Math 6320
Assignment 3
February 25, 2009

In these problems K always denotes a field.

1. If $[K(a) : K]$ is an odd integer, prove that $K(a^2) = K(a)$.

2. Let $a \in K(X) \setminus K$, where $K(X)$ is the field of rational functions in one variable over the field K. Prove that $K(X)$ is an algebraic extension of the field $K(a)$.

3. Let $f(X) \in K[X]$ be an irreducible polynomial, where K has characteristic $p > 0$. Show that $f(X) = g(X^{p^e})$, where $g(X) \in K[X]$ is an irreducible polynomial with distinct roots. Deduce that every root of $f(X)$ has the same multiplicity p^e.

4. For each of the polynomials below, determine the extension degree of the splitting field:

   (a) $X^5 - 7 \in \mathbb{Q}[X]$.

   (b) $X^p - X - 1 \in \mathbb{F}_p[X]$.

5. Let K be a field of characteristic $p > 0$, and consider the field of rational functions in two variables $M = K(X, Y)$ and its subfield $L = K(X^p, Y^p)$. Show that $[M : L] = p^2$. Prove that the extension $L \subset M$ has no primitive element.

6. Find the minimal polynomial of $\sqrt{2} + \sqrt{5}$ over $\mathbb{Q}$.

7. Let $K = \mathbb{Z}/2\mathbb{Z}(s, t)$ (the field of rational functions in two variables over $\mathbb{Z}/2\mathbb{Z}$, and let α be a root of the polynomial $X^4 + sX^2 + t$. Show that $K[\alpha]$ is not separable over K but that it contains no nontrivial purely inseparable extensions of K.

8. Let F be a finite field. Show that every element of F can be written as the sum of two squares in F.