Math 3150-1 Assignment 6 Solutions

Section 3.9

1. The solution, as in Example 1 in the text, is the double sum of $E_{mn} \sin m\pi x \sin n\pi y$, where in this case

$$E_{mn} = \frac{-4}{m^2\pi^2 + n^2\pi^2} \int_0^1 \int_0^1 x \sin m\pi x \sin n\pi y dy dx.$$ 

We have

$$\int_0^1 \sin n\pi y dy = (-1/n\pi) \cos n\pi y\bigg|_0^1 = (-1/n\pi)((-1)^n - 1),$$

which is $2/(n\pi)$ if $n$ is odd and 0 if $n$ is even.

$$\int_0^1 x \sin m\pi x dx = \frac{1}{m^2\pi^2} \sin m\pi x - \frac{x}{m\pi} \cos m\pi x\bigg|_0^1 = \frac{(-1)^{m+1}}{m\pi}.$$ 

Putting these together in a double series, we obtain

$$u(x, y) = \frac{8}{\pi^4} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^m \sin m\pi x \cos(2k + 1)\pi y}{m(2k + 1)(m^2 + (2k + 1)^2)}.$$ 

2. This problem is similar to the previous one, but now

$$E_{mn} = \frac{-4}{m^2\pi^2 + n^2\pi^2} \int_0^1 \int_0^1 \sin 2\pi x \sin m\pi x \sin n\pi y dy dx.$$ 

We know that

$$\int_0^1 \sin 2\pi x \sin m\pi x dx = 0$$

unless $m = 2$, in which case it is $\frac{1}{2}$. Thus $E_{mn} = 0$ for $m \neq 2$ and

$$E_{2n} = \frac{-2}{4\pi^2 + n^2\pi^2} \int_0^1 \sin n\pi y dy,$$

and as we have seen the integral gives $2/n\pi$ if $n$ is odd and 0 if $n$ is even. Thus the solution is

$$u(x, y) = \frac{-4}{\pi^3} \sum_{k=0}^{\infty} \frac{\sin 2\pi x \sin(2k + 1)\pi y}{(2(2k + 1)((2k + 1)^2 + 4)).}$$
Section 4.1

1. In polar coordinates we have
\[ u(r, \theta) = \frac{x}{x^2 + y^2} = \frac{r \cos \theta}{r^2} = \frac{\cos \theta}{r}. \]
The Laplacian is
\[ \nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = \frac{2 \cos \theta}{r^3} + \frac{1}{r} \left( \frac{-\cos \theta}{r^2} \right) + \frac{1}{r^2} \left( \frac{-\cos \theta}{r} \right) = (2 - 1 - 1) \frac{\cos \theta}{r^3} = 0. \]
This function satisfies Laplace’s equation.

3. Here \( u(r, \theta) = 1/r \). So
\[ \nabla^2 u = u_{rr} + \frac{1}{r} u_r = \frac{2}{r^3} + \frac{1}{r} \left( -\frac{1}{r^2} \right) = \frac{1}{r^3}. \]
This is not a solution of Laplace’s equation.

11. a. Since \( \frac{\partial^2 (cu + dv)}{\partial x^2} = c \frac{\partial^2 u}{\partial x^2} + d \frac{\partial^2 v}{\partial x^2} \) for any constants \( c \) and \( d \) and functions \( u \) and \( v \), and similarly for \( y \), we have \( \nabla^2 (cu + dv) = c \nabla^2 u + d \nabla^2 v \) for any constants \( c \) and \( d \). Thus if \( u \) and \( v \) are harmonic, we have
\[ \nabla^2 (\alpha u + \beta v) = \alpha \nabla^2 u + \beta \nabla^2 v = \alpha \cdot 0 + \beta \cdot 0 = 0, \]
so \( \alpha u + \beta v \) is harmonic.

