of $\mathcal{M}_\bullet$. We summarize the important properties on the map τ, proofs of which can again be found in the above references.

The assignment $M_\bullet \mapsto \tau(M_\bullet)$ induces a map

$$G_0^Z(Y) \otimes \mathbb{Q} \to CH_*(Z)$$

from the Grothendieck group of bounded complexes of coherent $\mathcal{O}_Y$-modules supported within Z, tensored with $\mathbb{Q}$, to the Chow group, which is an isomorphism of $\mathbb{Q}$-vector spaces (recall that we are using the rational Chow group here). As noted in [17], Corollary 3.13, our hypothesis, that Y is quasi-projective over a Noetherian ring, implies that the group $G_0(Y)$ coincides with the ("usual") Grothendieck group of the exact category of coherent $\mathcal{O}_Y$-modules.

If $Y = \mathrm{Spec}(A)$ and $\mathcal{M}_\bullet$ is simply the module $A/\mathfrak{p}$ in degree zero, then

$$\tau(A/\mathfrak{p}) = [A/\mathfrak{p}] + \tau_{i-1}(A/\mathfrak{p}) + \dots ,$$

where i is the dimension of $A/\mathfrak{p}$. In other words, $\tau(A/\mathfrak{p})$ is equal to the class of $A/\mathfrak{p}$ up to lower dimensional terms. If $Y = \mathrm{Spec}(A)$ and A is local, and if $\mathcal{M}_\bullet$ has homology of finite length, then $\tau(\mathcal{M}_\bullet)$ is an element of $CH_0(\mathrm{Spec}(A/\mathfrak{m})) \cong \mathbb{Q}$, and we have

$$\tau(\mathcal{M}_\bullet) = (\chi(\mathcal{M}_\bullet))[A/\mathfrak{m}] = \left(\sum (-1)^i \mathrm{length}\,(H_i(\mathcal{M}_\bullet))\right)[A/\mathfrak{m}].$$

The connection with local Chern characters comes through the local Riemann-Roch theorem. This theorem states the following: let $\mathcal{F}_\bullet$ be a perfect complex on Y, supported on Z, and let $\mathcal{M}_\bullet$ be a bounded complex of coherent sheaves on Y, supported on W. Then the tensor product $\mathcal{F}_\bullet \otimes \mathcal{M}_\bullet$ is also a bounded complex of coherent sheaves, and we have

$$\tau(\mathcal{F}_\bullet \otimes \mathcal{M}_\bullet) = \mathrm{ch}(\mathcal{F}_\bullet)(\tau(\mathcal{M}_\bullet)).$$

This is an equality in the the Chow group of the support of $\mathcal{F}_\bullet \otimes \mathcal{M}_\bullet$, which is just $Z \cap W$. In other words, we have a commutative diagram

$$\begin{array}{ccc} K_0^Z(Y) \otimes G_0(W) \otimes \mathbb{Q} & \stackrel{1\otimes\tau}{\to} & K_0^Z(Y) \otimes CH_*(W) \\ \downarrow & & \downarrow \\ G_0(Z\cap W) \otimes \mathbb{Q} & \stackrel{\tau}{\to} & CH_*(Z \cap W) \end{array}$$

where $K_0^Z(Y) \otimes G_0(W) \to G_0(Z \cap W)$ is induced by tensoring with strict perfect complexes (this is a particular case of [17], (3.15.5)).

We work out what this theorem says when $Y = \mathrm{Spec}\,(A)$, $\mathcal{M}_\bullet = A/\mathfrak{p}$ in degree zero, and $\mathcal{F}_\bullet$ is a perfect complex with homology supported at the maximal ideal of A. In this case, if $i = \dim\,(A/\mathfrak{p})$, identifying $CH_0(\mathrm{Spec}\,(A/\mathfrak{m}))$ with $\mathbb{Q}$, we have

$$\chi(\mathcal{F}_\bullet \otimes A/\mathfrak{p}) = \mathrm{ch}\,(\mathcal{F}_\bullet)(\tau(\mathcal{M}_\bullet)) =$$
$$\mathrm{ch}_{\,i}(\mathcal{F}_\bullet)([A/\mathfrak{p}]) + \mathrm{ch}_{\,i-1}(\mathcal{F}_\bullet)(\tau_{i-1}(A/\mathfrak{p})) + \cdots + \mathrm{ch}_{\,0}(\mathcal{F}_\bullet)(\tau_0(A/\mathfrak{p})).$$

We now return to the example described in the previous section. Let $A = k[[X,Y,Z,W]]/(XY - ZW)$ and let M be the module of finite length and finite projective dimension constructed by Dutta, Hochster, and MacLaughlin described there. Let $\mathcal{F}_\bullet$ be a free resolution of M, and let $\mathfrak{p}$ be the prime ideal (X,Z). Then we have

$$\chi(M, A/\mathfrak{p}) = \chi(\mathcal{F}_\bullet \otimes A/\mathfrak{p}) = -1 = \mathrm{ch}\,(\mathcal{F}_\bullet)(\tau(A/\mathfrak{p})).$$

In this case it can be shown that $\tau(A/\mathfrak{p}) = [A/\mathfrak{p}]$.

Thus, using the Riemann-Roch formula we can express the the failure of the vanishing conjecture of intersection multiplicities for nonregular rings in terms of the nonvanishing of local Chern characters on elements of low dimension (in this case dimension 2 rather than 3). The advantage is that it is then possible to use this example to construct other examples of behaviour of Euler characteristics (see Roberts [14], Chapter 13). We give one example of a question which arose from these developments.

If A is a local integral domain of dimension d, and if $\mathcal{F}_\bullet$ is a perfect complex with homology of finite length, then, applying the local Riemann-Roch formula to A itself, we have

$$\chi(\mathcal{F}_\bullet) = \mathrm{ch}\,(\mathcal{F}_\bullet)\tau(A) = \mathrm{ch}_{\,d}(\mathcal{F}_\bullet))[A] + \sum_{i<d} \mathrm{ch}_{\,i}(\mathcal{F}_\bullet)\tau_i(A),$$

where d is the dimension of A. It was asked when the sum on the right reduced to the first term, which is called the *Dutta multiplicity* of $\mathcal{F}_\bullet$ (see Kurano [6] and Dutta [1]; in positive characteristic the Dutta multiplicity is expressed as a limit over powers of the Frobenius map as introduced by Dutta). The Dutta multiplicity is equal to the Euler characteristic if A is a complete intersection, since in that case we have $\tau(A) = [A]$. They are not equal in general, and a counterexample in which A is Cohen-Macaulay can by constructed using the counterexample to vanishing described above. It was asked whether such an example existed for a Gorenstein ring.

There are two issues here. One is whether $\tau_i(A) = 0$ for $i < d$, and the other is the existence of a perfect complex $\mathcal{F}_\bullet$ for which $\mathrm{ch}\,(\mathcal{F}_\bullet)(\tau_i(A)) \neq 0$. An example of a Gorenstein ring A of dimension 5 with $\tau_3(A)) \neq 0$ was constructed by Kurano [7]. An example of such a Gorenstein ring together with a perfect complex $\mathcal{F}_\bullet$ with $\mathrm{ch}\,(\mathcal{F}_\bullet)(\tau_3(A)) \neq 0$ was later constructed by Miller and Singh [10]. Part of their construction involved finding 55 by 55 matrices with appropriate rank conditions similar to the method used by Dutta, Hochster, and McLaughlin in their example.

