Math 2280  
Spring 2008  
Test 2

You will have 50 minutes to complete this test. You may not use a calculator or any other electronic device on this test. You will be graded on your work and not necessarily on your answer, so be sure to show as much of your reasoning as possible in your responses. Each question is worth 10 points. If there are two parts, each part will be graded out of five points. Good luck!

1. Find bases for the image and kernel of the matrix

\[ A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 4 & 1 \\ 7 & 9 & 3 \end{bmatrix}. \]

We have that

\[ A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 4 & 1 \\ 7 & 9 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & -4 & -15 \\ 0 & -5 & -25 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{7}{2} \\ 0 & 1 & \frac{15}{4} \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{rref}(A). \]

This means that \( A \) is invertible and therefore \( \text{im}(A) = \mathbb{R}^3 \). We have that a basis for the image of \( A \) is

\[ \left\{ \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix} \right\}. \]

We also know that \( \ker(A) = \{ \mathbf{0} \} \), so technically there is no basis for \( \ker(A) \), but if you put that \( \mathbf{0} \) is a basis for \( \ker(A) \), that will be okay.

2. Find the coordinate vector \([\vec{x}]_{\mathcal{B}}\) of \( \vec{x} \) with respect to the basis \( \mathcal{B} = \{ \vec{v}_1, \vec{v}_2 \} \), where

\[ \vec{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}. \]

We have that

\[ [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \]

where

\[ c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{x}, \]

which is equivalent to the augmented system

\[ \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 1 \\ 0 & -1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & -1 & -4 \end{bmatrix} \Rightarrow c_1 = 3, \ c_2 = 4. \]
Therefore, 

\[ [x]_B = \begin{bmatrix} 3 \\ 4 \end{bmatrix} . \]

3. Find a basis for all polynomials \( f(t) \) in \( P_3 \) such that \( f(1) = 0 \) and 

\[ \int_{-1}^{1} f(t) \, dt = 0. \]

If \( f(t) \in P_3 \), then we can write \( f(t) = a + bt + ct^2 + dt^3 \) for some choice of constants \( a, b, c, \) and \( d \). If \( f(1) = 0 \), then 

\[ a + b + c + d = 0. \]

If

\[ \int_{-1}^{1} f(t) \, dt = 0 \]

then

\[ 0 = \left. \int_{-1}^{1} (a + bt + ct^2 + dt^3) \, dt = at + \frac{1}{2} bt^2 - \frac{1}{3} ct^3 + \frac{1}{4} dt^4 \right|_{-1} \]

\[ = a + \frac{1}{2} b + \frac{1}{3} c + \frac{1}{4} d - \left( -a + \frac{1}{2} b - \frac{1}{3} c + \frac{1}{4} d \right) = 2a + \frac{2}{3} c. \]

If we put these two equations into matrix form, then the coefficient matrix \( A \) is

\[ A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & \frac{2}{3} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -\frac{4}{3} & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{5}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 1 \end{bmatrix}. \]

The solution vector to the corresponding homogeneous system is

\[
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} c \\ \frac{2}{3} c - d \\ c \\ d \end{bmatrix} = c \begin{bmatrix} -\frac{5}{3} \\ \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix},
\]

where \( c \) and \( d \) can be chosen arbitrarily. Therefore,

\[ \left\{ f(t) \in P_3 : f(1) = \int_{-1}^{1} f(t) \, dt = 0 \right\} = \left\{ -\frac{5}{3} c + \left( \frac{2}{3} c - d \right) t + ct^2 + dt^3 : c, d \in \mathbb{R} \right\}, \]

i.e. we can write any polynomial in our set as

\[ -\frac{5}{3} c + \left( \frac{2}{3} c - d \right) t + ct^2 + dt^3 = c \left( -\frac{5}{3} + \frac{2}{3} t + t^2 \right) + d \left( -t + t^3 \right). \]

Notice that the coefficients of the polynomials here are exactly the entries of the vectors above. The vectors above are a basis for \( \text{ker}(A) \) and they are also the coordinate vectors of these polynomials with respect to the basis \( \{1, t, t^2, t^3\} \). This means that these two polynomials,
$-\frac{5}{3} + \frac{2}{3}t + t^2$ and $-t + t^3$ are linearly independent (because their coordinate vectors are) and they obviously span the given set, so they are a basis and the dimension of our set is 2.

4. Determine if the linear transformation $T : P_2 \to P_2$ defined by $T(a + bt + ct^2) = a - bt + ct^2$ is an isomorphism.

One way to do this problem is to think in terms of coordinates. If $f(t) = a + bt + ct^2$ is a polynomial in $P_2$, then its coordinate vector with respect to the standard basis $B = \{1, t, t^2\}$ for $P_2$ is

$$[f]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$  

The mapping in question takes

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ to } \begin{bmatrix} a \\ -b \\ c \end{bmatrix},$$

i.e. it changes the sign of the second component of the coordinate vector. It is easy to see that the matrix for this mapping is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  

Clearly this matrix is invertible (it is diagonal with nonzero diagonal entries), so $T$ is also invertible and therefore an isomorphism.

5. In the plane $V$ defined by the equation $2x_1 + x_2 - 2x_3 = 0$, consider the bases

$$U = \{\vec{u}_1, \vec{u}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}$$

and

$$B = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} \right\}.$$  

Find the change of basis matrix $S$ from $B$ to $U$.