b. Let \( u(x, y) = v(x, y) = x \). Then \( \nabla^2 u = \frac{\partial^2 (x)}{\partial x^2} + \frac{\partial^2 (x)}{\partial y^2} = 0 \), so \( u \) and \( v \) are harmonic. But \( uv(x, y) = x^2 \), and \( \nabla^2 (x^2) = 2 + 0 = 2 \), so \( uv \) is not harmonic. (There are many more examples).

c. Suppose that \( u \) and \( u^2 \) are both harmonic. The key to this problem is to compute the Laplacian of \( u^2 \) as follows.
\[ \frac{\partial (u^2)}{\partial x} = 2u \frac{\partial u}{\partial x} \]
by the chain rule. Thus using the product rule we get
\[ \frac{\partial^2 (u^2)}{\partial x^2} = 2 \left( \frac{\partial u}{\partial x} \right)^2 + 2u \frac{\partial^2 u}{\partial x^2}. \]
Adding the similar expression for the second partial with respect to \( y \) gives
\[ \nabla^2 (u^2) = 2 \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right) + 2u \nabla^2 u. \]
Now if $\nabla^2 u$ and $\nabla^2 (u^2)$ are both zero, we have

$$2 \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right) = 0.$$ 

This implies that $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are both zero, so $u$ is constant.

Section 4.2

1. Since $a = 2$, $c = 1$, $f(r) = 0$, and $g(r) = 1$, we will have a solution of the form

$$u(r, t) = \sum_{n=1}^{\infty} B_n J_0((\alpha_n/2)r) \sin((\alpha_n/2)t),$$

where

$$B_n = \frac{2}{4(\alpha_n/2)J_1(\alpha_n)^2} \int_0^{\alpha_n} J_0((\alpha_n/2)r)rdr = \frac{1}{\alpha_n J_1(\alpha_n)^2} \int_0^{\alpha_n} J_0((\alpha_n/2)r)rdr.$$ 

To work out the integral we make the substitution $s = (\alpha_n/2)r$ and get

$$\int_0^{\alpha_n} J_0((\alpha_n/2)r)rdr = \int_0^{\alpha_n} J_0(s) \left( \frac{2}{\alpha_n} \right) s \left( \frac{2}{\alpha_n} \right) ds = \frac{4}{\alpha_n^2} \int_0^{\alpha_n} sJ_0(s)ds.$$ 

Using the fact that $\int sJ_0(s) = sJ_1(s)$, this is equal to

$$\frac{4}{\alpha_n^2} \left[ sJ_1(s) \right]_0^{\alpha_n} = \frac{4}{\alpha_n} J_1(\alpha_n).$$ 

Thus the solution is

$$u(r, t) = \sum_{n=1}^{\infty} \frac{4}{\alpha_n^2 J_1(\alpha_n)} J_0((\alpha_n/2)r) \sin((\alpha_n/2)t).$$

We now consider the problem of graphing the function given by the first 5 terms of this series in Maple for various values of $t$.

First define a function $a(n)$ that gives the $n$th zero of $J_0$ for an integer $n$. This is done by the command:

```maple
a:= n->evalf(BesselJZeros(0,n));
```

One can divide the definition of the function we need into steps or do it all at once. Here is a way to do it in one step:

```maple
u:=(r,t) -> add((4/(a(n)^2*2*BesselJ(1,a(n))))
*BesselJ(0,(a(n)/2)*r)*sin((a(n)/2)*t),n=1..5);
```
This is the function we want. One way to graph it is to graph a cross-section; this has the advantage that one can put the graph for several values of $t$ on one graph and is simpler. It makes it simpler to define a function that draws the graph for a given $t$.

$$H := t \rightarrow \text{plot}(u(r,t), r=-2..2);$$

We can now plot the cross-sections for $t = 0.5, 1, 1.5,$ and $2$ with

```plaintext
display({H(0.5), H(1), H(1.5), H(2)}, insequence=false);
```

The first graphs in the "Graphs" section do this also between 2.5 and 4.

To give a three dimension graph use cylinderplot. The command for the picture at $t = 1$ is

```plaintext
cylinderplot([r, s, u(r, 1)], r=0..2, s=0..2*Pi);
```

This graph is also shown in the graph section.