In this paper we show that the localization exact sequence in K-theory implies the existence of examples of this type. In addition to the example of Dutta, Hochster, and McLaughlin, whose existence was proven by this method by Levine, it proves the existence of the example of Miller and Singh and shows that one exists also over the Gorenstein ring of Kurano.

After these preliminaries, we state the first problem considered in this paper more precisely. Let A be a Cohen-Macaulay local ring, and let $\widetilde{\mathcal{C}}$ be the category of modules of finite length and finite projective dimension. For each $M \in \widetilde{\mathcal{C}}$, we have a local Chern character $\mathrm{ch}\,(\mathcal{F}_\bullet)$, where $\mathcal{F}_\bullet$ is a free resolution of M. As described above, if we identify $CH_0(\mathrm{Spec}\,(A/\mathfrak{m}))$ with $\mathbb{Q}$, the local Chern character defines a map

$$\mathrm{ch}_{\widetilde{\mathcal{C}}} : K_0(\widetilde{\mathcal{C}}) \to \mathrm{Hom}\,(CH_*(\mathrm{Spec}\,(A)), \mathbb{Q}) = CH_*(\mathrm{Spec}\,A)^\vee.$$

More generally, if A is any Noetherian local ring, let $\mathcal{C}$ be the category of perfect complexes of A-modules with punctual support; in a similar fashion, we then obtain a homomorphism

$$\mathrm{ch}_{\mathcal{C}} : K_0(\mathcal{C}) \to CH_*(\mathrm{Spec}\,A)^\vee.$$

If A is Cohen-Macaulay, then there is a canonical isomorphism $K_0(\mathcal{C}) \to K_0(\widetilde{\mathcal{C}})$ (Proposition 2 below), under which these two maps become identified.

In certain cases, including those mentioned in this section, we determine the rational span of the image of the map

$$\mathrm{ch}_{\mathcal{C}} : K_0(\mathcal{C}) \to CH_*(\mathrm{Spec}\,A)^\vee.$$

This gives many more examples of modules with negative intersection multiplicities and makes it easier to find whether such examples exist over a given ring of the specified type.

To relate the category $\mathcal{C}$ with modules of finite length and finite projective dimension, we use Proposition 2 below (as noted earlier, this is a special case of [17], Ex. 5.7). We first prove a lemma which contains the main construction used in the proof.

Lemma 1. *Let A be a Cohen-Macaulay ring, and let $\mathcal{F}_\bullet$ be a complex of A-modules with homology of finite length. Then for each integer i, there exists a module M of finite length and finite projective dimension with projective resolution $\mathcal{K}_\bullet$ and a map from $\mathcal{K}_\bullet[i]$ to $\mathcal{F}_\bullet$ which induces a surjection on homology in degree i.*

Proof: We will construct a map $f_\bullet$ from a direct sum of Koszul complexes $\mathcal{K}_\bullet$ on sequences of parameters shifted by i to $\mathcal{F}_\bullet$ which induces a surjection on homology of degree i. Since A is Cohen-Macaulay, $\mathcal{K}_\bullet$ is a resolution of a module M in $\widetilde{\mathcal{C}}$.

We first map a free module onto $H_i(\mathcal{F}_\bullet)$. Since $H_i(\mathcal{F}_\bullet)$ has finite length, we can choose a system of parameters $x_1, \dots, x_d$ which annihilates $H_i(\mathcal{F}_\bullet)$ and obtain a map from a direct sum of copies of $A/(x_1, \dots, x_d)$ onto $H_i(\mathcal{F}_\bullet)$. We wish to show that if we replace the x_i by suitably high powers x_i^n, we can extend this map to a map from the direct sum of the Koszul complexes on $x_1^n, \dots, x_d^n$ to $\mathcal{F}_\bullet$. It suffices to show this for each Koszul complex separately. Let $\mathcal{K}_\bullet(x_1, \dots, x_d)$ be the Koszul complex on $x_1, \dots, x_d$, and assume that we have a map from $\mathcal{K}_\bullet(x_1, \dots, x_d)$ to $\mathcal{F}_\bullet$ defined up to degree k. If $\mathcal{F}_\bullet$ were exact we could extend the map to the next step, but we only know that the homology of $\mathcal{F}_\bullet$ has finite length. For each $i_1, \dots, i_j$

with $j \leq k$ let $a_{i_1,\dots,i_j}$ be the image of $e_{i_1} \wedge \cdots \wedge e_{i_j}$ in $\mathcal{F}_\bullet$. If we replace each $a_{i_1,\dots,i_j}$ by $x_{i_1}^{n-1} \cdots x_{i_j}^{n-1} a_{i_1,\dots,i_j}$, we can use these elements to define a map from the Koszul complex on $x_1^n, \dots, x_d^n$ to $\mathcal{F}_\bullet$. Since $\mathcal{F}_\bullet$ has homology of finite length, for n sufficiently large the image of the map in degree k will be contained in the boundaries, and we can extend $f_\bullet$. □

Proposition 2. *Let A be a Cohen-Macaulay local ring, let $\widetilde{\mathcal{C}}$ be the category modules of finite length and finite projective dimension, and let $\mathcal{C}$ be the category of perfect complexes with homology of finite length. We have a map from $K_0(\widetilde{\mathcal{C}})$ to $K_0(\mathcal{C})$, which is an isomorphism.*

Proof: The map from $K_0(\widetilde{\mathcal{C}})$ to $K_0(\mathcal{C})$ is defined by taking projective resolutions. The remainder of the proof is devoted to constructing an inverse $K_0(\mathcal{C}) \to K_0(\widetilde{\mathcal{C}})$. We will define a map from the free abelian group $\mathcal{F}$ on isomorphism classes of strict perfect complexes to $K_0(\widetilde{\mathcal{C}})$, and verify (i) that quasi-isomorphic complexes have the same class, and (ii) that we have the required relation between the images of complexes fitting into a short exact sequence of strict perfect complexes (note that such an exact sequence is automatically term-wise split exact since all modules are projective over the ring A). For a complex $F_\bullet$ in $\mathcal{C}$, we denote the image of $F_\bullet$ in $K_0(\widetilde{\mathcal{C}})$ under this map by $[F_\bullet]$.

If $F_\bullet$ is a strict perfect complex, let $i \leq k$ be integers such that $H_i(F_\bullet)$ and $H_k(F_\bullet)$ are non-zero, and $H_j(F_\bullet) = 0$ for $j < i$ or $j > k$. We define the image in $K_0(\widetilde{\mathcal{C}})$ of any such $F_\bullet$, by induction on $k-i$. If $k = i$, then we define the image of $F_\bullet$ to be $(-1)^i[H_i(F_\bullet)]$; this assignment depends only on the quasi-isomorphism class of $F_\bullet$, and will ensure that the induced composite map $K_0(\widetilde{\mathcal{C}}) \to K_0(\mathcal{C}) \to K_0(\widetilde{\mathcal{C}})$ is the identity. Further, given any exact sequence of strict perfect complexes of this special type, we have a short exact sequence in $\widetilde{\mathcal{C}}$ of their unique non-zero homologies, and hence the desired relation between their images in $K_0(\widetilde{\mathcal{C}})$. Note that the class of a shifted complex $F_\bullet[n]$ is $(-1)^n$ times that of $F_\bullet$.

