Math 6320
Assignment 2
February 11, 2009

1. Determine all maximal ideals of the ring $\mathbb{Z}[x]/(120; x^3+1)$.

2. Let A be a Noetherian integral domain. Show that A is a PID if and only if every pair of nonzero elements a and b of A have a greatest common divisor that can be written in the form $ra+sb$ for some r and s in A.

3. Let $f(X) \in K[X]$ be an irreducible polynomial, where K has characteristic $p > 0$. Let E be a field containing K. Show that if $f(X)$ has a multiple root in E, then we can write $f(X) = g(x^p)$, where $g(x) \in K[x]$.

4. Show that $X^5 - 5X + 1$ is irreducible over $\mathbb{Q}$.

5. Show that $\mathbb{Z}[X_1, \ldots, X_n]$ is a free $\mathbb{Z}[s_1, \ldots, s_n]$-module with basis $\{X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n}\}$ with $0 \leqslant k_i \leqslant n-i$ ($s_1, \ldots s_n$ are the elementary symmetric polynomials.)

6. Show that the polynomial $(X-1)(X-2)\cdots(X-n)-1$ is irreducible over $\mathbb{Q}$ for $n \geqslant 1$. (Hint: consider its values at $1, 2, \ldots, n$.)

7. Show that $1+X$ has a square root in $\mathbb{Q}[[X]]$ but not in $\mathbb{Q}[X]$.

8. Show that the *abc* conjecture implies that there are infinitely many primes p with $2^{p-1} \not\equiv 1$ modulo p^2.