Math 6320
Assignment 5
April 1, 2009

1. A field K of positive characteristic p is *perfect* if the Frobenius map $x \longrightarrow x^p$ is an isomorphism (which is true if and only if it is surjective). Prove that K is perfect if and only if every algebraic extension of K is separable.

2. Compute the Galois group of the polynomial $X^4 + 4X^2 + 2 \in \mathbb{Q}[X]$.

3. For which positive integers n does a primitive nth root of unity have degree 2 over $\mathbb{Q}$?

4. Let E/K be an extension of finite fields. Show that the norm and trace from E to K are surjective.

5. If $f(X) \in \mathbb{Q}[X]$ is irreducible with roots $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, prove that $\alpha_i - \alpha_j \notin \mathbb{Q}$ for all $i \neq j$.

6. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$. Let α be an element of $\overline{\mathbb{Q}}$ which is not in $\mathbb{Q}$, and let E be an extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$ maximal with respect to the property that $\alpha \notin E$. Show that every finite extension of E is cyclic.

7. If $a, b \in \mathbb{Q}$ satisfy $a^2 + b^2 = 1$, use Hilbert's Theorem 90 to prove that

$$a = \frac{s^2 - t^2}{s^2 + t^2} \qquad \text{and} \qquad b = \frac{2st}{s^2 + t^2} \qquad \text{for some } s, t \in \mathbb{Q}\,.$$

   This shows that the only right angled triangles with integer sides are

$$d(s^2 - t^2), \quad 2dst, \quad d(s^2 + t^2), \quad \text{for } d, s, t \in \mathbb{Z}\,.$$

8. If $n = p^r$ for a prime number p and ω is a primitive nth root of unity, show that the norm $N_{\mathbb{Q}}^{E}(1 - \omega) = p$, where $E = \mathbb{Q}[\omega]$.