Now suppose $k > i$. Assume by induction that, for all complexes with vanishing homologies in degrees $< m$ and $> m+(k-i-1)$, for some m, we have already defined their images in $K_0(\widetilde{\mathcal{C}})$, and shown that (i) the image of such a complex depend only on its quasi-isomorphism class, and shifting by n corresponds to multiplication by $(-1)^n$, and (ii) for any exact sequence $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ of strict perfect complexes of this special type, the classes in $K_0(\widetilde{\mathcal{C}})$ satisfy the desired relation $[B_\bullet] = [A_\bullet] + [C_\bullet]$.

Now let $F_\bullet$ be a complex with homologies $H_j(F_\bullet)$ vanishing outside the range $i \leq j \leq k$, with non-zero i-th and k-th homologies. Using Lemma 1, let $K_\bullet$ be a finite free resolution of some module $M \in \widetilde{\mathcal{C}}$, and let $f : K_\bullet[-i] \to F_\bullet$ be a map which induces a surjection $M \twoheadrightarrow H_i(K_\bullet)$. Then $H_j(\mathrm{Cone}(f)) = 0$ for $j \leq i$ and $j > k$; hence by induction, we have already defined the image $[\mathrm{Cone}(f)]$ in $K_0(\widetilde{\mathcal{C}})$. Since there is a term-wise split exact sequence of complexes $0 \to F_\bullet \to \mathrm{Cone}(f) \to K_\bullet[-i-1] \to 0$, we are forced to define

$$[F_\bullet] = [\mathrm{Cone}(f)] + (-1)^i[M] \in K_0(\widetilde{\mathcal{C}}).$$

We first show that this definition is independent of the choices of M, $K_\bullet$ and f. Suppose N is another such module, and assume first that $M \hookrightarrow N$ and that the following properties hold.

(1) We can lift this inclusion to a term-wise split injective map $h : K_\bullet \to L_\bullet$ between the given resolution of M and a free resolution $L_\bullet$ of N (then $\operatorname{coker} h$ is a finite free resolution of N/M).
(2) There is a map of complexes $g : L_\bullet[-i] \to F_\bullet$ extending f.

There is then a short exact sequence of strict perfect complexes

$$0 \to \operatorname{Cone}(f) \to \operatorname{Cone}(g) \to \operatorname{coker}(h)[-i-1] \to 0$$

which gives a relation in $K_0(\widetilde{\mathcal{C}})$

$$\begin{aligned}[\operatorname{Cone}(f)] + (-1)^i[M] &= ([\operatorname{Cone}(g)] + (-1)^{i-1}[N/M]) + (-1)^i[M] \\ &= [\operatorname{Cone}(g)] + (-1)^i[N].\end{aligned}$$

Hence $(M, K_\bullet, f)$ and $(N, L_\bullet, g)$ determine the same element of $K_0(\widetilde{\mathcal{C}})$. Now given two arbitrary choices $(M, K_\bullet, f)$ and $(M', K'_\bullet, f')$, we can take $N = M \oplus M'$, $L_\bullet = K_\bullet \oplus K'_\bullet$, $g = f + f'$. Now both $(M, K_\bullet, f)$ and $(M', K'_\bullet, f')$ determine the same element of $K_0(\widetilde{\mathcal{C}})$ as $(N, L_\bullet, g)$.

Next, since quasi-isomorphism of strict perfect complexes coincides with homotopy equivalence, if $(M, K_\bullet, f)$ is a choice as above for $F_\bullet$, then for any quasi-isomorphic complex $F'_\bullet$, there is a map of complexes $a : F_\bullet \to F'_\bullet$ realizing this quasi-isomorphism, and $(M, K_\bullet, a \circ f)$ is a choice as above for $F'_\bullet$. Since the cones over f and $a \circ f$ are quasi-isomorphic, the class in $K_0(\widetilde{\mathcal{C}})$ associated to $F_\bullet$ as above depends only on the quasi-isomorphism class of $F_\bullet$. That shifting by n corresponds to multiplication by $(-1)^n$ is also clear by construction.

It remains to verify that if $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is a short exact sequence of strict perfect complexes of the above type, then $[B_\bullet] = [A_\bullet] + [C_\bullet]$ in $K_0(\widetilde{\mathcal{C}})$. Since the image of any complex in $K_0(\widetilde{\mathcal{C}})$ depends only on its quasi-isomorphism class, we may use a mapping cone argument to "rotate the exact triangle" if necessary; hence, we may assume without loss of generality that for some indices $i < k$ (with the chosen value for $k - i$, on which parameter we are inducting), all 3 complexes have vanishing H_j for $j < i$ and $j > k$. Making a shift, we may in fact assume further that $i = 0$.

We claim that it is possible to find a short exact sequence of modules

$$0 \to M_A \to M_B \to M_C \to 0$$

in $\widetilde{\mathcal{C}}$, a corresponding short exact sequence of finite free resolutions

$$0 \to K(A)_\bullet \to K(B)_\bullet \to K(C)_\bullet \to 0$$

and a diagram of maps of complexes

$$\begin{tikzcd} 0 \arrow[r] & K(A)_\bullet \arrow[r] \arrow[d, "f"] & K(B)_\bullet \arrow[r] \arrow[d, "g"] & K(C)_\bullet \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A_\bullet \arrow[r] & B_\bullet \arrow[r] & C_\bullet \arrow[r] & 0 \end{tikzcd}$$

such that the induced maps on H_0, namely $M_A \to H_0(A_\bullet)$, $M_B \to H_0(B_\bullet)$, $M_C \to H_0(C_\bullet)$ are all surjective.

Assuming this claim, note that there is an induced short exact sequence of complexes

$$0 \to \operatorname{Cone}(f) \to \operatorname{Cone}(g) \to \operatorname{Cone}(h) \to 0$$

and by the induction hypothesis, we have

$$[\mathrm{Cone}(g)] = [\mathrm{Cone}(f)] + [\mathrm{Cone}(h)]$$

in $K_0(\widetilde{\mathcal{C}})$. Since we also have a relation $[M_B] = [M_A] + [M_C]$ in $K_0(\widetilde{\mathcal{C}})$, it follows that

$$[B_\bullet] = [\mathrm{Cone}(g)] + [M_B] = [\mathrm{Cone}(f)] + [\mathrm{Cone}(h)] + [M_A] + [M_C] = [A_\bullet] + [C_\bullet].$$

