

Homological Conjectures for Rings of Mixed Characteristic

Paul Roberts

October 20, 2005

We consider various questions for rings of mixed characteristic that are current topics of research.

1. Homological Conjectures
2. Local cohomology
3. Results of Heitmann and Faltings
4. Almost zero modules
5. Fontaine rings

Homological Conjectures

We consider the Monomial Conjecture, which states that for a commutative Noetherian local ring R with system of parameters x_1, \dots, x_d we have

$$x_1^t \cdots x_d^t \notin (x_1^{t+1}, \dots, x_d^{t+1}).$$

This is actually easy but useless in characteristic zero. It is proven using the Frobenius map in positive characteristic. It is not known in mixed characteristic for dimension larger than 3.

There are several related conjectures that are equivalent to or follow from this one; one of their common properties is that they are related to Cohen-Macaulay properties and local cohomology.

Cohen-Macaulay rings and Local Cohomology

We recall that elements x_1, \dots, x_d form a *regular sequence* if $ax_i \in (x_1, \dots, x_{i-1})$ implies $a \in (x_1, \dots, x_{i-1})$ for $i = 1, \dots, d$. A ring is *Cohen-Macaulay* if every system of parameters is a regular sequence.

If R is Cohen-Macaulay, the Monomial Conjecture is an easy exercise. We recall the characterization of Cohen-Macaulay rings in terms of local cohomology.

Let x_1, \dots, x_d be a system of parameters, and let \mathfrak{m} be the maximal ideal of R . The i th *local cohomology* of R , denoted $H_{\mathfrak{m}}^i(R)$, is the limit of homology of Koszul complexes $K_{\bullet}(x_1^n, \dots, x_d^n)$.

Equivalently, it is the cohomology of the complex

$$0 \rightarrow R \rightarrow \prod R_{x_i} \rightarrow \prod R_{x_1 x_j} \rightarrow \cdots \rightarrow R_{x_1 x_2 \cdots x_d} \rightarrow 0.$$

R is Cohen-Macaulay if and only if $H_{\mathfrak{m}}^i(A) = 0$ for $i < d$.

Results of Heitmann and Faltings

Heitmann's results: Let R be a Noetherian local normal domain of dimension 3 of mixed characteristic p . Let p, x, y be a system of parameters for R . If $p^N a \in (x, y)$ for some N , then for all positive integers n we have $p^{1/n} a \in (x, y)S$ for some finite extension S of R .

One interpretation: "full extended plus closure has the colon-capturing property" ; i.e. for a valuation v , if $ax_i \in (x_1, \dots, x_{i-1})$ then ηa in (x_1, \dots, x_{i-1}) for elements η of with $v(\eta)$ arbitrarily small.

This is enough to prove the Monomial Conjecture.

Two ingredients in the proof:

1. The local cohomology $H_{\mathfrak{m}}^2(R)$ is a module of finite length.
2. Divisibility properties of binomial coefficients.

Faltings' results: Let S be a complete local normal domain of mixed characteristic and dimension at least 3. Let R be a power series ring $V[[X_2, \dots, X_d]]$ over a discrete valuation ring V such that S is a finite extension of R . Let

$$R_{\infty} = \cup_n R[p^{1/p^n}, X_2^{1/p^n}, \dots, X_d^{1/p^n}].$$

Let S_{∞} be the normalization of the ring generated by R_{∞} and S in some field extension.

There is a length function defined on R_∞ -modules such that, for example, $R_\infty/\mathfrak{m}_R R_\infty$ has length 1.

Assume S becomes étale over R after inverting p .

Then $H_{\mathfrak{m}_R}^2(S_\infty)$ has length zero.

Almost zero modules

Both of these approaches involve infinite extensions (at least to get small roots of p). Let R^+ denote the absolute integral closure of R , i.e. the integral closure of R in the algebraic closure of its quotient field.

Let v be a valuation on R^+ that is positive on \mathfrak{m}_R . We say that a module M is *almost zero* if for all m in M , for all $\epsilon > 0$, there is an element $a \in A$ with $v(a) < \epsilon$ and $am = 0$.

Then Heitmann's theorem implies that in dimension 3, $H_{\mathfrak{m}}^2(R^+)$ is almost zero.

A general question: for any regular local ring R of dimension d , is $H_{\mathfrak{m}}^i(R^+)$ almost zero for $i < d$? Hochster and Huneke proved that it is actually zero in positive characteristic. (It is easy to prove it is almost zero in this case).

K. Shimomoto investigated whether the image of $H_{\mathfrak{m}}^2(S)$ in $H_{\mathfrak{m}}^2(R^+)$ for a finite extension S of R has length zero. He proved an inequality that shows that length decreases going to R^+ (without any assumptions on anything being étale).

We next describe an approach via the Fontaine ring.

The Fontaine ring of R^+

We define

$$E(R^+) = \varprojlim R_n,$$

where $R_n = R^+/pR^+$ for all n , and the map from R_{n+1} to R_n sends \bar{r} to \bar{r}^p , i.e. it is the Frobenius map on R^+/pR^+ .

$E(R^+)$ is the *Fontaine ring* of R_∞ .

Facts about $E(R^+)$:

1. $E(A)$ can be defined for any ring A ; but it is interesting mainly for rings of mixed characteristic containing a lot of p th roots.
2. $E(R^+)$ is a perfect ring of characteristic p .
3. Let $\langle p \rangle = (p, p^{1/p}, p^{1/p^2}, \dots)$. Then
$$E(R^+)/\langle p \rangle E(R^+) \cong R^+/pR^+.$$

For $i = 2, \dots, d$, let $T_i = (X_i, X_i^{1/p}, X_i^{1/p^2}, \dots)$.

Then $E(R^+)$ contains the power series ring $k[[\langle p \rangle, T_2, \dots, T_d]]$, which we denote T .

One can reconstruct the p -adic completion

$$\hat{R}^+ = \varprojlim R^+ / p^n R^+$$

as the quotient of the ring of Witt vectors on $E(R^+)$ by a principal ideal

$$\hat{R}^+ = W(E(R^+)) / (\langle p \rangle - p).$$

Recall that the ring of Witt vectors $W(E(R^+))$ is a ring of mixed characteristic p such that $W(E(R^+)) / pW(E(R^+)) = E(R^+)$.

This reduces question to whether $E(R^+)$ is Cohen-Macaulay or almost Cohen Macaulay.

Although $E(R^+)$ is a ring of characteristic p that contains a power series ring T , it is not integral over T ; if it were we would know immediately that it was almost Cohen-Macaulay.

To study $E(R^+)$ we need to express it as a union of subrings that are more understandable. In particular, they need to be constructed from subrings of R^+ and have a lot of p^n th roots. This leads to the study of homomorphic images of rings of the form

$$\cup_n V[[p^{1/p^n}, Y_2^{1/p^n}, \dots, Y_m^{1/p^n}]]$$

Clearly R^+ is a union of such subrings, and $E(R^+)$ is a union of $E(A)$ where A runs over these rings.

One main problem is to find the integral closure of a ring of this type.

An example: $V[X, Y, Z, W]/(XY - ZW)[\eta]$ with $\eta^2 = \prod_{i=1}^4 (X - \alpha_i Z)$. Its integral closure is generated by $(W/X)_\eta$ and $(W^2/X^2)_\eta$ and it is a non-Cohen-Macaulay integrally closed domain.

The local cohomology is generated by one element. The corresponding element becomes zero in R^+ in characteristic p and almost zero in characteristic zero. It is still not clear what happens to it in mixed characteristic.