Math 3150-1 Assignment 7 Solutions

Section 4.2

4. Since the initial condition is \( f(r) = 0 \) and \( g(r) = J_0(\alpha_3 r) \), which is a function that occurs as a factor in a product solution with \( \lambda = \alpha_3 \) (since \( a = 1 \)), the solution will be a multiple of \( J_0(\alpha_3 r) \sin(\alpha_3 t) \) (since \( c = 1 \)). The partial derivative of \( J_0(\alpha_3 r) \sin(\alpha_3 t) \) is \( \alpha_3 J_0(\alpha_3 r) \cos(\alpha_3 t) \), which when evaluated at \( t = 0 \) gives \( \alpha_3 J_0(\alpha_3 r) \). Thus the solution is

\[
\frac{1}{\alpha_3} J_0(\alpha_3 r) \sin(\alpha_3 t).
\]

If you use the formulas on page 202, it is clear that \( A_n = 0 \), and

\[
B_n = \frac{2}{\alpha_n J_1^2(\alpha_n)} \int_0^1 J_0(\alpha_3 r) J_0(\alpha_n r) r dr.
\]

The orthogonality relations on page 252 say that this integral is zero except when \( n = 3 \), when it is \( \frac{1}{2} J_1^2(\alpha_3) \). Thus this gives the same answer.

7. In this case also \( a = 1 \) and \( c = 1 \), and as in problem 4 the initial condition \( f(r) = J_0(\alpha_3 r) \) says that \( A_n = 0 \) unless \( n = 3 \) and that \( A_3 = 1 \). The initial condition \( g(r) = 1 - r^2 \) implies that

\[
B_n = \frac{2}{\alpha_n J_1^2(\alpha_n)} \int_0^1 (1 - r^2) J_0(\alpha_n r) r dr.
\]

This integral can be evaluated following the method of example 2. First substitute \( s = \alpha_n r \) to get

\[
\frac{2}{\alpha_n J_1^2(\alpha_n)} \int_0^1 (1 - r^2) J_0(\alpha_n r) r dr = \frac{2}{\alpha_n^5 J_1^2(\alpha_n)} \int_0^{\alpha_n} (\alpha_n^2 - s^2) J_0(s) s ds.
\]

Then integrate by parts using \( u = \alpha_n^2 - s^2 \) and \( dv = J_0(s) s ds \) and use the formula \( \int J_0(s) s ds = J_1(s) s + C \) to get

\[
\frac{2}{\alpha_n^5 J_1^2(\alpha_n)} \int_0^{\alpha_n} (\alpha_n^2 - s^2) J_0(s) s ds = \frac{2}{\alpha_n^5 J_1^2(\alpha_n)} \left( \left[ (\alpha_n^2 - s^2) J_1(s) s \right]_0^{\alpha_n} + 2 \int_0^{\alpha_n} J_1(s) s^2 ds \right).
\]

The first term gives zero when evaluated at both 0 and \( \alpha_n \), so, using that \( \int J_1(s) s^2 ds = J_2(s) s^2 \), we get that the above expression is equal to

\[
A_n = \frac{4}{\alpha_n^5 J_1^2(\alpha_n)} \int_0^{\alpha_n} J_1(s) s^2 ds = \frac{4}{\alpha_n^5 J_1^2(\alpha_n)} \left[ J_2(s) s^2 \right]_0^{\alpha_n} = 1.
\]
\[
\frac{4J_2(\alpha_n)\alpha_n^2}{\alpha_n^2 J_1^2(\alpha_n)} = \frac{4J_2(\alpha_n)}{\alpha_n^3 J_1(\alpha_n)^2}.
\]

Thus the solution is

\[
u(r, t) = J_0(\alpha_3 r) + \sum_{n=1}^{\infty} \frac{4J_2(\alpha_n)}{\alpha_n^3 J_1(\alpha_n)^2} J_0(\alpha_n r).
\]

This is a satisfactory formula (it can be calculated on a computer, for instance), but it can be simplified using the formula

\[J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x)\]

for \(p = 1\) to give

\[
u(r, t) = J_0(\alpha_3 r) + \sum_{n=1}^{\infty} \frac{8}{\alpha_n^4 J_1(\alpha_n)} J_0(\alpha_n r).
\]

**Section 4.4**

1. Since \(\cos \theta\) is a function that occurs as a factor in a product solution with \(n = 1\), the solution is \(u(r, \theta) = r \cos \theta\), since \(a = 1\) this gives \(\cos \theta\) when \(r = a\). (You might notice that this can be written \(u(x, y) = x\).