To prove the claim, first choose $K(A)_\bullet \to M_A$ such that there is a map of complexes $K(A)_\bullet \to A_\bullet$ inducing a surjection $M_A \to H_0(A_\bullet)$. Next choose $K(C)_\bullet \to M_C$ such that there is a map of complexes $K(C)_\bullet \to B_\bullet$, so that the induced map of complexes $K(C)_\bullet \to C_\bullet$ induces a surjection on H_0 (for example, we may arrange that $M_C \to H_0(B_\bullet)$ is itself surjective). Now define $K(B)_\bullet = K(A)_\bullet \oplus K(C)_\bullet$, $M_B = M_A \oplus M_C$. There is an obvious map of complexes $K(A)_\bullet \oplus K(C)_\bullet \to B_\bullet$, by construction, and it is now trivial to verify that, for the split exact sequences

$$0 \to K(A)_\bullet \to K(B)_\bullet \to K(C)_\bullet \to 0,$$

$$0 \to M_A \to M_B \to M_C \to 0$$

we have all the claimed properties. □

3. Proof of Theorem 1

We now restrict attention to the situation described at the end of section 1. We have a graded ring R, and we have $Q = \mathrm{Proj}\,(R)$, $X = \mathrm{Proj}\,(R[T])$, and the point $p \in X$ defined by the maximal graded ideal of R. We further consider $\pi : \widetilde{X} \to X$, the blow up of X at p, with exceptional set $E = \pi^{-1}(p)$. There is a morphism $g : \widetilde{X} \to Q$, induced by the natural map $X \setminus \{p\} \to Q$ determined by the homomorphism of graded rings $R \to R[T]$. Let $i : Q \to X$ be the closed immersion determined by the graded surjection $R[T] \to R$ with kernel $TR[T]$. Let $\widetilde{i} : Q \to \widetilde{X}$ be the induced embedding. The morphism $g : \widetilde{X} \to Q$ makes $\widetilde{X}$ a $\mathbb{P}^1$-bundle over Q, of which E and $\widetilde{i}(Q)$ are disjoint sections. If $A = R_\mathfrak{m} = \mathcal{O}_{X,p}$ is the local ring obtained by localizing R at the graded maximal ideal, there is a natural (flat) morphism $j : \mathrm{Spec}\, A \to X$, which is an inverse limit of open immersions. We summarize this data in the following commutative diagram.

$$\begin{tikzcd}
& & Q & \\
E \arrow[urr, "\cong"] \arrow[rr, "\widetilde{j}"] \arrow[d] & & \widetilde{X} \arrow[u, "g"] \arrow[d, "\pi"] & Q \arrow[ul, "1_Q"'] \arrow[l, "\widetilde{i}"'] \arrow[dl, "i"] \\
\{p\} \arrow[r] \arrow[drr, "="'] & \mathrm{Spec}\, A \arrow[r, "j"] & X \arrow[d, "f"] & \\
& & \mathrm{Spec}\, k &
\end{tikzcd}$$

In this section we describe relations between the K-groups defined over these schemes, and use these relations to prove Theorem 1. As mentioned above, the main tool used here is the localization exact sequence from K-theory. This sequence was constructed in special cases by Levine (generalizing work of Quillen and others) and in a very general situation by Thomason and Trobaugh [17]. If Y is a scheme, Z, W closed subschemes, then the main result of [17] (Theorem 7.4) yields an exact sequence

$$\cdots K_{n+1}^{Z\setminus W}(Y\setminus W) \to K_n^{Z\cap W}(Y) \to K_n^Z(Y) \to K_n^{Z\setminus W}(Y\setminus W) \to \cdots$$

where $K_n^Z(Y)$ denotes the n-th algebraic K-group of the category of perfect complexes on Y supported in Z (for $n=0$ this K-group has the concrete presentation described earlier). Note that by excision, $K_n^Z(Y)$ depends only on a Zariski neighbourhood of Z in Y (in fact a stronger form of "excision" is valid for K-groups with supports; see [17], Proposition 3.19 and Theorem 7.1).

In particular, let $\mathcal{C}$ denote the category of perfect complexes on X supported at p. In this situation the theorem implies that we have an exact sequence

$$K_0(\mathcal{C}) \to K_0(X) \xrightarrow{j^*} K_0(X\setminus\{p\}). \tag{2}$$

We will combine this with the following result of Kurano which describes the Chow groups of localizations of graded rings.

Proposition 3. (Kurano [7]) *The composition*

$$CH_*(Q) \xrightarrow{g^*} CH_*(\widetilde{X}) \xrightarrow{\pi_*} CH_*(X) \xrightarrow{j^*} CH_*(Spec\, A)$$

is a graded surjection of degree 1, with kernel equal to the image of multiplication by $h = c_1(\mathcal{O}_Q(1))$.

□

Lemma 4. *The map $\phi : X\setminus\{p\} \to Q$ (obtained from g) induces an isomorphism on both K-groups and on Chow groups by pullback.*

Proof: The map ϕ makes $X\setminus\{p\} \to Q$ into a geometric line bundle, so this is a standard homotopy property for Chow groups ([3], Prop. 1.9) and, since Q is regular, for K-groups as well (this is a very special case of [16], Prop. 5.17). □

We now begin the proof of the theorem. We wish to determine the image of the map defined by the local Chern character from $K_0(\mathcal{C})$ to the dual of the Chow group of Q, and to show that it can be identified with a certain subgroup of the dual of the Chow group of Q modulo numerical equivalence. We do this in two main steps. We first construct a map from $K_0(\mathcal{C})$ to $K_0(Q)$ and show that it maps onto the kernel of the hyperplane section. We then show that the action by the local Chern character takes this to the image described in Theorem 1.

The map from $K_0(\mathcal{C})$ to $K_0(Q)$ is defined using π^*, which maps $K_0^{\{p\}}(X)$ to $K_0^E(\widetilde{X})$; as noted above, we can identify $K_0(\mathcal{C})$ with $K_0^{\{p\}}(X)$. Since E and $\widetilde{X}$ are non-singular, there is a natural map $K_0(E) \to K_0^E(\widetilde{X})$ which is an isomorphism. Thus we can combine this map with the isomorphism between E and Q to define a map from $K_0(\mathcal{C})$ to $K_0(Q)$ by taking the composition

$$K_0(\mathcal{C}) \to K_0^E(\widetilde{X}) \to K_0(E) \to K_0(Q).$$

The map $\widetilde{j}^*$ is defined by tensoring with $\mathcal{O}_E$ over $\mathcal{O}_{\widetilde{X}}$. Let Y be the blow up of Spec (R) at p. Then Y is an open subset of $\widetilde{X}$ containing E. Thus, using the short exact sequence

$$0 \to \mathcal{O}_Y(1) \to \mathcal{O}_Y \to \mathcal{O}_E \to 0,$$

we can identify the composition

$$K_0(E) \xrightarrow{\cong} K_0^E(\widetilde{X}) \to K_0(\widetilde{X}) \xrightarrow{\widetilde{j}^*} K_0(E)$$

with multiplication by $[\mathcal{O}_E] - [\mathcal{O}_E(1)]$. On the other hand, the composite

$$K_0(\mathcal{C}) \xrightarrow{\pi^*} K_0^E(\widetilde{X}) \to K_0(\widetilde{X}) \xrightarrow{\widetilde{j}^*} K_0(E)$$

is zero, since it factors through $K_0(\mathcal{C}) \to K_0(X) \to K_0(\{p\})$, which is 0. Hence the image of

$$\pi^* : K_0(\mathcal{C}) \to K_0^E(\widetilde{X}) \cong K_0(E)$$

lies in the kernel of multiplication by $[\mathcal{O}_E]-[\mathcal{O}_E(1)]$. Identifying $K_0(E)$ with $K_0(Q)$ using g, we may view this as the kernel of multiplication by $h_K = [\mathcal{O}_Q] - [\mathcal{O}_Q(-1)]$ on $K_0(Q)$, since $[\mathcal{O}_Q] - [\mathcal{O}_Q(1)] = -h_K \cdot [\mathcal{O}_Q(1)]$ in $K_0(Q)$ (the right side is a product for the ring structure in $K_0(Q)$), where $-[\mathcal{O}_Q(1)]$ is a unit in the ring $K_0(Q)$.

