Math 6320
Assignment 1
January 28, 2009

1. Let $F = \mathbb{Z}/2\mathbb{Z}(s,t)$, where s and t are indeterminates, so that F is a field of rational functions in two variables over the field of two elements. Show that the polynomial $f(X) = X^4 + sX^2 + t$ is irreducible over F.

2. Let $f(X)$ be a polynomial with rational coefficients. Show that $f(n)$ is an integer for all $n \in \mathbb{Z}$ if and only if $f(X)$ is a linear combination with integer coefficients of the polynomials $\binom{X}{k} = \dfrac{X(X-1)\cdots(X-k+1)}{k!}$ for some non-negative integers k.

3. Let F be a field with q elements, and let $f(X_1,\ldots,X_n)$ be a polynomial in n variables with coefficients in F. Assume that $f(0,\ldots,0) = 0$ and that the degree of f is less than n.

   a. Assume that $f(a_1,\ldots,a_n) \neq 0$ for all $(a_1,\ldots,a_n)$ with at least one a_i not zero. Show that every such $(a_1,\ldots,a_n)$ is a root of the polynomial $1 - f(X_1,\ldots,X_n)^{q-1}$.

   b. By comparing the degree of the reduced form of $1 - f(X_1,\ldots,X_n)^{q-1}$ with that of the polynomial $(1 - X_1^{q-1})\cdots(1 - X_n^{q-1})$, show that we must have $f(a_1,\ldots,a_n) = 0$ for some $(a_1,\ldots,a_n)$ with at least one a_i not zero.

4. Write the polynomial $X_1^3 + X_2^3 + X_3^3$ as a polynomial in the elementary symmetric polynomials $X_1 + X_2 + X_3, X_1X_2 + X_1X_3 + X_2X_3, X_1X_2X_3$.

5. Show that an element of a commutative ring A is nilpotent if and only if it is in every prime ideal of A.

6. Show that if x is a nilpotent element of A and u is a unit, then $u + x$ is a unit.

7. Let $f(X) = a_0 + a_1X + \cdots + a_nX^n$ be an element of $A[X]$. Show that $f(X)$ is a unit if and only if a_0 is a unit and a_i is nilpotent for $i = 1,\ldots,n$.

8. Let $f(X) = a_0 + a_1X + \cdots + a_nX^n$ be an element of $A[X]$. Show that $f(X)$ is a zero-divisor in $A[X]$ if and only if there is an element $a \neq 0$ in A such that $af(X) = 0$. [Hint: Choose a nonzero polynomial $g(X) = b_0 + b_1X + \cdots + b_mX^m$ of least degree such that $f(X)g(X) = 0$. Then $a_nb_m = 0$, show that this implies that $a_ng(X) = 0$. Then show by induction that $a_{n-r}g(X) = 0$ for $r = 0,\ldots,n$.]