Comparing cycles in positive and mixed characteristic (Preliminary report)

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The problem we consider is to try to set up a correspondence between cycles over a regular local ring of mixed characteristic with cycles over a regular local ring of positive characteristic.

The motivation comes from Serre's conjectures on intersection mutiplicities, which are not completely known in mixed charateristic.

The method is to attempt to apply a construction of Fontaine that goes between rings of positive and of mixed characteristic.

Let $R = V[[X_2, \ldots, X_d]]$, where V is a discrete valuation ring ring with maximal ideal generated by the prime number p and perfect residue field k.

R is a regular local ring of mixed characteristic of dimension d.

Let $S = k[[T_1, \ldots, T_d]]$.

S is a regular local ring of positive characteristic of dimension d.

We would like to set up a correspondence between cycles over R and S. We show what we can actually do in this direction.

The Fontaine ring of R

Let $R_\infty = \cup_n R[[p^{1/p^n}, X_1^{1/p^n}, \ldots, X_d^{1/p^n}]]$. We then define

$$E(R_\infty) = \varprojlim R_n,$$

where $R_n = R_\infty/pR_\infty$ for all n, and the map from R_{n+1} to R_n sends $\overline{r}$ to $\overline{r}^p$, i.e. it is the Frobenius map.

$E(R_\infty)$ is the *Fontaine ring* of R_∞.

We denote an element of $E(R_\infty)$ by $(r_0, r_1, \ldots)$ with $r_i \in R_\infty$ (taken modulo p) and $r_{i+1}^p \equiv r_i$ modulo p.

Obvious facts:

1. $E(R_\infty)$ is a perfect ring of characteristic p.

2. The construction is functorial.

Some not so obvious facts:

1. $E(R_\infty)$ contains k.

2. Let

   $T_1 = (p, p^{1/p}, p^{1/p^2}, \ldots)$,

   $T_i = (X_i, X_i^{1/p}, X_i^{1/p^2}, \ldots)$ for $i = 2, \ldots, d$.

   Then $E(R_\infty)$ contains the power series ring $k[[T_1, \ldots, T_d]]$, which we denote S.

   Let S_∞ be the perfect closure of S, that is, the ring obtained by adjoining all p^nth roots of elements of S. Then

$$E(R_\infty) \cong \varprojlim S_\infty / T_1^n S_\infty.$$

Going back from $E(R_\infty)$ to R_∞

We can't quite reconstruct R_∞ from $E(R_\infty)$, but we can reconstruct the p-adic completion of R_∞,

$$\hat{R}_\infty = \varprojlim R_\infty / p^n R_\infty.$$

The first step is to take the ring of Witt vectors $W(E(R_\infty))$; since $E(R_\infty)$ is a perfect ring of positive characteristic, this is a reasonable operation.

Then: There is a surjective map

$$W(E(R_\infty)) \to \hat{R}_\infty$$

with kernel generated by $T_1 - p$. Thus the p-adic completion of R_∞ can be recovered from $E(R_\infty)$.

We now reformulate the question: is there a correspondence between cycles on $E(R_\infty)$. and cycles on $\hat{R}_\infty$?

The correspondence: to a prime ideal $\mathfrak{p}$ in $\hat{R}_\infty$ we take the kernel of $E(\hat{R}_\infty)$ to $E(\hat{R}_\infty/\mathfrak{p})$ and take the cycle it defines.

Question: does it actually define a cycle, can the kernel be decomposed into a finite sum of prime ideals?

To a prime ideal in $E(R_\infty)$ we take the associated prime ideal in the ring of Witt vectors then divide by $T_1 - p$. Again we have the question of whether we get a finite sum of prime ideals.

The question remains whether this gives an interesting correspondence.

Two simple examples

Let $R = V[[X_2, X_3, X_4]]$

First, take the ideal generated by $X_2 - X_3$. This is a prime ideal in R, but not in R_∞ as it is divisible by $X_2^{1/p^n} - X_3^{p^n}$ for all n. This corresponds to the ideal in $E(R_\infty)$ generated by $T_2 - T_3$.

Second, take the ideal generated by $X_2 + X_3 + X_4$. This ideal is again clearly prime in R, and it is not hard to show that it remains prime in R_∞. It is not clear, however, that it remains prime in $\widehat{R}_\infty$; modulo p it of course has p^nth roots for all p.

If one is really going to study intersection conjectures. For this to give anything one would have to study

$$\widehat{R}_\infty/\mathfrak{p}$$

where $\mathfrak{p}$ is a minimal prime ideal over the ideal generated by an Eisenstein polynomial in R.