Thus the map

$$K_0(\mathcal{C}) \to K_0^E(\widetilde{X}) \cong K_0(E) \cong K_0(Q)$$

has image contained in $\ker(K_0(Q) \xrightarrow{\cdot h_K} K_0(Q))$. We claim that the image of $K_0(\mathcal{C})$ actually coincides with this kernel. Since the embedding of Q in X is defined by a Cartier divisor, it induces a homomorphism $i_* : K_0(Q) \to K_0(X)$. By an argument similar to that used above for the embedding of E in $Y \subseteq \widetilde{X}$, the composition

$$K_0(Q) \xrightarrow{i_*} K_0(X) \to K_0(X \setminus \{p\}) \xrightarrow{i^*} K_0(Q)$$

is multiplication by $[\mathcal{O}_Q] - [\mathcal{O}_Q(-1)] = h_K$. Hence the exact sequence (2) implies that

$$i_* \left(\ker(K_0(Q) \xrightarrow{\cdot h_K} K_0(Q))\right) \subset \operatorname{image} K_0(\mathcal{C}) \to K_0(X).$$

Thus, since there is a commutative diagram

$$\begin{tikzcd}
K_0(\mathcal{C}) \arrow[r] \arrow[dr] & K_0(\widetilde{X}) & K_0(Q) \arrow[l, "\widetilde{i}^*"'] \arrow[dl, "i_*"] \\
& K_0(X) \arrow[u, "\pi^*"] &
\end{tikzcd}$$

we can deduce that

$$\widetilde{i}_* \left(\ker(K_0(Q) \xrightarrow{\cdot h_K} K_0(Q))\right) \subset \operatorname{image} \left(K_0(\mathcal{C}) \to K_0^E(\widetilde{X}) \cong K_0(E) \cong K_0(Q)\right).$$

It remains to relate this embedding by $\widetilde{i}_*$ with the previously defined map given by $\widetilde{j}_*$, and we do this by showing that the composite

$$\left(\ker(K_0(Q) \xrightarrow{\cdot h_K} K_0(Q))\right) \xrightarrow{\widetilde{i}_*} (\ker K_0(E) \xrightarrow{\cdot([\mathcal{O}_E]-[\mathcal{O}_E(1)])} K_0(E))$$
$$\xrightarrow{\cong} (\ker K_0(Q) \xrightarrow{\cdot h_K} K_0(Q))$$

is the identity. We have

$$\widetilde{i}_*(a) = g^*(a) \cdot [\mathcal{O}_{\widetilde{i}(Q)}],$$

and

$$\widetilde{j}_*(a) = g^*(a) \cdot [\mathcal{O}_{\widetilde{j}(E)}].$$

The result now follows from the formula in $K_0(\widetilde{X})$

$$[\mathcal{O}_{\widetilde{i}(Q)}] - [\mathcal{O}_{\widetilde{j}(E)}] = g^*(h_K)\cdot u$$

where $u \in K_0(\widetilde{X})$ is a unit (since $\mathcal{O}_{\widetilde{X}}(\widetilde{i}(Q) - \widetilde{j}(E)) = g^*\mathcal{O}_Q(1)$ in $\operatorname{Pic}(\widetilde{X})$, as $g : \widetilde{X} \to Q$ is the $\mathbb{P}^1$-bundle associated to $\mathcal{O}_Q \oplus \mathcal{O}_Q(1)$).

Thus we have constructed a surjection

$$\alpha : K_0(\mathcal{C}) \to K_0(Q)^{h_K=0} = \ker(K_0(Q) \xrightarrow{\cdot h_K} K_0(Q)).$$

We must show that this map is compatible with the action of local Chern characters, or, in other words, that for the surjection $\beta : CH_*(Q) \to CH_*(\operatorname{Spec} A)$, the pairing

$$K_0(\mathcal{C}) \otimes CH_*(Q) \to CH_*(\operatorname{Spec} k) = \mathbb{Q},$$

$$a \otimes b \mapsto \operatorname{ch}^{\{p\}}_{\mathrm{Spec A}}(\beta(b))$$

is identified with

$$a \otimes b \mapsto (f\circ\pi)_*(\operatorname{ch}_{\widetilde{X}}(\widetilde{i}_*\alpha(a))(g^*b)).$$

This compatibility comes from the commutativity of the following larger diagram.

$$\begin{tikzcd}
K_0(\mathcal{C}) \otimes CH_*(\operatorname{Spec} A) \arrow[drr, "\operatorname{ch}^{\{p\}}_X"] & & \\
K_0(\mathcal{C}) \otimes CH_*(X) \arrow[u, "1\otimes j^*"] \arrow[dr, "\operatorname{ch}^{\{p\}}_X"'] & & \\
K_0(\mathcal{C}) \otimes CH_*(\widetilde{X}) \arrow[u, "1\otimes\pi_*"] \arrow[d, "\pi^*\otimes 1"] & CH_*(\operatorname{Spec} k) = \mathbb{Q} & \\
K_0^E(\widetilde{X}) \otimes CH_*(\widetilde{X}) \arrow[r, "\operatorname{ch}^E_{\widetilde{X}}"'] \arrow[d, "\mathrm{nat}\otimes 1"] & CH_*(E) \arrow[u, "\pi_*"] \arrow[d, "\widetilde{j}_*"] & \\
K_0(\widetilde{X}) \otimes CH_*(\widetilde{X}) \arrow[r, "\operatorname{ch}_{\widetilde{X}}"'] & CH_*(\widetilde{X}) \arrow[uu, bend right=60, "(f\circ\pi)_*"'] &
\end{tikzcd}$$

In this diagram the central trapezoid commutes by the projection formula, and the rest commutes by the bivariant nature of the local Chern character, together with its functorial properties as proved in [3], Chapters 17 and 18 (here "nat" denotes the natural map $K_0^E(\widetilde{X}) \to K_0(\widetilde{X})$).

We now relate the local Chern character of $\widetilde{i}^*(\alpha(a))$ with the intersection pairing on Q. From the Riemann-Roch theorem for the inclusion $\widetilde{i} : Q \to \widetilde{X}$, we have that

$$\operatorname{ch}_{\widetilde{X}}(\widetilde{i}_*\alpha(a)) = \widetilde{i}_*(\operatorname{ch}_Q(\alpha(a))\cdot \operatorname{td}(\mathcal{O}_Q(1))^{-1})$$

$$= \widetilde{i}_*(\operatorname{ch}_Q(\alpha(a))\cdot \frac{1-e^{-h}}{h}),$$

where $\operatorname{td}(\mathcal{O}_Q(1))$ denotes the Todd class of the line bundle $\mathcal{O}_Q(1)$. Hence the above pairing becomes

$$a \otimes b \mapsto (f\circ\pi)_*(\widetilde{i}_*\operatorname{ch}_Q(\alpha(a))\cdot \frac{1-e^{-h}}{h})(g^*(b))$$

$$
\begin{aligned}
&= (f \circ \pi \circ \widetilde{i})_*(\mathrm{ch}_Q(\alpha(a)) \cdot \frac{1-e^{-h}}{h})(\widetilde{i}^* g^*(b)) \\
&= (f \circ \pi \circ \widetilde{i})_*(\mathrm{ch}_Q(\alpha(a)) \cdot \frac{1-e^{-h}}{h})(b)).
\end{aligned}
$$