From the lecture, we have that

$$S = \begin{bmatrix} [\vec{v}_1]_U & [\vec{v}_2]_U \end{bmatrix}.$$  

Since $\vec{v}_1 = \vec{u}_1$, it is clear that

$$[\vec{v}_1]_U = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$  

To find the $U$ coordinates of the vector $\vec{v}_2$, we solve the augmented matrix equation

$$\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vdots & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & \vdots & 3 \\ 2 & -2 & \vdots & 0 \\ 2 & 1 & \vdots & 3 \end{bmatrix}.$$
\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & -6 & -6 \\
0 & -3 & -3
\end{bmatrix} \sim 
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix} \Rightarrow [\vec{v}_2]_* = [1 \ 1].
\]

Therefore,
\[
S = \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}.
\]

6. Find the QR factorization of the matrix
\[
A = \begin{bmatrix}
5 & 3 \\
4 & 6 \\
2 & 7 \\
2 & -2
\end{bmatrix}.
\]

Let us denote the columns of \( A \) by \( \vec{v}_1 \) and \( \vec{v}_2 \). Recall that
\[
A = QR = \begin{bmatrix}
\vec{u}_1 & \vec{u}_2
\end{bmatrix} \begin{bmatrix}
\|\vec{v}_1\| & \vec{u}_1 \cdot \vec{v}_2 \\
0 & \|\vec{v}_2\|
\end{bmatrix},
\]
where \( \vec{u}_1 \) and \( \vec{u}_2 \) are obtained from \( \vec{v}_1 \) and \( \vec{v}_2 \) through the Gramm-Schmidt process. First we do the Gramm-Schmidt portion of the work:
\[
\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{25 + 16 + 4 + 4}} \begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix}
\]
\[
\vec{v}_2 = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{bmatrix} 3 \\ 6 \\ 7 \\ -2 \end{bmatrix} - \frac{1}{7} (15 + 24 + 14 - 4) \begin{bmatrix} 5 \\ 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 7 \\ -2 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \\ -4 \end{bmatrix}
\]
\[
\vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{4 + 4 + 25 + 16}} \begin{bmatrix} -2 \\ 2 \\ 5 \\ -4 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -2 \\ 2 \\ 5 \\ -4 \end{bmatrix}.
\]

Therefore,
\[
Q = \frac{1}{7} \begin{bmatrix} 5 & -2 \\ 4 & 2 \\ 2 & 5 \\ 2 & -4 \end{bmatrix}, \quad R = \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}.
\]

7. (a) If \( A \) and \( B \) are orthogonal, is \( B^{-1}AB \) orthogonal as well?
(b) If \( A \) and \( B \) are symmetric, is \( AB^2A \) symmetric as well?
Justify your answer on both parts.

(a) We have
\[(B^{-1}AB)(B^{-1}AB)^T = B^{-1}ABB^T A^T (B^{-1})^T = B^{-1}(B^{-1})^T = B^T (B^T)^T = B^T B = I.\]
This shows that \(B^{-1}AB\) is orthogonal.

(b) We have
\[(AB^2A)^T = A^T (B^2)^T A^T = A^T (BB)^T A^T = A^T B^T B^T A^T = ABBA = AB^2A.\]
This shows that \(AB^2A\) is symmetric.

8. Find the least squares solution of the system \(A\vec{x} = \vec{b}\), where
\[
A = \begin{bmatrix}
 1 & 1 \\
 1 & 0 \\
 0 & 1
\end{bmatrix}
\text{ and } \vec{b} = \begin{bmatrix}
 3 \\
 3 \\
 3
\end{bmatrix}.
\]
To find the least squares solution \(\vec{x}^*\), we solve the normal equations \(A^T A \vec{x}^* = A^T \vec{b}\). We have that
\[
A^T A = \begin{bmatrix}
 1 & 1 & 0 \\
 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
 1 & 1 \\
 1 & 0 \\
 0 & 1
\end{bmatrix}
= \begin{bmatrix}
 2 & 1 \\
 1 & 2
\end{bmatrix}
\]
and
\[
A^T \vec{b} = \begin{bmatrix}
 1 & 1 & 0 \\
 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
 3 \\
 3 \\
 3
\end{bmatrix}
= \begin{bmatrix}
 6 \\
 6
\end{bmatrix}.
\]
The augmented matrix for the normal equations is
\[
\begin{bmatrix}
 2 & 1 & 6 \\
 1 & 2 & 6
\end{bmatrix}
\sim
\begin{bmatrix}
 1 & \frac{1}{2} & 3 \\
 0 & \frac{3}{2} & 3
\end{bmatrix}
\sim
\begin{bmatrix}
 1 & 0 & 2 \\
 0 & 1 & 2
\end{bmatrix}
\Rightarrow \vec{x}^* = \begin{bmatrix}
 2 \\
 2
\end{bmatrix}.
\]

9. Find the determinant of the matrix
\[
A = \begin{bmatrix}
 0 & 2 & 3 & 4 \\
 0 & 0 & 4 \\
 1 & 2 & 3 & 4 \\
 0 & 0 & 3 & 4
\end{bmatrix}.
\]
To compute the determinant, we will expand along the third row, since it is almost completely zeros.
\[
\det(A) = 4 \cdot \det \begin{bmatrix}
 0 & 2 & 3 \\
 1 & 2 & 3 \\
 0 & 0 & 3
\end{bmatrix}
= 4 \cdot 3 \cdot \det \begin{bmatrix}
 0 & 2 \\
 1 & 2
\end{bmatrix}
= 4 \cdot 3 \cdot (0 - 1) = -24.
\]
10. Find the area of the parallelogram defined by
\[
\begin{bmatrix}
3 \\
7
\end{bmatrix}
\text{ and }
\begin{bmatrix}
8 \\
2
\end{bmatrix}.
\]

We form a matrix $A$ with these two vectors as columns,
\[
A = \begin{bmatrix}
3 & 8 \\
7 & 2
\end{bmatrix}.
\]

We proved in class that the volume $V$ of the parallelogram is given by
\[
V = |\det(A)| = |6 - 56| = |-50| = 50.
\]