Math 6320
Assignment 6
April 15, 2009

1. Let K be a field of characteristic not equal to 2 and let $E = K[\sqrt{a}]$, where a is an element of K that is not a square in K. Show that the only elements of K that are squares in E are of the form b^2 or $b^2 a$ for some $b \in K$.

2. Show directly (without using Kummer Theory) that if $p_1, \dots, p_n$ are distinct primes in $\mathbb{Z}$, then $\mathbb{Q}[\sqrt{p_1}, \dots, \sqrt{p_n}]$ is Galois with Galois group $(\mathbb{Z}/2\mathbb{Z})^n$.

3. Show that for each $n \geqslant 0$ there is a Galois extension of $\mathbb{Q}[i]$ with Galois group $(\mathbb{Z}/4\mathbb{Z})^n$.

4. Use the method of reduction modulo a prime number to construct an irreducible quartic polynomial $f(X)$ over $\mathbb{Q}$ such that the Galois group of $f(X)$ contains a 3-cycle.

5. Let E be the normal closure of an extension $\mathbb{Q}[\alpha]$ of $\mathbb{Q}$. Show that for any prime p dividing $[E : \mathbb{Q}]$ there is a subfield F of E with $[E : F] = p$ and $E = F[\alpha]$.

6. Let $R = k[X, Y]$, where k is a field, and let M be the ideal (X, Y) considered as an R-module. Show that there is an exact sequence
$$0 \longrightarrow R \longrightarrow R \oplus R \longrightarrow M \longrightarrow 0.$$

7. Use Problem 6 to compute $M \otimes_R M$ and find a nonzero element a of $M \otimes_R M$ such that $Xa = Ya = 0$.

8. Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ as rings.