Now $f \circ \pi \circ \widetilde{i}$ is just the structure morphism $Q \to \operatorname{Spec} k$. Since Q is non-singular and projective over k, there is an element of $CH_*(Q)$, which (by a slight abuse of notation) we also denote $\mathrm{ch}_Q(\alpha(a))$, and an element h similarly representing the hyperplane section, such that the above expression is

$$
(f \circ \pi \circ \widetilde{i})_* \left(\mathrm{ch}_Q(\alpha(a)) \cdot \frac{1-e^{-h}}{h} \cdot b \right),
$$

where the expression in parentheses is the intersection product in the Chow group of Q. On applying $(f \circ \pi \circ \widetilde{i})_*$, the induced pairing is the intersection number of the three classes $\mathrm{ch}_Q(\alpha(a))$, $\frac{1-e^{-h}}{h}$ and b.

The Chern character on Q yields an isomorphism of rings $K_0(Q) \otimes \mathbb{Q} \cong CH^*(Q)$, under which multiplication by h_K on $K_0(Q) \otimes \mathbb{Q}$ is transported to multiplication by $\mathrm{ch}_Q(h_K) = h \cdot (\text{unit})$ on $CH^*(Q)$. Since $\frac{1-e^{-h}}{h}$ is also a unit in $CH^*(Q)$, the image of

$$
\begin{aligned}
K_0(\mathcal{C}) \otimes \mathbb{Q} &\to CH^*(Q), \\
a &\mapsto \mathrm{ch}_Q(\alpha(a)) \cdot \frac{1-e^{-h}}{h},
\end{aligned}
$$

is just $\ker : CH^*(Q) \xrightarrow{\cdot h} CH^*(Q)$.

The intersection pairing on Q defines a map from $CH_*(Q)$ to its dual, and by definition its image vanishes on elements numerically equivalent to zero so gives a map from $CH_*(Q)$ to $CH^*_{\mathrm{num}}(Q)^\vee$. Let W denote $\ker : CH^*(Q) \xrightarrow{\cdot h} CH^*(Q)$ as in the statement of Theorem 1. Combining the results of the previous two paragraphs, we see that the image of the map from $K_0(\mathcal{C}) \otimes \mathbb{Q}$ in $CH_*(Q)^\vee$ given by the local Chern character is the image of W in $CH^*_{\mathrm{num}}(Q)^\vee$.

To conclude the proof, we note that by Proposition 3, we can identify the dual of the Chow group of A with the kernel of h on the dual of $CH_*(Q)$. It is clear that h vanishes on the image of W in $CH^*_{\mathrm{num}}(Q)^\vee$, and hence the image of W in $CH^*_{\mathrm{num}}(Q)^\vee$ is a subspace of $CH_*(A)^\vee$. Thus we have shown that this subspace is the image of $K_0(\mathcal{C})$ under the map defined by the local Chern character.

□

4. Some examples

Using Theorem 1, the problem of finding examples of nonvanishing intersection multiplicities for our specified types of rings reduces to the problem of computing Chow groups. While this problem is in general difficult, there are many cases in which it is straightforward. Many examples were constructed by Kurano in [7]. In particular, he defined determinantal rings for which, using the identification by the Segre embedding, Q is of the form $\mathbb{P}^m \times \mathbb{P}^n$. He showed that the Chow group of A in this case is $\mathbb{Q}[a]/(a^{m+1})$, where m and n are chosen so that $m \leq n$. Letting $m = n = 2$ the graded ring is Gorenstein and he showed that $\tau_3(A) \neq 0$. Corollary 1, combined with Proposition 2, implies that there exists a module of finite length and finite projective dimension such that $\mathrm{ch}(F_\bullet)(\tau_3(A)) \neq 0$, and hence such that the

Dutta mutiplicity is not equal to the ordinary multiplicity. We remark that the module constructed by Miller and Singh can also be shown to exist by this method.

Corollary 1, combined with Proposition 2, similarly implies the existence of "exotic" modules of finite length and finite projective dimension when Q is any homogeneous space for a semi-simple algebraic group, at least one of whose Betti numbers is > 1.

Similarly, if $R = k[X,Y,Z,W]/(f(X,Y,Z,W))$ where f is a homogeneous cubic polynomial for which $\mathrm{Proj}\,(R)$ is regular, $\mathrm{Proj}\,(R)$ is a cubic surface, and its homology in degree 2 is of dimension 7 and is algebraic. Hence there are numerous examples in this case.

We also give an example where one can conclude a positive result. If R is of even dimension and a complete intersection (graded) then it comes from a projective scheme Q of odd dimension, say n. The only cohomology which is not generated by a power of the hyperplane is in dimension n, and thus it cannot be algebraic. Thus the Chow group modulo numerical equivalence is generated by the hyperplane section and vanishing must hold for modules of finite length.

In most of the above examples, rational, homological and numerical equivalence coincide for cycles on the smooth projective variety Q, and the intersection pairing on $CH^*(Q)$ is a perfect pairing between finite dimensional vector spaces. If the intersection pairing on $CH_*(Q)$ is not perfect, the situation is more complicated.

For example, suppose R is defined over $\bar{\mathbb{Q}}$, so that Q and h are also defined over $\bar{\mathbb{Q}}$. Then we claim that, assuming several well-known conjectures in the theory of algebraic cycles, one has

$$W = \ker(CH^*_{\mathrm{num}}(Q) \xrightarrow{\cdot h} CH^*_{\mathrm{num}}(Q)).$$

Indeed, we can first reduce to the case $k = \mathbb{C}$. Assuming Grothendieck's Standard Conjectures, one knows that (a) (co)homological and numerical equivalence coincide for the rational Chow group, where "cohomology" refers to singular cohomology with rational coefficients, and (b) if $\dim Q = d$, then the inverse Hard Lefschetz isomorphisms $H^{d+i}(Q) \to H^{d-i}(Q)$, between rational cohomology groups, are induced by algebraic correspondences. Note that multiplication by h on $H^*(Q)$ is injective on $H^j(Q)$ for $j < d$, by the Hard Lefschetz theorem. Now suppose $a \in H^{d+i}(Q)$, for some $i \geq 0$, is the cohomology class of an algebraic cycle, such that $a \cdot h$ is numerically equivalent to 0, or equivalently, that $a \cdot h = 0$ in $H^{d+i+2}(Q)$. From the Hard Lefschetz theorem, and the Standard Conjectures (i.e. (b) above), we can find an algebraic cycle class $b \in H^{d-i}(Q)$ (in fact a primitive cohomology class), such that $a = b \cdot h^i$. A priori the cohomology class b is defined over $\mathbb{C}$, but since Q is defined over $\bar{\mathbb{Q}}$, any cycle class can be specialized to an algebraically equivalent (and hence homologically equivalent) class defined over $\bar{\mathbb{Q}}$. Thus b is represented by an algebraic cycle class $\widetilde{b} \in CH^{(d-i)/2}(Q)$ which is defined over $\bar{\mathbb{Q}}$. Our assumptions imply that the class $h^{i+1} \cdot \widetilde{b} \in CH^{(d+i)/2+1}(Q)$ is homologous to 0, and so defines (via an Abel-Jacobi map) a class in the corresponding intermediate Jacobian (tensored with $\mathbb{Q}$) associated to $H^{d+i+1}(Q)$. Again by Hard Lefschetz, multiplication by h^{i+1} yields an isomorphism $H^{d-i-1}(Q) \to H^{d+i+1}(Q)$, which (by the Standard Conjectures) preserves the images of algebraic cycles. Hence we can find $c \in CH^i(Q)$, defined over $\bar{\mathbb{Q}}$, such that c is homologically trivial, and $h^{i+1} \cdot c$ and $h^{i+1} \cdot \widetilde{b}$ have the same image in the intermediate Jacobian associated to $H^{d+i+1}(Q)$.