3. In this case we have to find the Fourier series for the periodic function of period \(2\pi\) given by \(f(\theta) = \frac{1}{2}(\pi - \theta)\) for \(0 < \theta < 2\pi\). The periodic extension of this function is odd, so \(a_n = 0\) for all \(n\). We have

\[
b_n = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2} (\pi - \theta) \sin n\theta d\theta.
\]

We integrate by parts using \(u = \pi - \theta\) and \(dv = \frac{1}{2} \sin n\theta d\theta\). Then \(du = -d\theta\) and \(v = -\frac{1}{2n} \cos n\theta\), so the above integral becomes

\[
\frac{1}{\pi} \left( -\left(\pi - \theta\right) \frac{1}{2n} \cos n\theta \right)_0^{2\pi} + \int_0^{2\pi} \frac{1}{2} \sin n\theta d\theta.
\]

Since \(\sin n\theta\) is periodic of period \(2\pi\) the integral on the right is 0, and we are left with

\[
\frac{1}{\pi} \left(-\left(\pi - \theta\right) \frac{1}{2n} \cos n\theta \right)_0^{2\pi} = \frac{1}{2\pi n} \left(-(-\pi)(1) - (-\pi)(1)\right) = \frac{2\pi}{2\pi n} = \frac{1}{n}.
\]

Thus we get

\[
b_n = \frac{1}{\pi} \frac{1}{n} = \frac{1}{n}.
\]
Hence the solution to the Dirichlet problem is

\[ u(r, \theta) = \sum_{n=1}^{\infty} \frac{r^n \sin n\theta}{n}. \]

7. The problem is to find the isotherms in problem 3, and for this we use the formula

\[ \sum_{n=1}^{\infty} \frac{r^n \sin n\theta}{n} = \tan^{-1} \left( \frac{r \sin \theta}{1 - r \cos \theta} \right) \]

discussed in the text. Since \( x = r \cos \theta \) and \( y = r \sin \theta \), the isotherms are given by setting \( \tan^{-1}(y/(1-x)) \) equal to a constant, and this amounts to the same as letting \( y/(1-x) = c \) for some constant \( c \). This gives the set of straight lines through the point \((1, 0)\).

Section 4.7

Bessel’s equation of order \( p \) is

\[ x^2 y'' + xy' + (x^2 - p^2)y = 0, \]

and the associated Bessel function \( J_p(x) \) is

\[ J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + p + 1)} \left( \frac{x}{2} \right)^{2k+p}. \]

If \( p \) is an integer \( \Gamma(k + p + 1) \) is equal to \( (k + p)! \).

1. The equation \( x^2 y'' + xy' + (x^2 - 9)y = 0 \) is Bessel’s equation of order 3 and the first three terms of the solution are

\[ J_3(x) = \frac{1}{3!} \frac{x^3}{2^3} - \frac{1}{4!} \frac{x^5}{2^5} + \frac{1}{2!5!} \frac{x^7}{2^7} - \cdots \]

or

\[ \frac{x^3}{2 \cdot 4 \cdot 6} - \frac{x^5}{(2) \cdot (2 \cdot 4 \cdot 6 \cdot 8)} + \frac{x^7}{(2 \cdot 4) \cdot (2 \cdot 4 \cdot 6 \cdot 8 \cdot 10)} - \cdots. \]

2. The equation \( x^2 y'' + xy' + x^2 y \) is Bessel’s equation of order 0 and the first three terms of its series are

\[ J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \cdots \]