Now we further assume the Bloch-Beilinson conjecture, which states (in a crude form) that for rational classes defined over $\bar{\mathbb{Q}}$ on smooth projective $\bar{\mathbb{Q}}$-varieties, the cycle class map to Deligne-Beilinson cohomology is injective modulo torsion; equivalently, for cycles homologous to 0, the Abel-Jacobi map is injective modulo torsion. This implies, in our situation, that $h^{i+1} \cdot (\widetilde{b} - c) = 0$ in the rational Chow group $CH^*(Q)$. Hence $\widetilde{a} = h^i \cdot (\widetilde{b} - c)$ is an algebraic cycle representing the cohomology class a, such that $h \cdot \widetilde{a} = 0$. In other words, in the above situation, we have proved that

$$\ker \left(CH^*_{\text{num}}(Q) \xrightarrow{\cdot h} CH^*_{\text{num}}(Q) \right) \subset W.$$

Similar reasoning applies to the case of pairs (Q, h) defined over $\bar{\mathbb{F}}_p$, the algebraic closure of a finite field, where conjecturally the cycle map to étale cohomology is injective.

On the other hand, if we omit the assumption that Q and h are defined over $\bar{\mathbb{Q}}$ or $\bar{\mathbb{F}}_p$, the inclusion

$$V \subset \ker \left(CH^*_{\text{num}}(Q) \xrightarrow{\cdot h} CH^*_{\text{num}}(Q) \right)$$

may be strict. The following simple example was given to the second author by H. Esnault and E. Viehweg. Let X be a smooth projective surface, say over $\mathbb{C}$, with $q = \dim \operatorname{Pic}^0(X) = 0$, and $p_g = \dim H^0(X, \Omega^2_{X/\mathbb{C}}) \neq 0$. For example, one could take X to be a smooth hypersurface of degree ≥ 4 in $\mathbb{P}^3_{\mathbb{C}}$. By Mumford's infinite dimensionality theorem for 0-cycles, there exist points $p, q \in X$ such that $[p] - [q] \neq 0$ in $CH^2(X)$ (conjecturally, one cannot do this with X defined over $\bar{\mathbb{Q}}$ and with algebraic points $p, q \in X$). Choose a smooth sufficiently ample curve C on X passing through p and q (in some projective embedding of X, take $C = X \cap H$ where H is a general hypersurface of sufficiently large degree passing through p and q). Now take Q to be the blow-up $\pi : Q \to X$ of p and q. Let $E_p = \pi^{-1}(p)$, $E_q = \pi^{-1}(q)$ be the exceptional divisors for π. Let D be the strict transform of C (that is, D is the closure in Q of $\pi^{-1}(C \setminus \{p, q\})$). Since C is smooth, so is D, and D intersects each of E_p, E_q transversally at 1 point. Since C was assumed to be sufficiently ample, the curve D is an ample divisor on Q. Hence we can find a projective embedding of Q in which the hyperplane divisor class is $n[D]$ for some $n > 0$ (we can even arrange that $n = 1$).

Since p_g and q are birational invariants for smooth projective surfaces, $\operatorname{Pic}^0(Q) = 0$, and rational equivalence coincides with numerical equivalence for $\mathbb{Q}$-divisors (divisors with rational coefficients) on Q. Now $h \cdot ([E_p] - [E_q]) = n([p'] - [q'])$ where $p' = D \cap E_p$, $q' = D \cap E_q$. Since this intersection cycle is a 0-cycle of degree 0, it is numerically equivalent to 0. However $\pi_*([p'] - [q']) = [p] - [q] \neq 0$ in $CH^2(X)$. Hence, in this example, $[E_p] - [E_q]$ determines an element of

$$\ker \left(CH^*_{\text{num}}(Q) \xrightarrow{\cdot h} CH^*_{\text{num}}(Q) \right)$$

which does not lie in the image of W. In fact, if we assume X is a general hypersurface of degree ≥ 4 in $\mathbb{P}^3_{\mathbb{C}}$, then $\operatorname{Pic} X = \mathbb{Z}$; further, the rank of $CH^2(X)$ is infinite, from Mumford's theorem, so we may assume the classes of $[C]^2$, $[p]$ and $[q]$ are linearly independent in $CH^2(X)$. In this case one sees that $CH^1(Q) \to CH^2(Q)$, given by multiplication by h, is injective. By Corollary 2, this implies that vanishing must hold for modules of finite length over the corresponding vertex ring $R_{\mathfrak{m}}$, even though multiplication by h has a non-trivial kernel on $CH^*_{\text{num}}(Q)$.

5. The Vanishing Theorem

In this section we prove Theorem 2, the vanishing theorem for intersection multiplicities of a pair of perfect complexes $A_\bullet$, $B_\bullet$ over a local ring R with an isolated singularity, provided the supports intersect at the closed point of $\operatorname{Spec} R$, but do not intersect properly (*i.e.*, the dimensions of the supports add up to $< \dim R$). It clearly suffices to prove this for strict perfect complexes $A_\bullet$, $B_\bullet$. We may also assume without loss of generality that R is complete.

Let $X = \operatorname{Spec} R$, and let $Y = \operatorname{supp} A_\bullet$ and $Z = \operatorname{supp} B_\bullet$. Then $\dim Y + \dim Z < d = \dim X$, while $Y \cap Z = \{p\}$, where $p \in X$ is the closed point. Let $\mathcal{C}$ be the category of strict perfect complexes with support at $\{p\}$. Then we have classes $[A_\bullet] \in K_0^Y(X)$, $[B_\bullet] \in K_0^Z(X)$. The assignment

$$[A_\bullet] \otimes [B_\bullet] \mapsto [A_\bullet \otimes B_\bullet] \in K_0^{\{p\}}(X) = K_0(\mathcal{C})$$

defines a pairing

$$K_0^Y(X) \otimes K_0^Z(X) \to K_0(\mathcal{C}),$$

such that on composing with the natural map

$$K_0(\mathcal{C}) \to G_0^{\{p\}}(X) \cong G_0(\{p\}) = \mathbb{Z},$$

the induced pairing $K_0^Y(X) \otimes K_0^Z(X) \to \mathbb{Z}$ is just the intersection product

$$[A_\bullet] \otimes [B_\bullet] \mapsto \chi(A_\bullet \otimes B_\bullet) \in \mathbb{Z}.$$

It suffices to prove that the pairing

$$K_0^Y(X) \otimes K_0^Z(X) \to K_0(\mathcal{C})$$

vanishes, whenever $Y, Z \subset X$ are closed subschemes with $Y \cap Z = \{p\}$, and $\dim Y + \dim Z < d = \dim X$. It suffices to prove this after tensoring with $\mathbb{Q}$.

Following the argument in [5], the vanishing of this pairing is deduced from properties of the Adams operations. It is shown in [4], [5] that for any closed subscheme S of a scheme T (which is, say, quasi-projective over a Noetherian regular ring), there are Adams operations $\psi_k : K_0^S(T) \to K_0^S(T)$ for each $k \geq 1$, satisfying expected properties (see [5], §4). In particular, if $S_1 \subset S_2 \subset T$ are closed subschemes, the natural map $K_0^{S_1}(T) \to K_0^{S_2}(T)$ commutes with the Adams operations; if $f : T' \to T$ is any morphism, $S' = f^{-1}(S)$, then the natural pull-back map $f^* : K_0^S(T) \to K_0^{S'}(T')$ also commutes with the Adams operations. In particular, given $S_1 \subset S_2 \subset T$ as above, the maps in the localization exact sequence

$$K_0^{S_1}(T) \to K_0^{S_2}(T) \to K_0^{S_2 \setminus S_1}(T \setminus S_1)$$

commute with the Adams operations. Finally, the pairings $K_0^{S_1}(T) \otimes K_0^{S_2}(T) \to K_0^{S_1 \cap S_2}(T)$ commute with the Adams operations, for any closed subschemes $S_1, S_2 \subset T$.

Let $F^p K_0^Y(X)$ denote the sum of the images of maps $K_0^{Y'}(X) \to K_0^Y(X)$ where $Y' \subset Y$ runs over all closed subschemes which have codimension $\geq p$ in X. Thus $F^d K_0^Y(X) = \operatorname{image} K_0(\mathcal{C}) \to K_0^Y(X)$. Clearly each $F^p K_0^Y(X)$ is stable under all the Adams operations, so there are induced operations on the subquotients $F^p K_0^Y(X)/F^q K_0^Y(X)$ for $q \geq p$.

Lemma 5. *Suppose $X = \operatorname{Spec} R$ has a singular locus $S \neq \emptyset$. Then for any $k > 1$, the Adams operator ψ_k on $(K_0^Y(X)/(\operatorname{image} K_0^{Y \cap S}(X)) \otimes \mathbb{Q}$, has eigenvalues k^i for some i with* $\operatorname{codim} Y \leq i \leq d-1$.

Proof: Clearly $F^dK_0^Y(X) \subset \text{image}\, K_0^{Y\cap S}(X)$. Hence it suffices to show that ψ_k acts as multiplication by k^p on

$$\frac{F^pK_0^Y(X)}{F^{p+1}K_0^Y(X) + F^pK_0^Y(X) \cap (\text{image}\, K_0^{Y\cap S}(X))}$$

for $1 \leq p \leq d-1$. The assertion for p follows from a similar assertion with Y replaced by an arbitrary closed subscheme Z which has codimension p in X, since $F^pK_0^Y(X)$ is generated by the images of all such $K_0^Z(X)$. We thus reduce to the case $p = \operatorname{codim} Y$, so that $F^pK_0^Y(X) = K_0^Y(X)$. Then, as in the proof of Lemma 5.2 of [5], a direct limit of localization sequences yields an exact sequence

$$0 \to (F^{p+1}K_0^Y(X) + \text{image}\, K_0^{Y\cap S}(X)) \to K_0^Y(X) \to \bigoplus_{x\in(Y\setminus S)\cap X^{(p)}} K_0^{\{x\}}(\mathcal{O}_{x,X})$$

where $X^{(p)}$ is the set of points of codimension p in X (since $X = \operatorname{Spec} R$ is not regular, we may not have surjectivity of the last arrow, unlike the situation in *loc. cit.*).

The local rings $\mathcal{O}_{x,X}$ above are each regular of dimension p. Hence $K_0^{\{x\}}(\mathcal{O}_{x,X}) = \mathbb{Z}$ generated by a Koszul complex giving a minimal free resolution of the residue field; as in the proof of Proposition 5.3 of [5], ψ_k acts on this class by multiplication by k^p. $\square$

Lemma 6. *The intersection pairing*

$$K_0^Y(X) \otimes K_0^Z(X) \to K_0(\mathcal{C})$$

factors through a pairing between quotients

$$K_0^Y(X)/F^dK_0^Y(X) \otimes K_0^Z(X)/F^dK_0^Z(X) \to K_0(\mathcal{C}).$$

Proof: Since the roles of Y and Z are symmetric, it suffices to show that the intersection pairing vanishes on $F^dK_0^Y(X) \otimes K_0^Z(X)$. Equivalently, the pairing $K_0(\mathcal{C}) \otimes K_0^Z(X) \to K_0(\mathcal{C})$ vanishes. The pairing $K_0(\mathcal{C}) \otimes K_0^Z(X) \to K_0(\mathcal{C})$ clearly factors through $K_0(\mathcal{C}) \otimes K_0(X) \to K_0(\mathcal{C})$. But since $\dim Z < d = \dim X$, and $K_0(X) = \mathbb{Z}$ through the rank homomorphism, the natural map $K_0^Z(X) \to K_0(X)$ vanishes. $\square$

In particular, Theorem 2 is trivial if $d = \dim R \leq 2$. Hence we may assume $d \geq 3$. We now restrict attention to the case when R has an isolated singularity.

Lemma 7. *Assume R has an isolated singularity. Then the Adams operation ψ_k on $K_0(\mathcal{C}) \otimes \mathbb{Q}$ has eigenvalues k^i for some integers i satisfying $1 \leq i \leq d$.*

Proof: Since $X = \operatorname{Spec} R$ where R is a local ring, we have $K_0(X) = \mathbb{Z}$. From the localization exact sequence

$$K_1(X) \to K_1(X \setminus \{p\}) \to K_0(\mathcal{C}) \to K_0(X) \to K_0(X \setminus \{p\})$$

we thus obtain a surjection $K_1(X \setminus \{p\}) \to K_0(\mathcal{C})$, compatible with the actions of Adams operations ([9]).

Since $X \setminus \{p\}$ is regular, of Krull dimension $d-1$, it has been shown by Soulé [15] that the action of ψ_k on $K_1(X \setminus \{p\}) \otimes \mathbb{Q}$ has eigenvalues k^i, where i must satisfy $1 \leq i \leq d$. $\square$

It follows from Lemma 5 that if $\dim Y + \dim Z < d = \dim X$, and $S = \{p\}$, then

$$\text{image}\,\big(K_0^Y(X)/F^dK_0^Y(X)) \otimes (K_0^Z(X)/F^dK_0^Z(X)) \otimes \mathbb{Q} \to K_0(\mathcal{C}) \otimes \mathbb{Q}\big)$$

is a subspace on which ψ_k has eigenvalues k^i, for each k, where i must satisfy $i \geq \operatorname{codim} Y + \operatorname{codim} Z > d$. This contradicts lemma 7, unless this image is 0, which is the desired conclusion.

It is not clear to us if the similar (stronger) vanishing result, the analog of Theorem 2 when $\dim S = 1$ (see [13]), can also be obtained by a similar argument.

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